A TREATISE

ON

PRACTICAL ASTRONOMY,

AS APPLIED TO

GEODESY AND NAVIGATION.

BY

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PREFACE.

THE following work is designed as a text-book for universities and technical schools, and as a manual for the field astronomer. The author has not sought after originality, but has attempted to present in a systematic form the most approved methods in actual use at the present time.

Each subject is developed as fully as the necessities of the case are likely to require; but as the work is designed to be a practical one, those methods and developments which have merely a theoretical or historic interest have been excluded.

Very complete numerical examples are given illustrative of all the prominent subjects treated. These have been selected with care from records of work actually performed, and will show what may be expected in circumstances ordinarily favorable.

Such auxiliary tables as are applicable only to special problems will be found in the body of the work; those which have a wider application are printed at the end of the volume.

The universal employment of the method of Least Squares in work of this kind has led to the publication of an introduction to the subject for the benefit of those readers who are not already familiar with it. This introduction develops the method with special reference to the requirements of

this particular class of work, and it has not been the design to make it exhaustive.

For the materials employed original papers and memoirs have been consulted whenever practicable. The illustrative examples have been drawn largely from the reports of the Coast and other government surveys. For most of the examples of sextant work, as well as for many valuable suggestions, the author is indebted to his friend and former colleague Prof. Lewis Boss. Much assistance has also been derived from the excellent works of Chauvenet, Brünnow, and Sawitsch.

Fully appreciating the difficulty of eliminating all mistakes from a work of this character, the author can only hope that this one may not prove to be disfigured by an undue number of such blemishes.

C. L. DOOLITTLE.

BETHLEHEM, PA., May 20, 1885.

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INTRODUCTION TO THE METHOD OF LEAST SQUARES.

1. When a quantity is determined by observation, the result can never be regarded otherwise than as an approximation to the true value. If a number of measurements of the same quantity are made with extreme care, no two of the values obtained will probably agree exactly; at the same time none of them will differ very widely from the true one. There is a limit to the precision of the most refined instrument, even when used by the most skilful observer, and therefore the determination of a quantity depending on instrumental measurement, however carefully made, must be imperfect. It becomes then a problem of great practical importance to determine how the mass of data resulting from observation shall be combined so as to give the best possible value of the quantity sought. The theory of probabilities furnishes the basis for such an investigation.*

2. Observations are liable to errors of three kinds:

First. Constant errors, or those which affect all observa-

* The reader is supposed to be familiar with the theory of probability as developed in the ordinary text-books on algebra. See, for instance, Davies' Bourdon, edition of 1874, p. 322, or Olney's University Algebra, p. 294.

tions of a given series alike. These may result from a variety of causes, such as errors in the instruments used, personal error of the observer, errors in the constants of refraction, parallax, etc., used in the reduction of observations. A proper investigation will generally show the magnitude of such errors, and consequently the necessary corrections—at least the more important ones. We shall suppose the data to which our discussion applies freed from such errors, as their investigation does not come within the scope of this subject.

Second. Mistakes, such as recording the wrong degree in measuring an angle, or the wrong hour in the clock reading. When such errors are large they are not likely to give much trouble, as their true nature appears at once. When they are small they may prove embarrassing. The present discussion does not apply to them, and we shall suppose that no undiscovered mistakes have been made.

Third. Errors which are purely accidental. It is to these that our present investigation applies.

At first sight it might seem that such purely accidental errors were entirely outside the sphere of mathematical investigation, but we shall see that they follow a very definite law, and that theory is verified in an exceedingly satisfactory manner by observation.

3. We shall assume as the basis of our investigation the following axioms:

I. *If we have a series of direct measurements of a quantity, all made with equal care, the most probable value of the quantity will be obtained by taking the arithmetical mean of the individual measurements.*

II. *Plus and minus errors will occur with equal frequency.*

III. *Small errors will occur with greater frequency than large ones.*

Various attempts have been made to prove the first of these as a proposition. All such proofs are more or less unsatisfactory, and for elementary purposes it is more expedient to assume its truth at once. The "most probable value" there mentioned must be understood as the value which most nearly represents the given data, and from the evidence furnished by this series of observations alone it is the best attainable approximation to the true value.

The principles are supposed in all cases to be applied to a large number of observations; the larger the number the more closely will the results correspond to the laws assumed.

The Law of Distribution of Error.

4. Let x be a quantity whose value is to be determined by observation either directly or indirectly.

Let M_1, M_2, M_3, . . . M_m be the individual values obtained.

Then regarding M_1 as a determination of the unknown quantity x, its error will be $(M_1 - x)$. Similarly, $(M_2 - x)$, $(M_3 - x)$, . . . $(M_m - x)$ will be the errors of the other observed values.

Let us write

$$(M_1 - x) = \Delta_1,\ (M_2 - x) = \Delta_2, \ldots (M_m - x) = \Delta_m. \quad (1)$$

Let y_1 = the probability of the occurrence of the error Δ_1;
y_2 = the probability of the occurrence of the error Δ_2;
. . .
y_m = the probability of the occurrence of the error Δ_m.

Then our second and third axioms assume a law as existing such that the probability of a given error occurring will be

a function of the magnitude of the error itself. We shall therefore have the equation

$$y = \varphi(\varDelta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which $\varDelta$ represents any error, and y the probability of its occurring.

If this reasoning seems obscure, a different application of the same logic may possibly assist in comprehending it. Suppose we have a large number of tickets in a lottery-wheel. Let a definite proportion of them be numbered 1, a certain other proportion respectively 2, 3, etc. Then the probability of drawing any given number from the wheel will be a function of the number itself—viz.:

Suppose 1 ticket in every 55 numbered 1;
2 tickets in every 55 numbered 2;
3 tickets in every 55 numbered 3;
. .
. .
10 tickets in every 55 numbered 10.

Then every ticket would have one of the numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and

The probability of drawing a 1 would be $\frac{1}{55}$;
The probability of drawing a 2 would be $\frac{2}{55}$;
The probability of drawing a 10 would be $\frac{10}{55}$.

Or if k represents any one of the numbers from 1 to 10 inclusive, the probability of drawing a k will be $\frac{k}{55} = f(k)$, or $y = f(k)$ is the equation which represents the probability of drawing a k.

If now we were ignorant of the relations existing between the successive numbers 1, 2, 3, etc., and the relative number

of tickets so marked, we could, by drawing a sufficiently large number of tickets from the wheel, determine it, at least approximately. In this case we have to determine the probability of a given event occurring, viz., that of drawing a ticket marked with any given number k. In the above problem we have also to discuss the probability of a certain event occurring, viz., that of the appearance of any given error Δ in any one of our observations taken at random.

The Curve of Probability.

5. In the equation $y = \varphi(\Delta)$, we can regard Δ as the abscissa, and y as the ordinate of a curve. From the laws previously assumed we at once infer that the general form of the curve will be that of the following figure. In the first place, as $+$ and $-$ errors are equally probable, it follows that the curve will be symmetrical with respect to the axis of y; and as small errors are more probable than large ones, it follows that the values of Δ near zero will correspond to large values of y, while as Δ becomes very large y becomes very small.

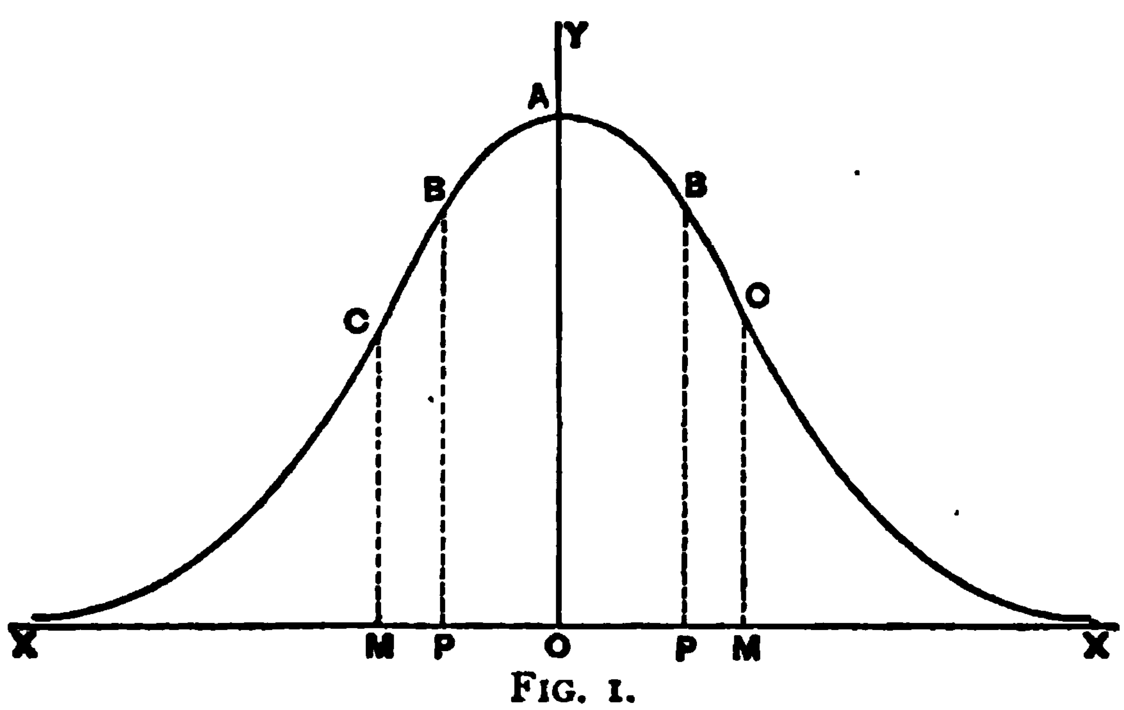

FIG. 1.

Practically Δ is not a continuous variable, and our locus, therefore, consists of a series of disconnected points. The

intervals between the different values of Δ will be equal to the smallest reading of the instrument with which the observations were made. The greater the degree of precision in the data, however, the more closely will our locus approach continuity; so by regarding it as a continuous curve we have a condition towards which we are constantly approximating as methods of observation become more and more refined.

Determination of the Function φ.

6. For the probability of an error Δ we have the equation

$$y = \varphi(\Delta);$$

and for an error $\Delta + \delta\Delta$,

$$y' = \varphi(\Delta + \delta\Delta).$$

The probability that an error falls between Δ and $\Delta + \delta\Delta$ will be the sum of all the probabilities between y and y'; or if $\delta\Delta$ is small, it will be nearly $\delta\Delta\varphi(\Delta)$. When $\delta\Delta$ becomes $d\Delta$, we have rigorously $y = \varphi(\Delta)d\Delta$ for the probability that an error falls between Δ and $\Delta + d\Delta$.* For the probability of an error falling between any finite limits, as for instance

* By way of illustration let us suppose the smallest unit of measure made use of in our observations to be $0''.1$, and that any given number of these units, as for instance 3, are represented by $\delta\Delta$. Then the errors between Δ and $\Delta + \delta\Delta$, including the latter, will be $(\Delta + 1)$, $(\Delta + 2)$, and $(\Delta + 3)$; and their respective probabilities, $y_1 = \varphi(\Delta + 1)$, $y_2 = \varphi(\Delta + 2)$, and $y_3 = \varphi(\Delta + 3)$. If now the limits between which the errors of our series lie extend to $\pm 10''$, we see that the probability y_1 will differ but little from y_3, and the sum of all the probabilities $y_1 + y_2 + y_3$ will differ but little from $3y$, or

$$\delta\Delta y = \varphi(\Delta)\delta\Delta.$$

$\pm a$, we shall have the sum of the probabilities for all values of Δ between $\pm a$, or

$$P = \int_{-a}^{+a} \varphi(\Delta) d\Delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When we extend the limits of integration so as to include all possible values of Δ, the probability becomes a certainty, which is expressed mathematically by unity. As, however, it is impossible to fix a finite limit to the value of Δ which shall be universal in its application, the limits in this case must be extended to $\pm \infty$, giving us the equation

$$1 = \int_{-\infty}^{+\infty} \varphi(\Delta) d\Delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From the foregoing we have

$$
\begin{array}{ll}
y_1 = \varphi(\Delta_1) & \text{for the probability of the error } \Delta_1; \\
y_2 = \varphi(\Delta_2) & \text{for the probability of the error } \Delta_2; \\
\cdot \quad \cdot & \qquad\qquad\qquad\qquad\qquad\qquad\quad \cdot \\
\cdot \quad \cdot & \qquad\qquad\qquad\qquad\qquad\qquad\quad \cdot \\
y_m = \varphi(\Delta_m) & \text{for the probability of the error } \Delta_m.
\end{array}
$$

If now $P =$ the probability that all these errors occur simultaneously, we have, from the theory of probabilities,

$$P = \varphi(\Delta_1)\varphi(\Delta_2)\varphi(\Delta_3) \quad . \quad . \quad . \quad \varphi(\Delta_m), \quad . \quad . \quad . \quad (5)$$

and the most probable value of the unknown quantity x will be that which makes the quantity P a maximum.

Taking the logarithms of both members of this equation, we have

$$\log P = \log \varphi(\Delta_1) + \log \varphi(\Delta_2) + \ . \ . \ . + \log \varphi(\Delta_m).$$

Differentiating this with respect to x, and placing the differential coefficient equal to zero, which is the condition of a maximum, we have

$$\frac{d(\log P)}{dx} = \frac{d[\log \varphi(\Delta_1)]}{d\Delta_1}\frac{d\Delta_1}{dx} + \frac{d[\log \varphi(\Delta_2)]}{d\Delta_2}\frac{d\Delta_2}{dx} + \ldots + \frac{d[\log \varphi(\Delta_m)]}{d\Delta_m}\frac{d\Delta_m}{dx} = 0.$$

From (1) we have

$$\frac{d\Delta_1}{dx} = \frac{d\Delta_2}{dx} = \ldots = \frac{d\Delta_m}{dx} = -1.$$

Substituting these values in the above equation, also for Δ_1, etc., their values $(M_1 - x)$, etc., it becomes

$$\frac{d \log \varphi(M_1 - x)}{d(M_1 - x)} + \frac{d \log \varphi(M_2 - x)}{d(M_2 - x)} + \ldots + \frac{d \log \varphi(M_m - x)}{d(M_m - x)} = 0. \quad (6)$$

This equation gives the means of determining x as soon as the form of the function φ is known, and this can best be determined by considering a particular case. As this function is strictly general, if we have once determined its form in a special case the result will be applicable to all cases.

We have assumed as an axiom that in the case of direct measurement of the quantity sought the most probable value will be the arithmetical mean of the individual measurements. This principle will furnish the basis for investigating the form of the function φ.

In case of direct measurement we have for the unknown quantity

$$x = \frac{M_1 + M_2 + \ldots + M_m}{m}, \quad \ldots \quad (7)$$

which may be written

$$(M_1 - x) + (M_2 - x) + \ldots + (M_m - x) = 0. \quad . \quad (8)$$

Equation (6) may be written

$$(M_1 - x)\left[\frac{d \log \varphi(M_1 - x)}{(M_1 - x)\, d(M_1 - x)}\right] + (M_2 - x)\left[\frac{d \log \varphi(M_2 - x)}{(M_2 - x)\, d(M_2 - x)}\right]$$
$$+ \ldots + (M_m - x)\left[\frac{d \log \varphi(M_m - x)}{(M_m - x)\, d(M_m - x)}\right] = 0. \quad . \quad (9)$$

Comparing equations (8) and (9), we see that since the quantities $(M_1 - x)$, $(M_2 - x)$, etc., are independent of each other, these equations can only be satisfied when the coefficients of $(M_1 - x)$, $(M_2 - x)$, etc., in (9) are respectively equal to the same constant quantity. We have therefore

$$\frac{d \log \varphi(M_1 - x)}{(M_1 - x)\, d(M_1 - x)} = \frac{d \log \varphi(M_2 - x)}{(M_2 - x)\, d(M_2 - x)}$$
$$= \ldots = \frac{d \log \varphi(M_m - x)}{(M_m - x)\, d(M_m - x)} = k. \quad (10)$$

Writing for $(M - x)$ in general Δ, we have

$$d \log \varphi(\Delta) = k\Delta d\Delta,$$

and, by integration, $\log \varphi(\Delta) = \frac{1}{2}k\Delta^2 + \log c,$

c being the constant of integration,

or $$\varphi(\Delta) = ce^{\frac{1}{2}k\Delta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From axiom III. it appears that as Δ increases this quantity must diminish, and this requires the exponent of e to be

negative. As Δ^2 cannot be negative, it follows that k must be so. Writing therefore $\frac{1}{2}k = -h^2$, our equation becomes

$$\varphi(\Delta) = ce^{-h^2\Delta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

7. Let us now consider the constant of integration c. This may be determined by substituting the value of $\varphi(\Delta)$ in (4), giving us

$$1 = \int_{-\infty}^{+\infty} ce^{-h^2\Delta^2} d\Delta,$$

a special form of the integral known as the gamma function. For the purpose of integrating the expression, place $h\Delta = t$. Then $d\Delta = \frac{dt}{h}$, and we have

$$1 = \int_{-\infty}^{+\infty} \frac{c}{h} e^{-t^2} dt = \frac{c}{h}\int_{-\infty}^{+\infty} e^{-t^2} dt.$$

As t in this expression is involved only in the quadratic form, we evidently have

$$\int_{-\infty}^{+\infty} e^{-t^2} dt = \int_{-\infty}^{0} e^{-t^2} dt + \int_{0}^{+\infty} e^{-t^2} dt = 2\int_{0}^{\infty} e^{-t^2} dt = 2A$$

(in which we write the integral equal to A for convenience).

In the definite integral $\int_0^\infty e^{-t^2} dt$ the value will be the same if we write another symbol instead of t. Therefore

$$\int_0^\infty e^{-t^2} dt = \int_0^\infty e^{-v^2} dv.$$

Multiplying both members of this equation by $\int_0^\infty e^{-t^2} dt$, we have

$$A^2 = \int_0^\infty \int_0^\infty e^{-(t^2+v^2)} dt dv.$$

In the second member of this equation write $v = tu$, $dv = tdu$. Then

$$A^2 = \int_0^\infty du \int_0^\infty e^{-t^2(1+u^2)} t dt.$$

But
$$\int e^{-t^2(1+u^2)} t dt = -\frac{e^{-t^2(1+u^2)}}{2(1+u^2)},$$

which between the given limits becomes $+\dfrac{1}{2(1+u^2)}$.

Therefore

$$A^2 = \frac{1}{2}\int_0^\infty \frac{du}{1+u^2} = \frac{1}{2}(\tan^{-1}\infty - \tan^{-1}0) = \frac{1}{4}\pi.$$

Therefore
$$A = \frac{\sqrt{\pi}}{2},$$

and we have
$$1 = \frac{c}{h}\sqrt{\pi}, \qquad \text{or} \qquad c = \frac{h}{\sqrt{\pi}},$$

and equation (12) becomes

$$y = \varphi(\Delta) = \frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In this equation the constant h will require further consideration; but if we assign any arbitrary value, as unity, to h we can readily construct the locus of the equation. It will at once appear that the general form will be that shown on page 5.

Condition of Maximum Probability.

8. Substituting in equation 5) the values of $\varphi(\Delta_1)$, $\varphi(\Delta_2)$, etc., from (13), it becomes

$$P = \left(\frac{h}{\sqrt{\pi}}\right)^m e^{-h^2(\Delta_1^2 + \Delta_2^2 + \ldots + \Delta_m^2)}. \quad . \quad . \quad . \quad (14)$$

From this equation we see that P will increase in value as the exponent of e diminishes, or P will be a maximum when $\Delta_1^2 + \Delta_2^2 + \ldots + \Delta_m^2$ is a minimum, thus giving us the important principle—

The most probable value of the unknown quantity is that which makes the sum of the squares of the residual errors a minimum.

From this principle comes the name *Method of Least Squares.*

The Measure of Precision.

9. Let us now consider the constant h.

Substituting in equation (3) the value of $\varphi(\Delta)$, we have for the probability of an error between the values $\pm a$

$$P = \int_{-a}^{+a} \frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2} d\Delta. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If we take another series of observations, we have the probability of an error between $\pm a'$

$$P' = \int_{-a'}^{+a'} \frac{h'}{\sqrt{\pi}} e^{-h'^2\Delta^2} d\Delta.$$

If these respective probabilities are equal we shall have

$$\int_{-a}^{+a} e^{-h^2\Delta^2} h d\Delta = \int_{-a'}^{+a'} e^{-h'^2\Delta^2} h' d\Delta,$$

which equation will be satisfied by making $ha = h'a'$, or

$$h : h' = a' : a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

We see from this equation that in two different series of observations h will have different values, these values being

to each other inversely as the errors to be ascribed with equal probability to each series. If, for instance, the errors of the first series are twice as great as those of the second, h will equal $\frac{1}{2}h'$. The constant h is therefore the measure of precision of the series of observations; and if its value could be determined from the observations themselves, we should by this means be able to know to what degree of confidence the data were entitled. This determination is possible,—at least approximately,—but for practical purposes it is more convenient to compare the relative accuracy of different series of observations by means of their respective probable errors, which will now be considered.

The Probable Error.

10. The probable error of any observation of a given series is a quantity such that if the errors committed be arranged according to their magnitude without reference to the algebraic sign, this quantity will occupy the middle place in the series. *It may therefore be defined as a quantity of such value that the probability of an error greater than this one is the same as the probability of one less.*

When we consider both plus and minus errors, we have from equation (15) the following expression for the probability of an error between $\pm a$, remembering that the probability between 0 and $+a$ is the same as between 0 and $-a$:

$$P = \frac{2h}{\sqrt{\pi}} \int_0^a e^{-h^2\Delta^2} d\Delta. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Let $r =$ the probable error.

The whole number of errors being represented by unity,

our definition of the probable error gives us the following equation:

$$\frac{1}{2} = \frac{2h}{\sqrt{\pi}}\int_0^r e^{-h^2\Delta^2}d\Delta, \quad \text{or} \quad \frac{1}{2} = \frac{2}{\sqrt{\pi}}\int_0^{hr} e^{-h^2\Delta^2}hd\Delta. \quad (18)$$

The solution of this equation will give us hr; so that if h is known r becomes known, and conversely.

11. It is evident that the equation for hr can only be solved approximately, as the expression $e^{-h^2\Delta^2}hd\Delta$ is not directly integrable. The only method of solution is to compute a series of numerical values of the integral for different values of the limit, hr, and then by interpolation determine that value which satisfies equation (18) with the necessary degree of precision.

Owing to the great importance of this integral, not only in this connection, but also in the theory of refraction, various methods have been developed for computing its numerical value. The most elementary of these consists in expanding $e^{-h^2\Delta^2} = e^{-t^2}$ ($h\Delta$ being written equal to t) into a series of ascending powers of t, by means of Maclaurin's formula, and integrating the separate terms of the series. This series converges rapidly for small values of t, and is therefore well adapted to numerical computation, but for large values of t it becomes diverging. For this case, as well as for the case where t is small, a series may be obtained by successive applications of the formula for integration by parts,

$$\int udv = uv - \int vdu,$$

by which means the expansion may be effected either in terms of ascending or descending powers of t. When an extensive series of values of the integral is required, as in computing a table of values for different values of the argu-

ment, t, the most simple process is to apply what is known as the method of *Mechanical Quadratures.*

As very complete tables of numerical values of this integral have been many times computed, we shall simply refer to the tabular quantities without entering more fully into the methods of computation. Table I. of this volume gives the values of $\int_0^t e^{-t^2}dt$ for values of t from 0 to ∞. We readily find from this table that the value of hr which satisfies equation (18) lies between .47 and .48. An interpolation readily gives

$$\left.\begin{aligned} hr &= 0.47694; \\ h &= \frac{.47694}{r}; \\ r &= \frac{.47694}{h}. \end{aligned}\right\} \quad \ldots \ldots \quad (19)$$

The Mean Error.

12. The probable error is not the only function of the errors which may be used for comparing the relative accuracy of different series of observations. Another quantity which may be used for this purpose, or as a convenient auxiliary for computing the probable error, is the *Mean Error.*

The Mean Error is a quantity whose square is the mean of the squares of the individual errors.

Let ε = the mean error. Then to determine the relation between ε and h, and consequently between ε and r, we proceed as follows: Let

Δ', Δ'', Δ''', etc. = the different errors which occur;
$\varphi(\Delta')$, $\varphi(\Delta'')$, $\varphi(\Delta''')$, etc. = their respective probabilities.

Then m being the whole number of errors, there will be a number expressed by the quantity $2m\varphi(\varDelta')$ (both $+$ and $-$ errors included) of the value $\varDelta'$, $2m\varphi(\varDelta'')$ of the value $\varDelta''$, etc., and in all

$$\varDelta_1 + \varDelta_2 + \varDelta_3 + \ldots \varDelta_m = 2m\varphi(\varDelta')\varDelta' + 2m\varphi(\varDelta'')\varDelta'' + 2m\varphi(\varDelta''')\varDelta''' + \text{etc.}$$

From the definition of the mean error ε we shall have

$$\varepsilon^2 = \frac{2m\varphi(\varDelta')\varDelta'^2 + 2m\varphi(\varDelta'')\varDelta''^2 + 2m\varphi(\varDelta''')\varDelta'''^2 + \text{etc.}}{m}$$
$$= 2\Sigma\varphi(\varDelta)\varDelta^2.$$

Expressing this by an integral, by the same method of reasoning as was used in deriving equation (3) we have

$$\varepsilon^2 = 2\int_0^\infty \frac{h}{\sqrt{\pi}}\varDelta^2 e^{-h^2\varDelta^2} d\varDelta.$$

This equation expresses a relation between ε and h. To effect the integration, let as before $h\varDelta = t$. Then $d\varDelta = \frac{dt}{h}$, and we have

$$\varepsilon^2 = \frac{2}{h^2\sqrt{\pi}}\int_0^\infty e^{-t^2}t^2 dt.$$

Integrating this by parts by placing $u = t$ and $dv = e^{-t^2}tdt$, and substituting in $\int udv = uv - \int vdu$, we find

$$\varepsilon^2 = \frac{2}{h^2\sqrt{\pi}}\left[-\left(\frac{t}{2e^{t^2}}\right)_{t=0}^{t=\infty} + \frac{1}{2}\int_0^\infty e^{-t^2}dt\right],$$

which readily gives $\qquad \varepsilon^2 = \frac{1}{2h^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$

Substituting the value of h from (19), we have

$$\left.\begin{aligned} \varepsilon &= 1.4826r; \\ r &= \ .6745\varepsilon. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

From these r is readily computed when we know ε, and vice versa.

The Mean of the Errors.

13. Another quantity which is much used as an auxiliary for computing r is The Mean of the Errors. This must not be confused with the mean error. It is thus defined:

The Mean of the Errors is the arithmetical mean of the different errors all taken with the positive sign.

Let η = the mean of the errors. Then to determine the relation between η and r we proceed in a manner similar to that followed in the previous section. As before, let

$\varDelta'$, $\varDelta''$, $\varDelta'''$, etc. = the individual errors.
$\varphi(\varDelta')$, $\varphi(\varDelta'')$, $\varphi(\varDelta''')$, etc. = their respective probabilities.

Then, the whole number of observations being m,

$$m\eta = 2m\,\varphi(\varDelta')\varDelta' + 2m\,\varphi(\varDelta'')\varDelta'' + 2m\,\varphi(\varDelta''')\varDelta''', \text{ etc.,}$$

from definition; and therefore

$$\eta = \frac{2m\varphi(\varDelta')\varDelta' + 2m\varphi(\varDelta'')\varDelta'' + 2m\varphi(\varDelta''')\varDelta''', \text{ etc.}}{m} = 2\Sigma\varphi(\varDelta)\varDelta.$$

Passing to the integral as before, m being supposed very large,

$$\eta = 2\int_0^\infty \frac{h}{\sqrt{\pi}}e^{-h^2\varDelta^2}\varDelta d\varDelta = \frac{1}{h\sqrt{\pi}}. \quad . \quad . \quad . \quad (22)$$

Substituting the value of h from (19),

$$\left.\begin{aligned} \eta &= 1.1829r; \\ r &= 0.8453\eta. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (23)$$

Equations (20) and (22) give us the following relations between ε and η, which we shall hereafter find convenient:

$$\left.\begin{aligned} \varepsilon &= \sqrt{\frac{\pi}{2}}\eta; \\ \eta &= \sqrt{\frac{2}{\pi}}\varepsilon. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (24)$$

Either of the quantities r, ε, or η may be used for comparing the relative accuracy of different series of observations, or of the quantities derived from them by computation. We shall, however, always use r for this purpose, making use of η and ε, when occasion serves, as convenient auxiliaries for computing the probable error r.

Precision of the Arithmetical Mean.

14. Although the arithmetical mean is the best value to be obtained from a series of equally good direct measurements, it will only be an approximation to the true value. It is therefore important to determine to what degree of confidence it is entitled. Let

$n_1, n_2, n_3, \ldots n_m = m$ individual measurements of the quantity x;
$\Delta_1, \Delta_2, \Delta_3, \ldots \Delta_m =$ the errors of each n respectively.

Then $\quad x = (n_1 - \Delta_1) = (n_2 - \Delta_2) = \ldots = (n_m - \Delta_m).$

Or taking the mean,

$$x = \frac{1}{m}(n_1 + n_2 + n_3 + \ldots + n_m) - \frac{1}{m}(\Delta_1 + \Delta_2 + \Delta_3 + \ldots + \Delta_m).$$

In this equation the first term of the second member is the arithmetical mean, and the second term is its error. As there is only one value of the arithmetical mean, this error will correspond, for this quantity, to our definition of the *mean error*. Therefore let

ε_0 = the mean error of the arithmetical mean;
r_0 = the probable error of the arithmetical mean.

Then

$$m^2\varepsilon_0^2 = (\Delta_1 + \Delta_2 + \Delta_3 + \ldots + \Delta_m)^2$$
$$= (\Delta_1^2 + \Delta_2^2 + \Delta_3^2 + \ldots \Delta_m^2) + 2(\Delta_1\Delta_2 + \Delta_1\Delta_3 + \ldots + \Delta_{m-1}\Delta_m).$$

Since from theory plus and minus errors will occur with equal frequency when the number of observations is large, the last term of this expression will vanish, or at least will become very small in comparison with the term preceding. Disregarding it and writing, in accordance with the notation of Gauss,

$$\Delta_1^2 + \Delta_2^2 + \Delta_3^2 + \ldots + \Delta_n^2 = [\Delta\Delta],$$

we have $m^2\varepsilon_0^2 = [\Delta\Delta]$.

But from definition, $m\varepsilon^2 = [\Delta\Delta]$.

Therefore $$\varepsilon_0 = \frac{\varepsilon}{\sqrt{m}}. \qquad (25)$$

Let h_0 = the measure of precision of the arithmetical mean. Then, from formulæ (19), (21), and (25),

$$\varepsilon_0 : \varepsilon = r_0 : r = h : h_0 = 1 : \sqrt{m}. \qquad (26)$$

That is, *the precision of a result obtained by direct measurement is directly as the square root of the number of measurements.*

Determination of the Probable Error.

15. From the foregoing principles we can now compute from the observations themselves the probable error of a quantity determined directly by observation.

As before, let $n_1, n_2, n_3, \ldots n_m$ = the individual measurements of a quantity x.

Let x_0 = the arithmetical mean of the n's;

$$v_1 = n_1 - x_0,\ v_2 = n_2 - x_0, \ldots v_m = n_m - x_0.$$

These quantities (v_1, v_2, etc.) are known as residuals, and must not be confounded with the true errors ($\varDelta_1$, $\varDelta_2$, etc.), from which they will always differ, unless x_0 is absolutely the true value of x.

Let the error of x_0 be δ. Then $x = x_0 + \delta$, and consequently

$$\varDelta_1 = v_1 - \delta,\ \varDelta_2 = v_2 - \delta,\ \ldots\ \varDelta_m = v_m - \delta,$$

and we shall have

$$[\varDelta\varDelta] = [vv] - 2[v]\delta + m\delta^2;$$

in which $[vv]^* = v_1^2 + v_2^2 + \ldots + v_m^2$

and $[v]^* = v_1 + v_2 + \ldots + v_m$.

Since x_0 is the arithmetical mean of the quantities n_1, n_2, etc., it follows that $[v] = 0$, and consequently

$$[\varDelta\varDelta] = [vv] + m\delta^2.$$

---

* Frequent use will be made hereafter of this symbol of summation, and it will require no further explanation.

δ being the error of the arithmetical mean, is unknown. A close approximation will, however, be obtained if we assume it equal to the mean error ε_0.* Then referring to (25), we have

$$m\delta^2 = m\varepsilon_0^2 = m\frac{\varepsilon^2}{m} = \varepsilon^2;$$

and since $[\Delta\Delta] = m\varepsilon^2$, we have

$$m\varepsilon^2 = [vv] + \varepsilon^2.$$

$$\left.\begin{aligned}
&\text{Therefore} && \varepsilon = \sqrt{\frac{[vv]}{m-1}};\\
&\text{and from (21),} && r = .6745\sqrt{\frac{[vv]}{m-1}}.\\
&\text{From (25) and (26),} && \varepsilon_0 = \sqrt{\frac{[vv]}{m(m-1)}};\\
& && r_0 = .6745\sqrt{\frac{[vv]}{m(m-1)}}.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (27)$$

Combining equations (27) and (24), we readily find

$$\left.\begin{aligned}
\varepsilon &= 1.2533\frac{[+v]}{\sqrt{m(m-1)}}; & r &= 0.8453\frac{[+v]}{\sqrt{m(m-1)}};\\
\varepsilon_0 &= 1.2533\frac{[+v]}{m\sqrt{m-1}}; & r_0 &= 0.8453\frac{[+v]}{m\sqrt{m-1}}.
\end{aligned}\right\} \quad (28)$$

In these expressions $[+v]$ represents the sum of the residuals all taken with the positive sign.

These simple formulæ (27) and (28) are of great practical value. When the number of observations is not large the values given by (27) will be a little more accurate than those

* From what precedes we see that this assumption would be rigorously true if the number of observations were infinite.

by (28), but when the number is large (28) will be sufficiently accurate for practical purposes, and the facility with which they are applied is something in their favor.

Probable Error of the Sum or Difference of Two or More Observed Quantities.

16. Let us next suppose the unknown quantity x, instead of being directly observed, to be the sum or difference of two or more quantities whose values are obtained by direct measurement; viz.:

Let $x = y_1 \pm y_2$, in which y_1 and y_2 are independent of each other and whose values are directly observed.

Let the individual errors of observation be—

$$\begin{array}{ll} \text{For } y_1, & \Delta_1', \Delta_1'', \dots \Delta_1^m; \\ \text{For } y_2, & \Delta_2', \Delta_2'', \dots \Delta_2^m. \end{array}$$

The errors of the individual determinations of x will then be

$$(\Delta_1' \pm \Delta_2'), (\Delta_1'' \pm \Delta_2''), \dots (\Delta_1^m \pm \Delta_2^m);$$

and if ε is the mean error of a determination of x, we shall have

$$m\varepsilon^2 = (\Delta_1' \pm \Delta_2')^2 + (\Delta_1'' \pm \Delta_2'')^2 + \dots + (\Delta_1^m \pm \Delta_2^m)^2.$$

Expanding and making use of the symbol for summation,

$$m\varepsilon^2 = [\Delta_1\Delta_1] \pm 2[\Delta_1\Delta_2] + [\Delta_2\Delta_2].$$

Let ε_1 and ε_2 = the mean errors of a measurement of y_1 and y_2 respectively. Then since, for reasons before explained,

the middle term ($[\Delta_1\Delta_2]$) may be regarded as vanishing in comparison with $[\Delta_1\Delta_1]$ and $[\Delta_2\Delta_2]$, we shall have

$$m\varepsilon^2 = m\varepsilon_1^2 + m\varepsilon_2^2,$$

or

$$\varepsilon = \sqrt{\varepsilon_1^2 + \varepsilon_2^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

In a manner precisely similar we may extend the method to the sum or difference of any number of observed quantities, so that in general if we have $x = y_1 \pm y_2 \pm \ldots \pm y_m$, the mean errors being respectively $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots \varepsilon_m$, we shall have

$$\varepsilon = \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \ldots + \varepsilon_m^2} = \sqrt{[\varepsilon\varepsilon]}. \quad . \quad . \quad (30)$$

Suppose next that we have $x = \alpha_1 y_1 \pm \alpha_2 y_2 \pm \ldots \pm \alpha_m y_m$, in which $\alpha_1, \alpha_2, \ldots \alpha_m$ are constants. If, as before, $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_m$ are the mean errors of $y_1, y_2, \ldots y_m$, then the mean errors of $\alpha_1 y_1, \alpha_2 y_2, \ldots \alpha_m y_m$ will be respectively $\alpha_1\varepsilon_1, \alpha_2\varepsilon_2, \ldots \alpha_m\varepsilon_m$, and the mean error of x

$$\varepsilon = \sqrt{\alpha_1^2\varepsilon_1^2 + \alpha_2^2\varepsilon_2^2 + \ldots + \alpha_m^2\varepsilon_m^2} = \sqrt{[\alpha^2\varepsilon^2]}. \quad . \quad (31)$$

Principle of Weights.

17. In the foregoing we have assumed all the observations considered to be equally trustworthy, or, as it is expressed technically, of equal weight. As will readily be seen, we shall frequently have occasion to combine observations of different weights. It is therefore important to ascertain how to treat them, so that each shall have its proper influence in determining the result.

Confining our discussion for the present to the case of a directly observed quantity, the most elementary form of the

problem will be that where the quantities combined are themselves the arithmetical means of several observations of the weight unity. Thus, suppose the quantity x to be determined from m' such observations; the most probable value of x' will then be

$$x' = \frac{n_1' + n_2' + n_3' + \ldots + n_{m'}'}{m'}.$$

From a second, third, etc., series of m'', m''', etc., observations we have respectively

$$x'' = \frac{n_1'' + n_2'' + n_3'' + \ldots + n_{m''}''}{m''};$$

$$x''' = \frac{n_1''' + n_2''' + n_3''' + \ldots + n_{m'''}'''}{m'''}.$$

$$\begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \end{array}$$

Combining all these individual values, we have for the most probable value of x

$$x = \frac{(n_1'+n_2'+\ldots+n_{m'}')+(n_1''+n_2''+\ldots+n_{m''}'')+(n_1'''+n_2'''+\ldots+n_{m'''}''')+\ldots}{m'+m''+m'''+\ldots},$$

or $$x = \frac{m'x' + m''x'' + m'''x''' + \ldots}{m' + m'' + m''' + \ldots}. \quad . \quad . \quad . \quad (32)$$

The value of x will not be affected if we multiply both numerator and denominator of this fraction by any constant α; viz.,

$$x = \frac{\alpha m'x' + \alpha m''x'' + \alpha m'''x''' + \ldots}{\alpha m' + \alpha m'' + \alpha m''' + \ldots}, \quad . \quad (32)^*$$

in which we may regard am', am'', etc., as the respective weights of x', x'', etc. a may be integral or fractional. From this we see that the weights are simply relative quantities and are in no case to be regarded as absolute.

From the foregoing we have the following practical rule: *When observations are to be combined to which different weights are to be ascribed, the most probable value of the unknown quantity will be obtained by multiplying each observation by its weight, and dividing the sum of the products by the sum of the weights.*

It is clear that the difference of weights may result from a variety of causes other than the simple one considered above; as, for instance, one series of observations may be made with a more accurate instrument than another, or by a more skilled observer. Thus, for example, it may be the case that ten measurements made by one observer will have as much value as twenty made by another. If the weight of an observation of the first series be unity, one of the second would only be entitled to a weight of one half; or more generally,

Letting p = the weight of an observation of the second series,
Then $2p$ = the weight of an observation of the first series.

If then we have a series x_1, x_2, x_3, etc., of observations of the weights p_1, p_2, p_3, etc., and consequently

$$x = \frac{p_1x_1 + p_2x_2 + p_3x_3 + \dots}{p_1 + p_2 + p_3 + \dots}$$

as the most probable value of x, it is evident that, whatever may have been the cause of this difference of weight, we may consider each value x_1, x_2, etc., as derived from p_1, p_2, etc., individual observations of the weight unity. Let

ε = the mean error of an observation of the weight unity;
ε_1, ε_2, etc., the mean errors of x_1, x_2, etc.

$$\left.\begin{array}{ll}\text{Then from (25),} & \varepsilon_1 = \dfrac{\varepsilon}{\sqrt{p_1}}, \quad \varepsilon_2 = \dfrac{\varepsilon}{\sqrt{p_2}}, \text{ etc.,} \\ \text{or} & \varepsilon_1 : \varepsilon_2 = \sqrt{p_2} : \sqrt{p_1}, \quad \varepsilon_1 : \varepsilon_3 = \sqrt{p_3} : \sqrt{p_1}, \text{ etc.}\end{array}\right\} \quad . \quad . \quad (33)$$

The whole number of observations being equal to $p_1 + p_2 + p_3 + \ldots = [p]$ observations of the weight unity or of the mean error ε, we have for the mean error of x, from (25),

$$\varepsilon_0 = \frac{\varepsilon}{\sqrt{[p]}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The Probable Error when Observations have Different Weights.

18. The mean taken according to weights, as in equation (32) or (32)*, is sometimes called the *General Mean*. In order to derive the formula for the probable error in this case, let, as before, δ be the error of the general mean x_0; viz., $x - x_0 = \delta$. Then, the notation being as before, we have

$$\Delta_1 = v_1 - \delta, \quad \Delta_2 = v_2 - \delta, \quad \Delta_3 = v_3 - \delta, \text{ etc.}$$

The error Δ_1 belongs to x_1 and therefore appears p_1 times;
The error Δ_2 belongs to x_2 and therefore appears p_2 times;

. . .

. . .

Therefore $\quad [p\Delta\Delta] = [pvv] - 2[pv]\delta + [p]\delta^2.$

For the same reason as in previous cases $[pv]$ may be disregarded as being inappreciable in comparison with the other terms, when we have

$$[p\Delta\Delta] = [pvv] + [p]\delta^2.$$

Substituting for δ the mean error of x from (34), we have

$$[p\Delta\Delta] = [pvv] + \varepsilon^2.$$

Now, as x_1 is equivalent to p_1 observations of weight unity, there will be the equivalent of p_1 errors equal to Δ_1; and ε_1 being the mean error of x_1, we shall have

$$p_1\varepsilon_1^2 = p_1\Delta_1\Delta_1.$$

Whence from (33), $\varepsilon^2 = p_1\Delta_1\Delta_1.$

Similarly, $\varepsilon^2 = p_2\Delta_2\Delta_2 = p_3\Delta_3\Delta_3$, etc.

And m being the whole number of quantities, or observations, x_1, x_2, etc., we have

$$\begin{aligned} m\varepsilon^2 &= p_1\Delta_1\Delta_1 + p_2\Delta_2\Delta_2 + p_3\Delta_3\Delta_3, \text{ etc.} \\ &= [p\Delta\Delta]. \end{aligned}$$

Our equation therefore becomes $m\varepsilon^2 = [pvv] + \varepsilon^2$, from which

$$\left.\begin{aligned} \varepsilon &= \sqrt{\frac{[pvv]}{m-1}}; \\ \text{and from (34),} \qquad \varepsilon_0 &= \sqrt{\frac{[pvv]}{[p]\,(m-1)}}; \\ \text{and from (21),} \qquad r &= .6745\sqrt{\frac{[pvv]}{m-1}}, \\ r_0 &= .6745\sqrt{\frac{[pvv]}{[p]\,(m-1)}}. \end{aligned}\right\} \quad . \; . \; . \quad (35)$$

m in these formulæ is the number of individual observations, or quantities, x_1, x_2, etc., and must not be mistaken for the sum of the weights.

It will be evident upon a careful comparison of these expressions with the formulæ (27) that we should have reached

the same result by multiplying each quantity x_1, x_2, etc., by the square root of its weight, and then proceeding exactly as we have previously done with observations of equal weight.

We have therefore established the following rule which we may apply in combining observations of different weights:

First reduce all observations to a common unit of weight by multiplying each by the square root of its weight, then combine them precisely as if they had originally been of equal weight.

For examples of the application of the formulæ see pages 515 and 516.

General Remarks.

19. We have hitherto considered only those cases where the unknown quantity is derived in the simplest manner from observation, viz., by direct measurement or by the sum or difference of directly measured quantities.

Before proceeding to the more complex cases a few general remarks may not be out of place.

Equation (13), which represents the law of distribution of error, and on which the subsequent discussion is based, rests upon two hypotheses neither of which is ever fully realized in practice, viz., that the number of observations is infinite, and that they are entirely free from constant errors, i.e., errors which affect all alike. The formulæ deduced when applied to the cases which actually arise can give us only approximate results, although they will be the best attainable approximations from the given data. This is particularly to be borne in mind when the number of observations is small. The probable errors in such cases are apt to be entirely illusory, and in general are only reliable when the number of observations is large enough to exhibit approximately the law of distribution of error derived from the hypothesis of an infinite series of observations.

The second hypothesis mentioned above, viz., that constant errors do not exist in our data, can never be fully realized, and this fact is often the source of great annoyance and uncertainty in combining observations taken under different conditions. Such errors arise from a variety of causes, some easy to investigate and others not at all so. It is of very frequent occurrence that a result derived from a single series of observations will give a small probable error, and yet differ widely from that derived from a second series to all appearances equally good. It sometimes happens that computers who are puzzled by such occurrences attribute the difficulty to faults in the method, the truth being that they are due to the presence of a class of errors with which the method does not profess to deal.

The remedy for this difficulty is to vary as much as possible the conditions under which the observations are made, and in a manner calculated to eliminate as far as possible those constant errors which cannot be investigated.

Comparison of Theory with Observation.

20. The test of theory is its agreement with observed facts. We may in this manner test the truth of the law which we have derived for the distribution of errors.

We have the probability that an error falls between the limits $\pm a$ expressed by the equation

$$p = \int_{-a}^{+a} \frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2} d\Delta. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In accordance with the theory of probabilities, p here is a fraction which expresses the ratio of the number of errors

between $\pm a$ to the whole number. If then the number of observations is m, the number of errors between $\pm a$ will be

$$m\frac{h}{\sqrt{\pi}}\int_{-a}^{+a} e^{-h^2\Delta^2}d\Delta.$$

To test the law expressed by this formula we have only to compute the probable error of the series of observations under consideration by (27) or (28), and then h by (19). The value of the integral will then be obtained from Table I., and we shall be in possession of everything necessary for comparing the number of errors between any two limits as indicated by this formula with the number shown by the series of observations. Many such comparisons have been made, and always with satisfactory results, when the number of observations compared has been large. A perfect agreement is of course not to be looked for, as our formula has been derived on the theory of an infinite number of observations; and further, we are not in possession of the true errors for comparison with the formula, but the residuals instead, which will always differ from the errors unless we are in possession of the absolutely true value of the unknown quantity.

As an illustration of the above the following tabular statement gives the result of a comparison with theory of the errors of the observed right ascensions of Sirius and Altair. The example is given by Bessel in the *Fundamenta Astronomiæ*.

In a series of 470 observations by Bradley the probable error of a single observation was found to be $r = 0''.2637$, whence $h = 1.80865$. Therefore for the number of errors less than $''.1$ the argument of Table I. will be $t = h\Delta = .180865$. With this argument we find for the integral .20188, which multiplied by 470, the entire number of errors, gives 95 as

the number of errors less than ″.1. In a manner similar to this the following results were found:

Between	No. of Errors by Theory.	No. of Errors by Experience.
0″.0 and 0″.1	95	94
0″.1 and 0″.2	89	88
0″.2 and 0″.3	78	78
0″.3 and 0″.4	64	58
0″.4 and 0″.5	50	51
0″.5 and 0″.6	36	36
0″.6 and 0″.7	24	26
0″.7 and 0″.8	15	14
0″.8 and 0″.9	9	10
0″.9 and 1″.0	5	7
over 1″.0	5	8

This agreement is very satisfactory, but here, as in other similar examples, the larger errors occur a little more frequently than theory would indicate.

This is probably due to the fact that (unconsciously, perhaps) every observer will occasionally let an observation pass which is not up to the average standard of accuracy. Small mistakes will sometimes occur, also, which are not of sufficient magnitude to attract attention. A consideration of the matter has led to attempts on the part of Pierce of Harvard College and Stone of England to establish criteria for the rejection of such doubtful observations. Glaisher, on the other hand, proposed to overcome the difficulty by determining a system of weights which should give those observations which show large discrepancies less influence than those showing small ones.

This branch of the subject, however, is beyond the scope of the present work. It is an exceedingly delicate matter to deal with, and from its nature is probably incapable of a mathematical treatment which shall be entirely satisfactory.

Every computer occasionally feels compelled to reject

observations. This should always be done with extreme caution. As for the criteria for this purpose hitherto proposed, probably the most that can be said in their favor is that their use insures a uniformity in the matter, thus leaving nothing to the individual caprice of the computer.

Indirect Observations.

21. We have now investigated the simplest case of the determination of the unknown quantity by observation, viz., that when the quantity to be determined is measured directly. In the more general form of the problem the unknown quantities are connected with the observed quantities by an equation of the form

$$f(x, y, z, \ldots) = M,$$

M being given by observation, and x, y, z, etc., being the unknown quantities. This general form includes the case which we have previously investigated, where there was only one unknown quantity. Each observation furnishes an equation of this form; therefore a number of observations equal to that of the unknown quantities will completely determine their value.

This would leave nothing to be desired if the observations were perfect; but owing to the errors to which they are liable, the values of x, y, z, etc., will be more reliable the greater the number of observations on which they depend. If now we have four unknown quantities, x, y, z, and w, four observations will give us four equations from which the values of the unknown quantities may be determined. If we have more than four equations, we may determine values of the unknown quantities by combining any four of them. As the equations depend on observations more or less erroneous, we should thus obtain a variety of values for x, y, z, and w, all of them probably in error to some extent.

The problem then is this: Of all possible systems of values of the unknown quantities, to find that which most accurately represents all of the observations.

We shall confine ourselves to the consideration of linear equations; and as the problems in which we shall be more particularly interested do not give rise to equations of more than four unknown quantities, we shall limit our discussion to that number. It will be obvious, however, that it can be extended to any number.

Suppose we have the following system of equations:

$$\left.\begin{array}{l} a_1x + b_1y + c_1z + d_1w = n_1; \\ a_2x + b_2y + c_2z + d_2w = n_2; \\ a_3x + b_3y + c_3z + d_3w = n_3; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad . \quad . \quad . \quad . \quad (36)$$

in which x, y, z, and w are unknown quantities, a, b, c, d, etc., are coefficients given by theory, and n_1, n_2, n_3, etc., are quantities given by observation.

If now the data were perfect we should obtain the same values of x, y, z, and w by combining any four of these equations. Owing, however, to the errors of observation to which n_1, n_2, etc., are subject, it is not probable that a substitution of the true values of x, y, z, and w (if we knew them) would exactly satisfy any one of the equations.

Let v_1, v_2, v_3, etc., be the residuals obtained by substituting in equations (36) for x, y, z, and w their approximate values such that the following equations will be rigorously satisfied:

$$\left.\begin{array}{l} a_1x + b_1y + c_1z + d_1w = n_1 - v_1; \\ a_2x + b_2y + c_2z + d_2w = n_2 - v_2; \\ a_3x + b_3y + c_3z + d_3w = n_3 - v_3. \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad . \quad . \quad . \quad (37)$$

Now the most probable values of our unknown quantities will be those which make the sum of the squares of these residuals a minimum, viz.,

$$v_1^2 + v_2^2 + v_3^2 + \text{etc.} = f(x, y, z, w) \quad . \quad . \quad (38)$$

must be a minimum.

In these equations x, y, z, and w are supposed independent, therefore the differential coefficients with reference to each variable must separately be equal to zero to satisfy the conditions of a minimum. That is,

$$\frac{d[vv]}{dx} = 0, \qquad \frac{d[vv]}{dy} = 0, \qquad \frac{d[vv]}{dz} = 0, \qquad \frac{d[vv]}{dw} = 0.$$

Writing out these expressions in full, we have the following:

$$\left.\begin{aligned} v_1 \frac{dv_1}{dx} + v_2 \frac{dv_2}{dx} + v_3 \frac{dv_3}{dx} + \ldots &= 0; \\ v_1 \frac{dv_1}{dy} + v_2 \frac{dv_2}{dy} + v_3 \frac{dv_3}{dy} + \ldots &= 0; \\ v_1 \frac{dv_1}{dz} + v_2 \frac{dv_2}{dz} + v_3 \frac{dv_3}{dz} + \ldots &= 0; \\ v_1 \frac{dv_1}{dw} + v_2 \frac{dv_2}{dw} + v_3 \frac{dv_3}{dw} + \ldots &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad (39)$$

x, y, z, and w being independent, we have from (37),

$$\begin{aligned} &\frac{dv_1}{dx} = -a_1, \quad &\frac{dv_2}{dx} = -a_2, \quad &\frac{dv_3}{dx} = -a_3, \text{ etc.}; \\ &\frac{dv_1}{dy} = -b_1, \quad &\frac{dv_2}{dy} = -b_2, \quad &\frac{dv_3}{dy} = -b_3, \text{ etc.}; \\ &\frac{dv_1}{dz} = -c_1, \quad &\frac{dv_2}{dz} = -c_2, \quad &\frac{dv_3}{dz} = -c_3, \text{ etc.}; \\ &\frac{dv_1}{dw} = -d_1, \quad &\frac{dv_2}{dw} = -d_2, \quad &\frac{dv_3}{dw} = -d_3, \text{ etc.}; \end{aligned}$$

by means of which values equations (39) become

$$\left.\begin{array}{l} a_1v_1 + a_2v_2 + a_3v_3 + \ldots = 0; \\ b_1v_1 + b_2v_2 + b_3v_3 + \ldots = 0; \\ c_1v_1 + c_2v_2 + c_3v_3 + \ldots = 0; \\ d_1v_1 + d_2v_2 + d_3v_3 + \ldots = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (40)$$

Substituting for v_1, v_2, etc., their values from (37), we have for the first of these

$$\left.\begin{array}{l} a_1a_1x + a_1b_1y + a_1c_1z + a_1d_1w - a_1n_1 \\ + a_2a_2x + a_2b_2y + a_2c_2z + a_2d_2w - a_2n_2 \\ + a_3a_3x + a_3b_3y + a_3c_3z + a_3d_3w - a_3n_3 \\ + \quad . \quad\quad . \quad\quad . \quad\quad . \quad\quad . \\ \quad\; . \quad\quad . \quad\quad . \quad\quad . \quad\quad . \end{array}\right\} = 0.$$

The second of (40) becomes

$$\left.\begin{array}{l} a_1b_1x + b_1b_1y + b_1c_1z + b_1d_1w - b_1n_1 \\ + a_2b_2x + b_2b_2y + b_2c_2z + b_2d_2w - b_2n_2 \\ + a_3b_3x + b_3b_3y + b_3c_3z + b_3d_3w - b_3n_3 \\ + \quad . \quad\quad . \quad\quad . \quad\quad . \quad\quad . \\ \quad\; . \quad\quad . \quad\quad . \quad\quad . \quad\quad . \end{array}\right\} = 0,$$

and similarly for the remaining equations. Using Gauss' symbols of summation, we have therefore

$$\left.\begin{array}{l} [aa]x + [ab]y + [ac]z + [ad]w = [an]; \\ [ab]x + [bb]y + [bc]z + [bd]w = [bn]; \\ [ac]x + [bc]y + [cc]z + [cd]w = [cn]; \\ [ad]x + [bd]y + [cd]z + [dd]w = [dn]. \end{array}\right\} \quad . \quad . \quad (41)$$

These are called *Normal Equations*, and the values of the unknown quantities obtained by solving them will be the system of values which makes the sum of the squares of the residuals v_1, v_2, etc., a minimum, and therefore the most probable system of values. Equations (36) are called *Equations of*

Condition, or *Observation Equations*. An inspection of (41) gives us the following rule for solving a series of equations of condition:

Multiply each equation by the coefficient of x in that equation, then add together the resulting equations for a new equation, then multiply each equation by the coefficient of y in that equation, and, as before, form the sum of the resulting equations. Continue the process with the coefficients of each of the unknown quantities. The number of resulting Normal Equations will be equal to that of the unknown quantities, and the values of the unknown quantities deduced therefrom will be the most probable values.

It must be borne in mind that this process supposes the number of equations of condition to be greater than that of the unknown quantities. If it is less, this process will give us a number of equations equal to that of the quantities to be determined, but they will be indeterminate none the less than the original equations were, as can be easily shown.

Observations of Unequal Weight.

22. In deriving the normal equations from the equations of condition, we have regarded the latter as of equal weight. In the more general case the weights will be unequal.

In the equation $a_1x + b_1y + c_1z + d_1w = n_1$, if we suppose, as in (33), that p_1 represents the weight of an observation, viz., of n_1, that ε_1 is the mean error of n_1, and ε the mean error of an observation of weight unity, we have

$$\varepsilon_1 = \frac{\varepsilon}{\sqrt{p_1}}.$$

Multiplying the above equation by $\sqrt{p_1}$, we have

$$a_1\sqrt{p_1}x + b_1\sqrt{p_1}y + c_1\sqrt{p_1}z + d_1\sqrt{p_1}w = n_1\sqrt{p_1}, \tag{42}$$

an equation in which the mean error of the absolute term $n_1\sqrt{p_1}$ is ε, and the weight unity. In the same manner we multiply each equation by the square root of its weight, thus reducing them all to the same unit of weight, when we proceed precisely as before in forming the normal equations.

Computation of the Coefficients.

23. The method of forming the normal equations is now fully explained; the work of computation, however, is somewhat laborious, especially when the number of equations of condition is large. It will therefore be important to arrange the work so that the numerous multiplications and additions may be performed with the least liability to error, and so that convenient checks may be applied for insuring accuracy in the results. The multiplications may be performed by logarithms, in which case a four-place table will give the necessary degree of precision, or Crelle's multiplication-table may be employed with advantage.* We shall also show how to perform the multiplications by the use of a table of squares.

Convenient proof-formulæ may be derived as follows: Let the sum of all the coefficients entering into each equation be formed in succession, and represent them by s with the proper subscript. Thus:

$$\left.\begin{array}{l} a_1 + b_1 + c_1 + d_1 - n_1 = s_1; \\ a_2 + b_2 + c_2 + d_2 - n_2 = s_2. \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad . \quad . \quad . \quad (43)$$

* Dr. A. L. Crelle's "Rechentafeln welche alles multipliciren und dividiren mit Zahlen unter Tausend" (Berlin, 1869).

Multiplying these sums by their respective a, b, c, etc., in succession, and adding the products, we shall have the following equations for checking the accuracy of the coefficients of the normal equations:

$$\left.\begin{array}{l} [aa] + [ab] + [ac] + [ad] - [an] = [as]; \\ [ab] + [bb] + [bc] + [bd] - [bn] = [bs]; \\ [ac] + [bc] + [cc] + [cd] - [cn] = [cs]; \\ [ad] + [bd] + [cd] + [dd] - [dn] = [ds]. \end{array}\right\} \quad . \quad (44)$$

This requires the computation of the additional terms $[as]$, $[bs]$, . . . and the agreement must come within the limit of error of the computation. These additional terms will be further useful for checking the accuracy of the solution of the normal equations, as will afterwards appear.

24. If it should happen that the coefficients of one unknown quantity in the equations of condition were much larger than those of another, considerable discrepancies might exist in the agreement of the proof-formulæ with the sums of the coefficients. It will generally be necessary practically to limit the computation to a certain number of decimals, when the products of the large quantities may introduce errors into the last places, where the products of the small quantities introduce none.

This difficulty is overcome by substituting for the unknown quantities other quantities which will make the coefficients of the same order of magnitude throughout. This is conveniently accomplished by selecting the largest coefficient with which an unknown quantity is affected and dividing each of the coefficients of this quantity by it. Thus, let α, β, γ, δ be the largest coefficients of the quantities x, y, z, w, respectively, which occur in the equations of condition, and let ν be the largest of the series of known quantities n_1, n_2,

$n_2, \ldots$ Then we may place the equations of condition in the following form:

$$\frac{a_1}{\alpha}(\alpha x) + \frac{b_1}{\beta}(\beta y) + \frac{c_1}{\gamma}(\gamma z) + \frac{d_1}{\delta}(\delta w) = \frac{n_1}{\nu};$$
$$\frac{a_2}{\alpha}(\alpha x) + \frac{b_2}{\beta}(\beta y) + \frac{c_2}{\gamma}(\gamma z) + \frac{d_2}{\delta}(\delta w) = \frac{n_2}{\nu};$$
$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

where the unknown quantities are (αx), (βy), ... and the values obtained in solving the equations will be in terms of ν. The equations will be made homogeneous by this process before beginning the work of forming the normal equations. The sums s_1, s_2, ... will be most convenient for the purpose to which they are applied, if they are formed from these homogeneous equations.

For the kind of problems which we shall have occasion to solve in the following pages there will seldom be a systematic difference in the magnitudes of the coefficients of the different unknown quantities of importance enough to render this operation necessary. In cases, however, where there is a marked difference in this respect it will be advisable to incur the slight additional labor involved, and in some cases it becomes a matter of considerable importance.

25. The formation of the normal equations with the accompanying proof-formulæ will therefore require the computation of the following quantities:

$$\begin{array}{rrrrrr} [aa] & [ab] & [ac] & [ad] & [an] & [as]; \\ & [bb] & [bc] & [bd] & [bn] & [bs]; \\ & & [cc] & [cd] & [cn] & [cs]; \\ & & & [dd] & [dn] & [ds]; \\ & & & & [nn] & [ns]. \end{array}$$

The latter will be employed for checking the final computation, as will be shown hereafter. As will be seen, there are twenty of these quantities required in a series of four equations. In general the number will be* $\frac{(n+2)(n+3)}{2} - 1$, where n is the number of unknown quantities.

Let a sheet of paper be ruled with a number of vertical columns represented by the above formula. In the first horizontal line will be the symbols of the products written in the columns below, viz., $[aa]$, $[ab]$, . . . and in the last line the sums of the products. If the results are correct the proof-equations (44) must be satisfied. The algebraic signs of the various products will demand special attention, as they form a very fruitful source of error.

If the application of the proof-formulæ is postponed until the conclusion of this part of the computation, the position of an error is often shown at once, since each sum, with the exception of the sum of the squares, is found in two different proof-equations. If two of the proof-formulæ fail to be satisfied, while the others prove true, the error is in the term common to both; while if only one equation fails to be satisfied, the error is in the quadratic term.

Before proceeding further it is recommended that the reader refer to the example found on page 329. The number of observation equations is twelve, each of which has been multiplied by the square root of its weight. The number of unknown quantities is three, the coefficients of which have no systematic difference in magnitude of sufficient importance to require the application of the process for rendering them homogeneous. The formation of the normal equations is found on page 330. The number of

* It is the sum of a series of terms in arithmetical progression minus 1; number of terms = $(n + 2)$; first term = 1; last term = $(n + 2)$.

unknown quantities being three, we require by the formula just given fourteen columns. It will be observed that the proof-formulæ are perfectly verified, as they should be in this case, no decimal terms having been neglected.

Computation of the Coefficients by a Table of Squares.

26. By whatever method the multiplications are performed a table of squares will be found very convenient for the quadratic terms. Terms of the form $[ab]$ may also be computed with such a table, as will appear from the following.

We have
$$\begin{aligned} a_1b_1 &= \tfrac{1}{2}\{(a_1 + b_1)^2 - a_1^2 - b_1^2\}; \\ a_2b_2 &= \tfrac{1}{2}\{(a_2 + b_2)^2 - a_2^2 - b_2^2\}; \\ a_3b_3 &= \tfrac{1}{2}\{(a_3 + b_3)^2 - a_3^2 - b_3^2\}; \\ &\cdot \qquad \cdot \qquad \cdot \qquad \cdot \\ &\cdot \qquad \cdot \qquad \cdot \qquad \cdot \\ [ab] &= \tfrac{1}{2}\{[(a + b)^2] - [aa] - [bb]\}. \quad . \quad . \quad (45) \end{aligned}$$

The quadratic terms $[aa]$, $[bb]$, ... will be computed in any case, so there will only be required in addition the terms of the form $[(a + b)^2]$. In case of four unknown quantities we shall require the following quadratic terms:

$$\left.\begin{array}{lllll} [aa] & [(a+b)^2] & [(a+c)^2] & [(a+d)^2] & [(a-n)^2]; \\ & [bb] & [(b+c)^2] & [(b+d)^2] & [(b-n)^2]; \\ & & [cc] & [(c+d)^2] & [(c-n)^2]; \\ & & & [dd] & [(d-n)^2]; \\ & & & [ss] & [nn]. \end{array}\right\} \quad (46)$$

The last two will be employed in checking this and the subsequent computation. Thus for the case of four unknown quantities we have sixteen terms of the above form, or, in general, $\dfrac{(n+1)(n+2)}{2} + 1$.

The equations having been multiplied by the square roots of their respective weights, and the coefficients made homogeneous if necessary, the computation will be carried out as shown in the following scheme:

	aa	bb	cc	...	nn	ss	$(a+b)^2$	$(a+c)^2$	...	$(a-n)^2$	$(b+c)^2$	...	...
1	a_1a_1	b_1b_1	c_1c_1	...	n_1n_1	s_1s_1	$(a_1+b_1)^2$	$(a_1+c_1)^2$	...	$(a_1-n_1)^2$	$(b_1+c_1)^2$	...	...
2	a_2a_2	b_2b_2	c_2c_2	...	n_2n_2	s_2s_2	$(a_2+b_2)^2$	$(a_2+c_2)^2$	...	$(a_2-n_2)^2$	$(b_2+c_2)^2$	...	...
3	a_3a_3	b_3b_3	c_3c_3	...	n_3n_3	s_3s_3	$(a_3+b_3)^2$	$(a_3+c_3)^2$	...	$(a_3-n_3)^2$	$(b_3+c_3)^2$	...	...
.	....	....	....	...	....	....	....	....	...	....	....	...	...
.	....	....	....	...	....	....	....	....	...	....	....	...	...
	$[aa]$	$[bb]$	$[cc]$	...	$[nn]$	$[ss]$	$[(a+b)^2]$	$[(a+c)^2]$	...	$[(a-n)^2]$	$[(b+c)^2]$	...	...
							$[aa]+[bb]$ $2[ab]$ $[ab]$	$[aa]+[cc]$ $2[ac]$ $[ac]$	...	$[aa]+[nn]$ $2[an]$ $[an]$	$[bb]+[cc]$ $2[bc]$ $[bc]$	...	...

In order to derive a convenient proof-formula we square both members of equations (43) and add

$$\left.\begin{aligned}&[ss] + 3\,\{[aa] + [bb] + [cc] + [dd] + [nn]\} = \\ &[(a+b)^2] + [(a+c)^2] + [(a+d)^2] + [(a-n)^2] \\ &\qquad + [(b+c)^2] + [(b+d)^2] + [(b-n)^2] \\ &\qquad\qquad + [(c+d)^2] + [(c-n)^2] \\ &\qquad\qquad\qquad + [(d-n)^2].\end{aligned}\right\}\quad(47)$$

For an example of the application of the above method the reader will turn to page 334, where the normal equations are computed from the equations of condition before referred to. This method possesses some advantages over that by direct multiplication; the most important of these is in the fact that the liability to error in algebraic signs is for the most part avoided. Care being taken in forming the sums $(a+b)$, $(a+c)$, etc., no further attention need be given to the algebraic signs until the coefficients of the normal equations are completed.

Solution of the Normal Equations.

27. In the solution of the normal equations the work should be arranged so that it may be conveniently reviewed for detecting errors in case such exist, and so that proof-formulæ may be applied at the various stages of progress.

The order in which the unknown quantities are determined is generally indifferent except in the case where the nature of the problem is such that one or more of them cannot be determined with accuracy from the equations. We may know in advance that we have a case of this kind, or it may be discovered in solving the equations.

It will be shown hereafter that the weight of any unknown quantity will be determined by arranging the solution in such a way that this quantity is determined first. The weight will then be represented by its coefficient in the last equation from which the others have been eliminated. If now this coefficient is very small it shows that this quantity cannot be well determined without additional data, and the solution must then be arranged so that the uncertainty in this quantity will have the least effect on the others. In case a preliminary computation shows that the weight of any unknown quantity is very small, the elimination will be repeated in such a way that this quantity is first determined. The values of the others will then be expressed in terms of this one. If then at any time additional data become available for determining this quantity, or if it is known from any other source, the other quantities become known also.

As such cases will seldom occur in the problems with which we shall have to deal, it will not be necessary to enter more fully into the matter at present.

28. In the elimination it will be convenient to employ the method of substitution, using a form of notation proposed by

Gauss. In developing the formulæ, we shall suppose as before the number of unknown quantities to be four. It will be a simple matter to extend or abridge them in case of a greater or less number.

The equations to be solved are

$$\left.\begin{aligned}
[aa]x + [ab]y + [ac]z + [ad]w &= [an]\,;\\
[ab]x + [bb]y + [bc]z + [bd]w &= [bn]\,;\\
[ac]x + [bc]y + [cc]z + [cd]w &= [cn]\,;\\
[ad]x + [bd]y + [cd]z + [dd]w &= [dn]\,.
\end{aligned}\right\} \quad . \quad . \quad (41)$$

From the first of these we have

$$x = \frac{[an]}{[aa]} - \frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z - \frac{[ad]}{[aa]}w\,; \quad . \quad . \quad (48)$$

which value being substituted in the remaining three equations, we shall have x eliminated. The first of the resulting equations will be

$$\left[[bb] - \frac{[ab]}{[aa]}[ab]\right]y + \left[[bc] - \frac{[ab]}{[aa]}[ac]\right]z$$
$$+ \left[[bd] - \frac{[ab]}{[aa]}[ad]\right]w = \left[[bn] - \frac{[ab]}{[aa]}[an]\right],$$

and similarly for the remaining two.

Let us now write

$$\left.\begin{aligned}
&[bb] - \frac{[ab]}{[aa]}[ab] = [bb\,1]\,; \qquad [bd] - \frac{[ab]}{[aa]}[ad] = [bd\,1]\,;\\
&[bc] - \frac{[ab]}{[aa]}[ac] = [bc\,1]\,; \qquad [bn] - \frac{[ab]}{[aa]}[an] = [bn\,1]\,;
\end{aligned}\right\} (49)$$

and for the coefficients of the second equation,

$$\left.\begin{array}{l}
[cc] - \frac{[ac]}{[aa]}[ac] = [cc\,1]; \quad [cn] - \frac{[ac]}{[aa]}[an] = [cn\,1]; \\
[cd] - \frac{[ac]}{[aa]}[ad] = [cd\,1]. \\
\text{Similarly for the third,} \\
[dd] - \frac{[ad]}{[aa]}[ad] = [dd\,1]; \quad [dn] - \frac{[ad]}{[aa]}[an] = [dn\,1].
\end{array}\right\} \quad (49)$$

Our three equations then become

$$\left.\begin{array}{l}
[bb\,1]y + [bc\,1]z + [bd\,1]w = [bn\,1]; \\
[bc\,1]y + [cc\,1]z + [cd\,1]w = [cn\,1]; \\
[bd\,1]y + [cd\,1]z + [dd\,1]w = [dn\,1].
\end{array}\right\} \quad . \quad . \quad . \quad (50)$$

In these the same symmetry of notation is preserved as in the normal equations, and it can easily be shown that the terms $[bb\,1]$, $[cc\,1]$, and $[dd\,1]$, which have the quadratic form, will always be positive.

From the first of (50) we have

$$y = \frac{[bn\,1]}{[bb\,1]} - \frac{[bc\,1]}{[bb\,1]}z - \frac{[bd\,1]}{[bb\,1]}w. \quad . \quad . \quad . \quad (51)$$

This is to be substituted in the second and third, and the following auxiliary coefficients computed:

$$\left.\begin{array}{l}
[cc\,1] - \frac{[bc\,1]}{[bb\,1]}[bc\,1] = [cc\,2]; \quad [cn\,1] - \frac{[bc\,1]}{[bb\,1]}[bn\,1] = [cn\,2]; \\
[cd\,1] - \frac{[bc\,1]}{[bb\,1]}[bd\,1] = [cd\,2]; \\
[dd\,1] - \frac{[bd\,1]}{[bb\,1]}[bd\,1] = [dd\,2]; \quad [dn\,1] - \frac{[bd\,1]}{[bb\,1]}[bn\,1] = [dn\,2];
\end{array}\right\} \quad (49)_1$$

which process gives us the following equations:

$$\left.\begin{aligned}[cc\,2]z + [cd\,2]w &= [cn\,2];\\ [cd\,2]z + [dd\,2]w &= [dn\,2].\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (52)$$

From the first of these,

$$z = \frac{[cn\,2]}{[cc\,2]} - \frac{[cd\,2]}{[cc\,2]}w. \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Substituting this in the second, and writing

$$[dd\,2] - \frac{[cd\,2]}{[cc\,2]}[cd\,2] = [dd\,3], \quad [dn\,2] - \frac{[cd\,2]}{[cc\,2]}[cn\,2] = [dn\,3], (49),$$

we have $$[dd\,3]w = [dn\,3]; \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

from which $$w = \frac{[dn\,3]}{[dd\,3]}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

z, y, and x can now readily be found by substituting successively in (53), (51), and (48).

The first equation in each of (41), (50), (52), and (54) are called elimination equations, and are here brought together for convenience of reference:

$$\left.\begin{aligned}[aa]x + [ab]y + [ac]z + [ad]w &= [an];\\ [bb\,1]y + [bc\,1]z + [bd\,1]w &= [bn\,1];\\ [cc\,2]z + [cd\,2]w &= [cn\,2];\\ [dd\,3]w &= [dn\,3].\end{aligned}\right\} \quad (56)$$

This is all that will be strictly necessary in case the weights and probable errors of the unknown quantities are not required.

Proof-Formulæ.

29. Convenient proof-formulæ for checking the accuracy of the successive auxiliary coefficients may be derived from the summation terms $[as]$, $[bs]$, . . . of equations (44).

Referring to these formulæ, let us write

$$[bs] - \frac{[ab]}{[aa]}[as] = [bs\,1].$$

Substituting for $[bs]$ and $[as]$ their values, this expression may be written in the form

$$[bs\,1] = \left[[bb] - \frac{[ab]}{[aa]}[ab]\right] + \left[[bc] - \frac{[ab]}{[aa]}[ac]\right] \\ + \left[[bd] - \frac{[ab]}{[aa]}[ad]\right] - \left[[bn] - \frac{[ab]}{[aa]}[an]\right].$$

Therefore, writing for the quantities in the brackets their values, we have

$$[bs\,1] = [bb\,1] + [bc\,1] + [bd\,1] - [bn\,1],$$

a formula by which the accuracy of the coefficients in the second member can be tested, and which requires the additional auxiliary quantity $[bs\,1]$.

Proceeding in a similar manner, we shall require for checking the computation at the end of the first stage of the elimination the following auxiliary quantities:

$$[bs\,1] = [bs] - \frac{[ab]}{[aa]}[as]; \qquad [cs\,1] = [cs] - \frac{[ac]}{[aa]}[as];$$

$$[ds\,1] = [ds] - \frac{[ad]}{[aa]}[as];$$

when we shall have the following proof-equations:

$$\left.\begin{aligned}[bs\,1] &= [bb\,1] + [bc\,1] + [bd\,1] - [bn\,1]\,;\\ [cs\,1] &= [bc\,1] + [cc\,1] + [cd\,1] - [cn\,1]\,;\\ [ds\,1] &= [bd\,1] + [cd\,1] + [dd\,1] - [dn\,1]\,.\end{aligned}\right\} \quad . \quad (57)$$

In the same manner we have, for checking the next step in the operation,

$$[cs\,2] = [cs\,1] - \frac{[bc\,1]}{[bb\,1]}[bs\,1]\,; \qquad [ds\,2] = [ds\,1] - \frac{[bd\,1]}{[bb\,1]}[bs\,1]:$$

$$\left.\begin{aligned}[cs\,2] &= [cc\,2] + [cd\,2] - [cn\,2]\,;\\ [ds\,2] &= [cd\,2] + [dd\,2] - [dn\,2]\,;\end{aligned}\right\} \quad . \quad . \quad . \quad (58)$$

and finally, $$[ds\,3] = [ds\,2] - \frac{[cd\,2]}{[cc\,2]}[cs\,2]\,;$$

$$[ds\,3] = [dd\,3] - [dn\,3]. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

The agreement of these two values of $[ds\,3]$ must be within the limits of error of the computation, and it furnishes a very accurate control over the accuracy of the computation up to this point.

30. After the values of x, y, z, w have been determined, a most thorough proof of the accuracy of the entire computation is obtained by means of the residuals, $v_1, v_2, \ldots$ obtained by substituting these values of x, y, z, w in the equations of condition, (37), p. 33, viz.:

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1w - n_1 &= -v_1\,;\\ a_2x + b_2y + c_2z + d_2w - n_2 &= -v_2\,;\\ a_3x + b_3y + c_3z + d_3w - n_3 &= -v_3\,.\\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot\\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot\end{aligned}\right\} \quad . \quad . \quad (37)$$

Multiplying these equations by $-v_1$, $-v_2$, $-v_3$, ... in order, adding, and writing, in accordance with the notation employed,

$$a_1v_1 + a_2v_2 + a_3v_3 + \ldots = [av],$$
$$\begin{array}{cccc} . & . & . & . \\ . & . & . & . \end{array}$$

we have

$$[nv] - [av]x - [bv]y - [cv]z - [dv]w = [vv];$$

but by equations (40),

$$[av] = 0, \qquad [bv] = 0, \qquad [cv] = 0, \qquad [dv] = 0.$$

Therefore $$[nv] = [vv]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Now multiply equations (37) by n_1, n_2, n_3 ... in order, and add, viz.:

$$[nn] - [an]x - [bn]y - [cn]z - [dn]w = [nv] = [vv]. \; (61)$$

By means of this equation $[vv]$ may also be computed as soon as x, y, z, w become known. But we have

$$x = \frac{[an]}{[aa]} - \frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z - \frac{[ad]}{[aa]}w. \quad . \quad . \quad (48)$$

Let this value be substituted in (61), and write

$$[nn] - \frac{[an]}{[aa]}[an] = [nn\,1];$$

also write $[bn\,1]$, $[cn\,1]$, etc., for their values, when we have

$$[nn\,1] - [bn\,1]y - [cn\,1]z - [dn\,1]w = [vv].$$

Let the same process be carried on for eliminating y, z, and

w in succession from this and the resulting equations. We shall have in all the following auxiliary quantities to compute:

$$[nn\,1] = [nn] - \frac{[an]}{[aa]}[an]; \quad [nn\,2] = [nn\,1] - \frac{[bn\,1]}{[bb\,1]}[bn\,1];$$
$$[nn\,3] = [nn\,2] - \frac{[cn\,2]}{[cc\,2]}[cn\,2]; \quad [nn\,4] = [nn\,3] - \frac{[dn\,3]}{[dd\,3]}[dn\,3].$$

Either of the following equations will then give the value of $[vv]$:

$$\left.\begin{array}{rl}
[nn] - [an]x - [bn]y - [cn]z - [dn]w &= [vv]; \\
[nn\,1] - [bn\,1]y - [cn\,1]z - [dn\,1]w &= [vv]; \\
[nn\,2] - [cn\,2]z - [dn\,2]w &= [vv]; \\
[nn\,3] - [dn\,3]w &= [vv]; \\
[nn\,4] &= [vv].
\end{array}\right\} \quad (62)$$

Only the last of these will generally be used.

31. The value of $[nn4] = [vv]$ can be derived from the summation quantities $[ns]$, $[ns\,1]$, etc., with very little additional labor. We have

$$[ns] = [an] + [bn] + [cn] + [dn] - [nn].$$

Let us write $\qquad [ns\,1] = [ns] - \dfrac{[an]}{[aa]}[as],$

and substitute in this expression for $[ns]$ and $[as]$ their values, when it may be placed in the following form:

$$[ns\,1] = \left[[bn] - \frac{[an]}{[aa]}[ab]\right] + \left[[cn] - \frac{[an]}{[aa]}[ac]\right]$$
$$+ \left[[dn] - \frac{[an]}{[aa]}[ad]\right] - \left[[nn] - \frac{[an]}{[aa]}[an]\right];$$

or what is the same thing,

$$[ns\,1] = [bn\,1] + [cn\,1] + [dn\,1] - [nn\,1].$$

Proceeding in a similar manner to form in succession the following auxiliary quantities, we have the series of equations by which the accuracy of the quantities $[bn\,1]$, $[cn\,1]$, ... $[nn\,4]$ may be verified:

$$\left.\begin{aligned}
[ns\,1] &= [ns] - \frac{[an]}{[aa]}[as]; \quad [ns\,2]=[ns\,1]-\frac{[bn\,1]}{[bb\,1]}[bs\,1];\\
[ns\,3] &= [ns\,2] - \frac{[cn\,2]}{[cc\,2]}[cs\,2]; \quad [ns\,4]=[ns\,3]-\frac{[dn\,3]}{[dd\,3]}[dd\,3].
\end{aligned}\right\} \quad (49)'$$

$$\left.\begin{aligned}
[ns\,1] &= [bn\,1] + [cn\,1] + [dn\,1] - [nn\,1];\\
[ns\,2] &= [cn\,2] + [dn\,2] - [nn\,2];\\
[ns\,3] &= [dn\,3] - [nn\,3];\\
[ns\,4] &= -[nn\,4].
\end{aligned}\right\} \quad (63)$$

Only the last of these equations will generally be required.

Form of Computation.

32. In computing the various auxiliary quantities which occur in the solution of a series of normal equations, the work should be arranged so that it may be carried through from beginning to end in a systematic manner in order to keep a general oversight of the results at the various stages of progress, and to apply conveniently the proof-formulæ. This will be the more important the greater the number of unknown quantities. The following scheme will be found to answer these requirements.

It will generally be found expedient to make the computation by the use of logarithms, but in some cases the computer may prefer to perform the multiplications and divisions by the aid of Crelle's table. In the following scheme we have

supposed logarithms used. A sheet of paper is first ruled with vertical columns, the number of which is greater by two than that of the unknown quantities. In the first horizontal line will be written in order the coefficients which are combined with a, viz., $[aa]$, $[ab]$, . . . $[an]$, $[as]$, and immediately below these their logarithms. Attention is directed to this line by means of the letter E in the margin, as it is the first of the elimination equations (56), and will be used for determining x after y, z, and w become known.

In the third line are the coefficients $[bb]$, $[bc]$, . . . $[bs]$, so placed that the letters combined with b fall in the same vertical column with the same letters combined with a, viz., $[bc]$ under $[ac]$, $[bd]$ under $[ad]$, etc.

In the fourth line of the first column is now written $\log \dfrac{[ab]}{[aa]}$, the value of which, as well as those of all the quantities in this column, must be carefully verified, as an error in this factor may not be detected by the proof-formula.

The $\log \dfrac{[ab]}{[aa]}$ is now written on the lower edge of a card and added in succession to the logarithms of $[ab]$, $[ac]$, . . . $[as]$, and as each addition is performed the natural number is taken from the logarithmic table and written in the place indicated in the scheme. With a little practice the computer will be able to make this addition mentally, and take from the table the corresponding number without writing down this logarithm. Thus we shall have

$$\frac{[ab]}{[aa]}[ab] \text{ written under } [bb];$$

$$\frac{[ab]}{[aa]}[ac] \text{ written under } [bc];$$

$$\begin{matrix} \cdot & & \cdot \\ \cdot & & \cdot \end{matrix}$$

$[aa]$ $\log [aa]$	$[ab]$ $\log [ab]$	$[ac]$ $\log [ac]$	$[ad]$ $\log [ad]$	$[an]$ $\log [an]$	$[as]$ $\log [as]$	E
$\log \frac{[ab]}{[aa]}$*	$[bb]$ $\frac{[ab]}{[aa]}[ab]$	$[bc]$ $\frac{[ab]}{[aa]}[ac]$	$[bd]$ $\frac{[ab]}{[aa]}[ad]$	$[bn]$ $\frac{[ab]}{[aa]}[an]$	$[bs]$ $\frac{[ab]}{[aa]}[as]$	
	$[bb\,1]$ $\log [bb\,1]$	$[bc\,1]$ $\log [bc\,1]$	$[bd\,1]$ $\log [bd\,1]$	$[bn\,1]$ $\log [bn\,1]$	$[bs\,1]$ $\log [bs\,1]$	I' E
$\log \frac{[ac]}{[aa]}$*		$[cc]$ $\frac{[ac]}{[aa]}[ac]$	$[cd]$ $\frac{[ac]}{[aa]}[ad]$	$[cn]$ $\frac{[ac]}{[aa]}[an]$	$[cs]$ $\frac{[ac]}{[aa]}[as]$	
$\log \frac{[bc\,1]}{[bb\,1]}$*		$[cc\,1]$ $\frac{[bc\,1]}{[bb\,1]}[bc\,1]$	$[cd\,1]$ $\frac{[bc\,1]}{[bb\,1]}[bd\,1]$	$[cn\,1]$ $\frac{[bc\,1]}{[bb\,1]}[bn\,1]$	$[cs\,1]$ $\frac{[bc\,1]}{[bb\,1]}[bs\,1]$	II
		$[cc\,2]$ $\log [cc\,2]$	$[cd\,2]$ $\log [cd\,2]$	$[cn\,2]$ $\log [cn\,2]$	$[cs\,2]$ $\log [cs\,2]$	III' E
$\log \frac{[ad]}{[aa]}$*			$[dd]$ $\frac{[ad]}{[aa]}[ad]$	$[dn]$ $\frac{[ad]}{[aa]}[an]$	$[ds]$ $\frac{[ad]}{[aa]}[as]$	
$\log \frac{[bd\,1]}{[bb\,1]}$*			$[dd\,1]$ $\frac{[bd\,1]}{[bb\,1]}[bd\,1]$	$[dn\,1]$ $\frac{[bd\,1]}{[bb\,1]}[bn\,1]$	$[ds\,1]$ $\frac{[bd\,1]}{[bb\,1]}[bs\,1]$	IV
$\log \frac{[cd\,2]}{[cc\,2]}$*			$[dd\,2]$ $\frac{[cd\,2]}{[cc\,2]}[cd\,2]$	$[dn\,2]$ $\frac{[cd\,2]}{[cc\,2]}[cn\,2]$	$[ds\,2]$ $\frac{[cd\,2]}{[cc\,2]}[cs\,2]$	V
			$[dd\,3]$ $\log [dd\,3]$	$[dn\,3]$ $\log [dn\,3]$	$[ds\,3]$	VI' E
$\log \frac{[an]}{[aa]}$*	$[nn]$ $\frac{[an]}{[aa]}[an]$	$[ns]$ $\frac{[an]}{[aa]}[as]$		$\log w$		
$\log \frac{[bn\,1]}{[bb\,1]}$*	$[nn\,1]$ $\frac{[bn\,1]}{[bb\,1]}[bn\,1]$	$[ns\,1]$ $\frac{[bn\,1]}{[bb\,1]}[bs\,1]$	VII			
$\log \frac{[cn\,2]}{[cc\,2]}$*	$[nn\,2]$ $\frac{[cn\,2]}{[cc\,2]}[cn\,2]$	$[ns\,2]$ $\frac{[cn\,2]}{[cc\,2]}[cs\,2]$	VIII			
$\log \frac{[dn\,3]}{[dd\,3]}$*	$[nn\,3]$ $\frac{[dn\,3]}{[dd\,3]}[dn\,3]$	$[ns\,3]$ $\frac{[dn\,3]}{[dd\,3]}[ds\,3]$	IX			
	$[nn\,4]$	$[ns\,4]$	X			

Proof-Equations.

I'. $[bs\,1] = \ldots [bd\,1] - [bn\,1]$;
II. $[cs\,1] = \ldots [cd\,1] - [cn\,1]$;
III'. $[cs\,2] = \ldots [cn\,2]$;
IV. $[ds\,1] = \ldots [dd\,1] - [dn\,1]$;
V. $[ds\,2] = \ldots [dn\,2]$;
VI'. $[ds\,3] =$
VII. $[ns\,1] = \ldots [dn\,1] - [nn\,1]$;
VIII. $[ns\,2] = \ldots [nn\,2]$;
IX. $[ns\,3] =$
X'. $[ns\,4] =$

Practically only those proof-equations which are distinguished by an accent will ordinarily be employed. The lines marked by an E in the margin give the logarithms of the coefficients of the elimination equations. The logarithms marked * must be carefully verified, since an error in one of these may escape detection by the proof-equation.

For the application to a numerical example see page 331.

and by subtraction,

$$[bb\,1],\ [bc\,1],\ [bd\,1],\ [bn\,1],\ [bs\,1].$$

These are the coefficients of the second elimination equation, and will be used for determining y after z and w have become known. The I in the margin refers to the proof-formula by which the values of these quantities will be verified.

It will not be necessary to proceed farther with this explanation, as a reference to the scheme in connection with the formulæ for the auxiliary quantities will show clearly the process. The elimination being completed, the quantities $[nn\,4]$ and $-[ns\,4]$ are computed as shown in the scheme, the agreement of which with each other and with $[vv]$, obtained by substituting the values of x, y, z, w in the equations of condition, furnishes a most thorough proof of the accuracy of the entire computation.

Weights of the Most Probable Values of the Unknown Quantities.

33. In case of a single unknown quantity determined by direct observation, the computation of the weight of the arithmetical mean was found to be very simple. In the case under consideration, where the equations to be solved contain several unknown quantities, the difficulty is greatly augmented.

In our equations of condition we have supposed the quantities observed to be n_1, n_2, n_3, etc. We have already shown that if the resulting equations of condition are not of equal weight, they may be made so by multiplying each by the square root of its respective weight. We shall therefore in investigating the weights of the unknown quantities assume the weight of each observation to be unity.

Let p_x, p_y, p_z, p_w, be the weights of x, y, z, and w respectively;
ε_x, ε_y, ε_z, ε_w, their mean errors.
Let ε be the mean error of an observation.

As all of our equations are linear, it is evident that if the elimination of the three unknown quantities x, y, and z be completely carried out, the resulting equation will give w as a linear function of n_1, n_2, n_3, etc. Similarly, if x, y, and w be eliminated, we shall have z expressed as a linear function of the same quantities, and so of each of the others.

We may therefore write

$$\left.\begin{aligned} x &= \alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3 + \text{etc.}; \\ y &= \beta_1 n_1 + \beta_2 n_2 + \beta_3 n_3 + \text{etc.}; \\ z &= \gamma_1 n_1 + \gamma_2 n_2 + \gamma_3 n_3 + \text{etc.}; \\ w &= \delta_1 n_1 + \delta_2 n_2 + \delta_3 n_3 + \text{etc.}; \end{aligned}\right\} \quad . \quad . \quad . \quad (64)$$

α, β, etc., being numerical coefficients and functions of a, b, etc.

We have now from (31), remembering the above notation,

$$\left.\begin{aligned} \varepsilon_x &= \varepsilon \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \text{etc.}} = \varepsilon \sqrt{[\alpha\alpha]}. \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \varepsilon_w &= \varepsilon \sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2 + \text{etc.}} = \varepsilon \sqrt{[\delta\delta]}. \end{aligned}\right\} \quad . \quad . \quad (65)$$

From (33), $$p_x = \frac{\varepsilon^2}{\varepsilon_x^2} = \frac{1}{[\alpha\alpha]} \cdots p_w = \frac{1}{[\delta\delta]}. \quad . \quad . \quad . \quad (66)$$

The weights therefore become known when we have the values of $[\alpha\alpha] \ldots [\delta\delta]$. For this purpose we must make use of the normal equations (41), which for convenience of reference are here rewritten:

$$\left.\begin{aligned}[][aa]x + [ab]y + [ac]z + [ad]w &= [an]\,;\\ [ab]x + [bb]y + [bc]z + [bd]w &= [bn]\,;\\ [ac]x + [bc]y + [cc]z + [cd]w &= [cn]\,;\\ [ad]x + [bd]y + [cd]z + [dd]w &= [dn]\,.\end{aligned}\right\} \quad . \quad . \quad (41)$$

Let us now assume the following system of equations:

$$\left.\begin{aligned}[][aa]Q + [ab]Q' + [ac]Q'' + [ad]Q''' &= 0\,;\\ [ab]Q + [bb]Q' + [bc]Q'' + [bd]Q''' &= 0\,;\\ [ac]Q + [bc]Q' + [cc]Q'' + [cd]Q''' &= 0\,;\\ [ad]Q + [bd]Q' + [cd]Q'' + [dd]Q''' &= 1\,.\end{aligned}\right\} \quad . \quad . \quad (67)$$

These equations will be possible, as there are four unknown quantities, Q, Q', Q'', and Q''', and four equations for determining their values; further, as the equations are of the first degree there will only be one system of values for Q, Q', etc.

Now let the normal equations be multiplied by Q, Q', Q'', and Q''', in their respective orders, and the resulting equations added. Then in consequence of (67) in the resulting equations the coefficients of x, y, and z will be zero, and that of w unity. Therefore we shall have

$$w = [an]Q + [bn]Q' + [cn]Q'' + [dn]Q'''. \quad . \quad (68)$$

We shall now show that $Q''' = [\delta\delta]$, and is therefore the reciprocal of the weight of w.

Let us expand the quantities contained in the brackets, equation (68), and compare the results with the last of equations (64). We thus find the following values of δ_1, δ_2, etc.:

$$\left.\begin{aligned}\delta_1 &= a_1Q + b_1Q' + c_1Q'' + d_1Q'''\,;\\ \delta_2 &= a_2Q + b_2Q' + c_2Q'' + d_2Q'''\,;\\ \delta_3 &= a_3Q + b_3Q' + c_3Q'' + d_3Q'''\,.\\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned}\right\} \quad . \quad . \quad . \quad (69)$$

Multiplying each of these by its a and then adding, then multiplying each by its b, c, and d successively and adding, we have by (67) the following equations:

$$\left.\begin{aligned} a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + \ldots &= [a\delta] = 0; \\ b_1\delta_1 + b_2\delta_2 + b_3\delta_3 + \ldots &= [b\delta] = 0; \\ c_1\delta_1 + c_2\delta_2 + c_3\delta_3 + \ldots &= [c\delta] = 0; \\ d_1\delta_1 + d_2\delta_2 + d_3\delta_3 + \ldots &= [d\delta] = 1. \end{aligned}\right\} \quad . \quad . \quad (70)$$

Now let each of (69) be multiplied by its δ and the results added. Then by (70) we have

$$\delta_1\delta_1 + \delta_2\delta_2 + \delta_3\delta_3 + \ldots = [\delta\delta] = Q'''. \quad \text{Q. E. D.} \quad (71)$$

The solution of equations (67) therefore determines the weight of w. In a precisely similar manner the weight of each of the unknown quantities may be determined. Thus, to determine the weight of x, we write for the second member of the first of (67) unity instead of zero, and write zero for the absolute term of each remaining equation. The resulting value of Q will be the reciprocal of the weight of x.

This process is simple enough in theory, but its application is laborious, as we must solve equations (67) separately for the weight of each unknown quantity. This does not involve so great an amount of labor as may at first appear, as much of the computation will already have been performed in the solution of the normal equations. It is easy, however, to derive a process which will generally be much more convenient. It is as follows:

34. In the solution of equations (41) by successive substitutions we found for the final equations in w—see (56)—

$$[dd\,3]w = [dn\,3].$$

We shall now show that the coefficient $[dd\,3] = \frac{1}{Q'''}$, and is therefore the weight of w.

For this purpose let us write equations (41) as follows:

$$\begin{aligned}
[aa]x + [ab]y + [ac]z + [ad]w - [an] &= A;\\
[ab]x + [bb]y + [bc]z + [bd]w - [bn] &= B;\\
[ac]x + [bc]y + [cc]z + [cd]w - [cn] &= C;\\
[ad]x + [bd]y + [cd]z + [dd]w - [dn] &= D.
\end{aligned}$$

Let us now suppose the equations solved by means of the auxiliaries Q, Q', Q'', and Q''', determined from (67), when we shall have

$$\begin{aligned}
w = [an]Q + [bn]Q' + [cn]Q'' + [dn]Q'''&\\
+ AQ + BQ' + CQ'' + DQ'''.& \qquad (72)
\end{aligned}$$

This will now be the same value of w as before obtained, if we make $A = B = C = D = 0$.

Let us now suppose the equations solved, as before, by substitution. Since in this process no new terms in D are introduced, the coefficient of D will not be changed in the final equation for w, and we shall have

$$[dd\,3]w = [dn\,3] + D + \text{terms in } A, B, \text{ and } C;$$

from which $w = \dfrac{[dn\,3]}{[dd\,3]} + \dfrac{D}{[dd\,3]} + \text{terms in } A, B, \text{ and } C.$

Now it is evident that the coefficients of $A, B, C,$ and D must be the same in this equation as in the value before obtained, equation (72). Therefore

$$Q''' = \frac{1}{[dd\,3]}. \qquad \text{Q. E. D.}$$

We therefore see that we can obtain the values of the unknown quantities from equations (41), and at the same time their respective weights, by arranging the elimination so that

each in succession shall come out last. The coefficient of the unknown quantity in the final equation will be its weight.

35. In solving a system of four equations like the above it is best to proceed as follows: Let w be determined, as above, by substitution in the order x, y, z. We then have w with its weight from

$$[dd\,3]w = [dn\,3].$$

Equations (56) then give successively z, y, and x.

Let now the elimination be performed in the opposite order, viz., w, z, y, when we have x with its weight from the equation

$$[aa\,3]x = [an\,3],$$

$[aa\,3]$ being the weight of x.

This value of x must agree with the former value within the limits of error of the computation, thus furnishing a convenient check to the accuracy of the computation.

For the weight of y and z we need not repeat the elimination, but proceed as follows:

Let us suppose the elimination performed in the order x, y, w, z. We shall then have the same auxiliary coefficients as in the first case, as far as those indicated by the numerals 1 and 2, and equations (52) will be the same as before; but as the elimination will now be performed in the order w, z, instead of z, w, we write them

$$[dd\,2]w + [cd\,2]z = [dn\,2];$$
$$[cd\,2]w + [cc\,2]z = [cn\,2].$$

From the first of these,

$$w = \frac{[dn\,2]}{[dd\,2]} - \frac{[cd\,2]}{[dd\,2]}z.$$

Substituting this in the second gives us for the coefficient of z

$$[cc\,3] = [cc\,2] - \frac{[cd\,2]}{[dd\,2]}[cd\,2] = p_z.$$

But we have $\quad [dd\,3] = [dd\,2] - \dfrac{[cd\,2]}{[cc\;2]}[cd\,2].$

From these two equations we find

$$[cc\,3] = [cc\,2]\frac{[dd\,3]}{[dd\,2]} = p_z$$

And in a similar manner,

$$[bb\,3] = [bb\,2]\frac{[aa\,3]}{[aa\,2]} = p_y.$$

We therefore have the following precepts and formulæ for computing the weights in the case of four normal equations:

$$\left.\begin{array}{l}
\text{First, perform the elimination in the order } x, y, z, w, \\
\qquad\qquad \text{then } \; p_w = [dd\,3]; \\
\qquad\qquad\qquad\;\; p_z = [cc\,2]\dfrac{[dd\,3]}{[dd\,2]}. \\
\text{Second, perform the elimination in the order } w, z, y, x, \\
\qquad\qquad \text{then } \; p_x = [aa\,3]; \\
\qquad\qquad\qquad\;\; p_y = [bb\,2]\dfrac{[aa\,3]}{[aa\,2]}.
\end{array}\right\} \quad (73)$$

The formulæ for the auxiliary coefficients for the second elimination may be derived from those for the first by simply interchanging the letters a and d and b and c. The process is so simple that it will be unnecessary to write them out in full.

Other Expressions for the Weights.

36. When the equations have been solved, as already explained, and the various checks applied, so that the computer is convinced that the results obtained are reliable, it may be undesirable to repeat the elimination merely for determining the weights of the first and second unknown quantities. We may derive convenient expressions for computing the weights in this case, as follows:

Suppose four solutions of the equations to be carried through so that each unknown quantity in turn is first determined, the order of the others remaining the same: we should then have each unknown quantity with its weight completely determined, as we have already seen. The solution of the equations for which we have given the complete formulæ is in the order d, c, b, a, where we have written the coefficients instead of the unknown quantities. If now we substitute the values of w, z, and y in the third, second, and first of equations (56) in order, we have finally the expression for x, which will be a fraction with the denominator

$$[aa]\,[bb\,1]\,[cc\,2]\,[dd\,3].$$

In the four solutions which we have supposed made, the unknown quantities last determined will be in succession $x, y, z,$

w, and the denominators of the expressions for their values will be as follows:

$$[aa]_d\,[bb\,1]_d\,[cc\,2]_d\,[dd\,3]_d\,;$$
$$[aa]_c\,[bb\,1]_c\,[dd\,2]_c\,[cc\,3]_c\,;$$
$$[aa]_b\,[cc\,1]_b\,[dd\,2]_b\,[bb\,3]_b\,;$$
$$[bb]_a\,[cc\,1]_a\,[dd\,2]_a\,[aa\,3]_a\,;$$

where the subscripts show which unknown quantity is first determined in each solution. As the elimination is performed by successive substitutions, no new factors being introduced, it follows that these expressions are equal to each other respectively.

It is evident that when the order of the elimination is changed so that a different quantity is first determined, the order of the others remaining the same as before, the values of the auxiliary coefficients $[bb\,1]$, $[cc\,2]$, etc., which do not contain the coefficient of this quantity will remain as before.

Suppose, as above, the unknown quantities to be determined in the order d, c, b, a. Now let a second solution be made in the order c, d, b, a; then all of the auxiliary coefficients as far as those designated by the numerals 1 and 2 will remain as before. In a third solution following the order b, d, c, a, the coefficients designated by the numeral 1 will have the same values as in the first case; while in a fourth determination in the order a, d, c, b, they will all differ from the first series of values.

Thus indicating by the subscripts only those coefficients which have values different from those given by the first elimination, we have the following equations:

$$[aa]\,[bb\,1]\,[cc\,2]\,[dd\,3] = [aa]\,[bb\,1]\,[dd\,2]\,\,[cc\,3]\,;$$
$$[aa]\,[bb\,1]\,[cc\,2]\,[dd\,3] = [aa]\,[cc\,1]\,[dd\,2]_b\,[bb\,3]\,;$$
$$[aa]\,[bb\,1]\,[cc\,2]\,[dd\,3] = [bb]\,[cc\,1]_a\,[dd\,2]_a\,[aa\,3].$$

We already have the weight of w. The weights of z, y, and x are given by these last equations, viz.:

$$\left.\begin{aligned}
p_w &= [dd\,3]; \\
p_z &= [cc\,3] = \frac{[cc\,2]}{[dd\,2]}[dd\,3]; \\
p_y &= [bb\,3] = [bb\,1]\frac{[cc\,2]}{[cc\,1]}\frac{[dd\,3]}{[dd\,2]_b}; \\
p_x &= [aa\,3] = [aa]\frac{[bb\,1]}{[bb]}\frac{[cc\,2]}{[cc\,1]_a}\frac{[dd\,3]}{[dd\,2]_a}.
\end{aligned}\right\} \quad . \quad . \quad . \quad (74)$$

In applying these formulæ the following additional auxiliary coefficients must be computed:

$$\left.\begin{aligned}
[dd\,2]_b &= [dd\,1] - \frac{[cd\,1]}{[cc\,1]}[cd\,1]; \\
[cc\,1]_a &= [cc] - \frac{[bc]}{[bb]}[bc]; \\
[cd\,1]_a &= [cd] - \frac{[bc]}{[bb]}[bd]; \\
[dd\,1]_a &= [dd] - \frac{[bd]}{[bb]}[bb]; \\
[dd\,2]_a &= [dd\,1]_a - \frac{[cd\,1]_a}{[cc\,1]_a}[cd\,1]_a.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (75)$$

In case of three unknown quantities the formulæ become

$$\left.\begin{aligned}
p_z &= [cc\,2]; \\
p_y &= [bb\,1]\frac{[cc\,2]}{[cc\,1]}; \\
p_x &= [aa]\frac{[bb\,1]}{[bb]}\frac{[cc\,2]}{[cc\,1]_a};
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

where $[cc\,1]_a$ has the value given above.

37. An elegant expression for the weights is obtained by making use of the determinant notation. Thus, referring to the normal equations (41),

$$w = \frac{\begin{vmatrix} [ab] & [bb] & [bc] \\ [ac] & [bc] & [cc] \\ [ad] & [bd] & [cd] \end{vmatrix}[an] + \begin{vmatrix} [aa] & [ab] & [ac] \\ [ac] & [bc] & [cc] \\ [ad] & [bd] & [cd] \end{vmatrix}[bn] + \begin{vmatrix} [aa] & [ab] & [ac] \\ [ab] & [bb] & [bc] \\ [ad] & [bd] & [cd] \end{vmatrix}[cn] + \begin{vmatrix} [aa] & [ab] & [ac] \\ [ab] & [bb] & [bc] \\ [ac] & [bc] & [cc] \end{vmatrix}[dn]}{\begin{matrix} [aa] & [ab] & [ac] & [ad] \\ [ab] & [bb] & [bc] & [bd] \\ [ac] & [bc] & [cc] & [cd] \\ [ad] & [bd] & [cd] & [dd] \end{matrix}}.$$

Q''', the reciprocal of the weight of w, given by equations (67), is the same as the value of w obtained from the above equation by making $[an] = [bn] = [cn] = 0$ and $[dn] = 1$.

Therefore writing $\varDelta$ for the complete determinant which forms the denominator of the above expression, D''' for the partial determinant formed by dropping the last horizontal line and last vertical column, D'' for the partial determinant formed by dropping the third horizontal line and third vertical column, and similarly D' and D for the other two, we have

$$\left.\begin{aligned} p_w &= \frac{\varDelta}{D'''}; & p_z &= \frac{\varDelta}{D''}; \\ p_y &= \frac{\varDelta}{D'}; & p_x &= \frac{\varDelta}{D}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (77)$$

A number of other forms may be derived for the weights, all of which involve about the same numerical operations as the above. In certain special cases different forms may be more convenient, but for our immediate purposes it will not be necessary to develop the subject further.

It may readily be seen from what precedes that the relative weights of the unknown quantities may be derived, even when the number of observations does not exceed the number of unknown quantities. No probable errors, however, can be determined in this case.

Mean Errors of the Unknown Quantities.

38. For determining the mean and probable error of an unknown quantity nothing further is required except the expression for the mean error of an observation. It is supposed that the equations of condition have been reduced to the common unit of weight by multiplying each equation when necessary by the square root of its weight.

The values of x, y, z, and w, as deduced above, are the most probable values as deduced from the given data. When substituted in the equations of condition the residuals v_1, v_2, v_3, etc., will not be the true errors unless the derived values x, y, z, and w are absolutely the true values, a condition not likely to be realized.

Let $(x+\delta x), (y+\delta y), (z+\delta z), (w+\delta w)$ be the true values; $\Delta_1, \Delta_2, \Delta_3, \ldots \Delta_m$, the true errors.

We shall then have two systems of equations, as follows:

$$\left.\begin{array}{l} a_1x + b_1y + c_1z + d_1w - n_1 = -v_1; \\ a_2x + b_2y + c_2z + d_2w - n_2 = -v_2; \\ a_3x + b_3y + c_3z + d_3w - n_3 = -v_3. \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad . \quad . \quad (78)$$

$$\left.\begin{array}{l} a_1(x+\delta x)+b_1(y+\delta y)+c_1(z+\delta z)+d_1(w+\delta w) - n_1 = -\Delta_1; \\ a_2(x+\delta x)+b_2(y+\delta y)+c_2(z+\delta z)+d_2(w+\delta w) - n_2 = -\Delta_2; \\ a_3(x+\delta x)+b_3(y+\delta y)+c_3(z+\delta z)+d_3(w+\delta w) - n_3 = -\Delta_3. \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} (79)$$

Let us multiply each of equations (78) by its v and add the resulting equations. Then by (40) the coefficients of x, y, z, and w will vanish, giving us the relation before derived,

$$[vn] = [vv]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

Proceeding in the same manner with (79), we find

$$[vn] = [v\Delta]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

Therefore $$[v\Delta] = [vv]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

In order to obtain an expression for the sum of the squares of the true errors, viz., $[\Delta\Delta]$, in terms of the sum of the squares of the residuals $[vv]$, let us first multiply each of equations (78) by its Δ and add the resulting equations; secondly, let us multiply each of (79) by its Δ and add in like manner. The results are as follows:

$$[a\Delta]x + [b\Delta]y + [c\Delta]z + [d\Delta]w - [n\Delta] = -[v\Delta] = -[vv];$$
$$[a\Delta](x + \delta x) + [b\Delta](y+\delta y) + [c\Delta](z + \delta z)$$
$$+ [d\Delta](w + \delta w) - [n\Delta] = -[\Delta\Delta].$$

Subtracting the first of these from the second, we obtain

$$[\Delta\Delta] = [vv] - [a\Delta]\delta x - [b\Delta]\delta y - [c\Delta]\delta z - [d\Delta]\delta w. \quad (83)$$

If we could now assume δx, δy, δz, and δw to vanish, we should obtain, since $m\varepsilon^2 = [\Delta\Delta]$ by definition,

$$\varepsilon^2 = \frac{[vv]}{m}.$$

This will give us a close approximation to the true value of ε when m is large.

For a more accurate determination of ε we must endeavor to find approximate values of $[a\Delta]\delta x$, $[b\Delta]\delta y$, etc. The true values are beyond our reach, but principles already established give us a means of approximation.

Multiplying each of equations (79) by its a, and adding, we have

$$\left.\begin{array}{l} [aa]x + [ab]y + [ac]z + [ad]w - [an] \\ + [aa]\delta x + [ab]\delta y + [ac]\delta z + [ad]\delta w \end{array}\right\} = -[a\Delta].$$

Comparing this with (41), we see that the first line is equal to zero.

Multiplying each equation of (79) by its b and adding, then in a similar manner by its c and d and adding, we have finally

$$\left.\begin{aligned}
[aa]\delta x + [ab]\delta y + [ac]\delta z + [ad]\delta w &= -[a\varDelta];\\
[ab]\delta x + [bb]\delta y + [bc]\delta z + [bd]\delta w &= -[b\varDelta];\\
[ac]\delta x + [bc]\delta y + [cc]\delta z + [cd]\delta w &= -[c\varDelta];\\
[ad]\delta x + [bd]\delta y + [cd]\delta z + [dd]\delta w &= -[d\varDelta].
\end{aligned}\right\} \qquad (84)$$

Comparing these with (41), we see that they are of precisely the same form, the unknown quantities being in this case δx, δy, δz, and δw, instead of x, y, z, and w, and the absolute terms having $-\varDelta$ in the place of n. The solution will therefore have the form—see (64)—

$$\left.\begin{aligned}
\delta x &= -(\alpha_1\varDelta_1 + \alpha_2\varDelta_2 + \alpha_3\varDelta_3 + \ldots);\\
\delta y &= -(\beta_1\varDelta_1 + \beta_2\varDelta_2 + \beta_3\varDelta_3 + \ldots);\\
\delta z &= -(\gamma_1\varDelta_1 + \gamma_2\varDelta_2 + \gamma_3\varDelta_3 + \ldots);\\
\delta w &= -(\delta_1\varDelta_1 + \delta_2\varDelta_2 + \delta_3\varDelta_3 + \ldots).
\end{aligned}\right\} \quad . \quad . \quad (85)$$

If we now write these values in (83), we shall have for $-[a\varDelta]\delta x$, etc., the following values:

$$\left.\begin{aligned}
-[a\varDelta]\delta x &= (a_1\varDelta_1 + a_2\varDelta_2 + a_3\varDelta_3 + \ldots)\\
&\qquad\qquad (\alpha_1\varDelta_1 + \alpha_2\varDelta_2 + \alpha_3\varDelta_3 + \ldots);\\
-[b\varDelta]\delta y &= (b_1\varDelta_1 + b_2\varDelta_2 + b_3\varDelta_3 + \ldots)\\
&\qquad\qquad (\beta_1\varDelta_1 + \beta_2\varDelta_2 + \beta_3\varDelta_3 + \ldots);\\
-[c\varDelta]\delta z &= (c_1\varDelta_1 + c_2\varDelta_2 + c_3\varDelta_3 + \ldots)\\
&\qquad\qquad (\gamma_1\varDelta_1 + \gamma_2\varDelta_2 + \gamma_3\varDelta_3 + \ldots);\\
-[d\varDelta]\delta w &= (d_1\varDelta_1 + d_2\varDelta_2 + d_3\varDelta_3 + \ldots)\\
&\qquad\qquad (\delta_1\varDelta_1 + \delta_2\varDelta_2 + \delta_3\varDelta_3 + \ldots).
\end{aligned}\right\} (86)$$

In regard to these products it is to be remarked that they must necessarily be positive, as our conditions require $[vv]$

to be a minimum. Any system of values of x, y, z and w, therefore, differing from those derived from the normal equations (41) must increase the sum of the squares of the residuals. Therefore $[\Delta\Delta] > [vv]$, and the terms following $[vv]$ in (83) must be positive.

Let us now perform the indicated multiplication in (86).

Confining ourselves to the last equation, since the form is the same for all, we can indicate the result as follows:

$$-[d\Delta]\delta w = d_1\delta_1\Delta_1\Delta_1 + d_2\delta_2\Delta_2\Delta_2 + d_3\delta_3\Delta_3\Delta_3 + \ldots + \Sigma k(\Delta_p\Delta_r).$$

The last term indicates the sum of all the terms formed by multiplying together different values of Δ, as $\Delta_1\Delta_2$, $\Delta_1\Delta_3$, ... $\Delta_{m-1}\Delta_m$. Now, since positive and negative errors occur with equal frequency when the number of equations of condition is very large, we may assume this term equal to zero.

Writing for $(\Delta_1\Delta_1)$, $(\Delta_2\Delta_2)$, etc., the mean value of those quantities, viz., ε^2, and placing for $[d\delta]$ its value from the last of (70), viz., $[d\delta] = 1$, we have

$$-[d\Delta]\delta w = \varepsilon^2.$$

In a manner precisely similar we find

$$-[a\Delta]\delta x = -[b\Delta]\delta y = -[c\Delta]\delta z = -[d\Delta]\delta w = \varepsilon^2.$$

Therefore equation (83) becomes

$$m\varepsilon^2 = [vv] + 4\varepsilon^2.$$

From which
$$\varepsilon = \pm\sqrt{\frac{[vv]}{m-4}}. \qquad (87)$$

In this case there are four unknown quantities. In general if the number of unknown quantities is μ, we shall have

$$\varepsilon = \pm\sqrt{\frac{[vv]}{m-\mu}}. \qquad (88)$$

With the values of p_x, p_y, p_z, and p_w computed by (73), we have finally

$$\varepsilon_x = \frac{\varepsilon}{\sqrt{p_x}}; \quad \varepsilon_y = \frac{\varepsilon}{\sqrt{p_y}}; \quad \varepsilon_z = \frac{\varepsilon}{\sqrt{p_z}}; \quad \varepsilon_w = \frac{\varepsilon}{\sqrt{p_w}}; \tag{89}$$

and the probable errors of x, y, z, and w will be obtained by multiplying these respectively by .6745.

We have now developed the subject as far as is necessary for our purposes. A complete example of the solution of a series of equations with three unknown quantities, together with the determination of their respective weights and probable errors, will be found in connection with article (191) of this volume.

INTERPOLATION.

39. In the Nautical Almanac are given various quantities, such as the right ascension and declination of the sun, moon, and planets, places of fixed stars, etc., which are functions of the time. This is assumed as the independent variable, or argument as it is termed by astronomers. The ephemeris gives a series of values of the function corresponding to equidistant values of the argument. In case of the moon, which moves rapidly, the position is given at intervals of one hour; the place of the sun is given at intervals of twenty-four hours; while the apparent places of the fixed stars vary so slowly that ten-day intervals are sufficiently small. When any of these quantities are required for a given time, this time will generally fall between two of the dates of the ephemeris—seldom coinciding with one of them; the required value must then be found by interpolation.

Interpolation in general is the process by which, having given a series of numerical values of any function of a quantity (or argument), the value of the function for any other value of the argument may be deduced without knowing the analytical form of the function.

We shall consider the subject more in detail than will be necessary for the simple purpose of using the ephemeris, on account of its importance in other directions.

In what follows we shall suppose the values of the function given for equidistant values of the argument, which will always be the case practically. Also the intervals must be

small enough, so that the function will be continuous between consecutive values of the argument.

Let $w =$ the interval of the argument.

$\ldots (T-3w), (T-2w), (T-w), (T), (T+w), (T+2w),$
$(T+3w), \ldots =$ the values of the argument.

The notation for the arguments, functions, and successive differences will be shown by the following scheme:

$$\left.\begin{array}{lllllll}
\text{Argument.} & \text{Function.} & \text{1st Difference.} & \text{2d Difference.} & \text{3d Difference} & \text{4th Difference.} & \text{5th Difference.} \\
T-3w & f(T-3w) & & & & & \\
 & & f'(T-\frac{5}{2}w) & & & & \\
T-2w & f(T-2w) & & f''(T-2w) & & & \\
 & & f'(T-\frac{3}{2}w) & & f'''(T-\frac{3}{2}w) & & \\
T-w & f(T-w) & & f''(T-w) & & f^{iv}(T-w) & \\
 & & f'(T-\frac{1}{2}w) & & f'''(T-\frac{1}{2}w) & & f^{v}(T-\frac{1}{2}w) \\
T & f(T) & & f''(T) & & f^{iv}(T) & \\
 & & f'(T+\frac{1}{2}w) & & f'''(T+\frac{1}{2}w) & & f^{v}(T+\frac{1}{2}w) \\
T+w & f(T+w) & & f''(T+w) & & f^{iv}(T+w) & \\
 & & f'(T+\frac{3}{2}w) & & f'''(T+\frac{3}{2}w) & & \\
T+2w & f(T+2w) & & f''(T+2w) & & & \\
 & & f'(T+\frac{5}{2}w) & & & & \\
T+3w & f(T+3w) & & & & &
\end{array}\right\} \quad (90)$$

The notation shows at once where each quantity belongs in the scheme. The first differences are formed by subtracting each function from the quantity immediately following it, the argument being the arithmetical mean of the arguments of the two functions. Similarly the second differences are formed by subtracting each quantity in the column of first differences from the one immediately below it, and so on for the successive orders of differences. It will be observed that the even orders of differences, f'', f^{iv}, etc., fall in the same horizontal lines with the functions themselves, and have the same arguments, while the odd orders, f', f''', etc., fall between those lines. The even differences all have integral arguments, and the odd differences fractional arguments.

The arithmetical mean of two consecutive differences is indicated by writing it as a function of the intermediate argument. For example:

$$f^{v}(T) = \tfrac{1}{2}[f^{v}(T-\tfrac{1}{2}w) + f^{v}(T+\tfrac{1}{2}w)];$$
$$f^{iv}(T+\tfrac{1}{2}w) = \tfrac{1}{2}[f^{iv}(T) + f^{iv}(T+w)].$$

. . . . .

40. Suppose now we set out from the function whose argument is T. Evidently,

$$\begin{aligned}
f(T+w) &= f(T)+f'(T+\tfrac{1}{2}w);\\
f(T+2w) &= f(T+w) + f'(T+\tfrac{3}{2}w)\\
&= f(T)+2f'(T+\tfrac{1}{2}w)+f''(T+w);\\
f(T+3w) &= f(T+2w)+f'(T+\tfrac{5}{2}w)\\
&= f(T)+3f'(T+\tfrac{1}{2}w)+3f''(T+w)+f'''(T+\tfrac{3}{2}w).
\end{aligned}$$

Proceeding in this manner, we readily discover the law of the series; viz., the coefficients are those of the binomial formula, and each successive function, f', f'', etc., is on the horizontal line drawn under the one which immediately precedes it. Thus we have the general formula

$$\begin{aligned}
f(T+nw) = f(T)+nf'(T+\tfrac{1}{2}w)&+\frac{n(n-1)}{1.2}f''(T+w)\\
&+\frac{n(n-1)(n-2)}{1.2.3}f'''(T+\tfrac{3}{2}w)\\
&+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}f^{iv}(T+2w)\\
&+\ldots \qquad (91)
\end{aligned}$$

If we assign integral values to n we obtain the tabular values, viz., $f(T+w)$, $f(T+2w)$, etc.; but the formula is not used for this purpose, but for interpolating between the tabular values, in which case n is fractional and must be expressed in terms of the interval of argument w as the unit.

41. A more convenient form may be given to this expression (91), as follows: We have

$$\begin{aligned}
f''(T+w) &= f''(T)+f'''(T+\tfrac{1}{2}w);\\
f'''(T+\tfrac{3}{2}w) &= f'''(T+\tfrac{1}{2}w)+f^{iv}(T)+f^{v}(T+\tfrac{1}{2}w);\\
f^{iv}(T+2w) &= f^{iv}(T)+2f^{v}(T+\tfrac{1}{2}w)+f^{vi}(T)+f^{vii}(T+\tfrac{1}{2}w).
\end{aligned}$$

. . . .

Substituting these values in (91) and reducing, we readily obtain

$$f(T+nw)=f(T)+nf'(T+\tfrac{1}{2}w)+\frac{n(n-1)}{1.2}f''(T)$$
$$+\frac{(n+1)\,n\,(n-1)}{1.2.3}f'''(T+\tfrac{1}{2}w)$$
$$+\frac{(n+1)(n)(n-1)(n-2)}{1.2.3.4}f^{iv}(T)+\ldots \quad (92)$$

The law of the series is obvious; viz., a factor is added to the numerator of each succeeding coefficient alternately after and before the other factors, the last factor of the denominator being the same as the order of differences. The successive differences are taken alternately below and above the horizontal line drawn immediately below the function from which we set out.

Formula (92) will be used for interpolating forward. For interpolating backward a better form may be derived by writing for $f'(T+\frac{1}{2}w), f'''(T+\frac{1}{2}w), \ldots$ their values in terms of $f'(T-\frac{1}{2}w), f'''(T-\frac{1}{2}w), \ldots$ viz.:

$$f'\,(T+\tfrac{1}{2}w)=f'\,(T-\tfrac{1}{2}w)+f''(T);$$
$$f'''(T+\tfrac{1}{2}w)=f'''(T-\tfrac{1}{2}w)+f^{iv}(T).$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Changing n at the same time into $-n$, since the formula is to be used for interpolating backwards, we readily find

$$f(T-nw)=f(T)-nf'(T-\tfrac{1}{2}w)+\frac{n(n-1)}{1.2}f''(T)$$
$$-\frac{(n+1)\,n\,(n-1)}{1.2.3}f'''(T-\tfrac{1}{2}w)$$
$$+\frac{(n+1)\,n\,(n-1)\,(n-2)}{1.2.3.4}f^{iv}(T). \quad . \quad . \quad (93)$$

42. In applying (92) and (93) it will be more convenient to write them as follows:

$$f(T+nw)=f(T)+n\Big\{f'(T+\tfrac{1}{2}w)+\frac{n-1}{2}\Big\{f''(T)$$
$$+\frac{n+1}{3}\Big\{f'''(T+\tfrac{1}{2}w)+\frac{n-2}{4}\Big\{f^{iv}(T)$$
$$+\frac{n+2}{5}\Big\{f^{v}(T+\tfrac{1}{2}w)\ldots\Big\}\Big\}\Big\}\Big\}\Big\}\quad . \quad . \quad (92)_1$$

$$f(T-nw)=f(T)-n\Big\{f'(T-\tfrac{1}{2}w)-\frac{n-1}{2}\Big\{f''(T)$$
$$-\frac{n+1}{3}\Big\{f'''(T-\tfrac{1}{2}w)-\frac{n-2}{4}\Big\{f^{iv}(T)$$
$$-\frac{r+2}{5}\Big\{f^{v}(T-\tfrac{1}{2}w)\ldots\Big\}\Big\}\Big\}\Big\}\Big\}\quad . \quad . \quad (93)_1$$

In $(92)_1$ and $(93)_1$ each difference is used to correct the one of the next lower order immediately preceding it, and the quantities to be multiplied will generally be small. In interpolating a value of the function corresponding to a value of the argument between T and $(T+\frac{1}{2}w)$, we use $(92)_1$ and set out from $f(T)$. If the argument is between $(T+\frac{1}{2}w)$ and $(T+w)$, we use $(93)_1$ and set out from $f(T+w)$.

When the interpolation is carried to any given order of differences, as the fifth, it is a little more accurate to take the arithmetical mean of the last differences, which fall immediately above and below the horizontal line drawn in the vicinity of the required function. Thus the last term of $(92)_1$ and $(93)_1$ would be $f^{v}(T)$.

43. For the quantities tabulated in the American Ephemeris it will only be necessary to carry the interpolation to second differences; but for computing ephemerides or tables

of any continuous function, much labor is saved by computing the quantity directly for a comparatively few dates and supplying the intermediate values by interpolation. If the function is of such a character that some order of differences, as the third, fourth, or any other, vanishes, this gives exact values for the interpolated quantities, and in fact the process may then be used for computing values of the function for any value whatever of the argument. It is on this principle that "tabulating engines" are constructed.

44. As an example of the application of (90), (92)$_1$, and (93)$_1$, we take from the American Ephemeris the following values of the moon's right ascension for intervals of 12 hours:

1883, July	$f = \alpha$.	f'.	f''.	f'''.	f^{iv}.	f^{v}.
	h. m. s.					
3d, 0^h	5 45 15.68					
		29 39.05				
12^h	6 14 54.73		− 27.08			
		29 11.97		− 6.91		
4th, 0^h	6 44 6.70		− 33.99		+ 2.01	
		28 37.98		− 4.90		− .06
12^h	7 12 44.68		− 38.89		+ 1.95	
		27 59.09		− 2.95		− .01
5th, 0^h	7 40 43.77		− 41.84		+ 1.94	
		27 17.25		− 1.01		− .16
12^h	8 8 1.02		− 42.85		+ 1.78	
		26 34.40		+ .77		− .33
6th, 0^h	8 34 35.42		− 42.08		+ 1.45	
		25 52.32		+ 2.22		− .33
12^h	9 0 27.74		− 39.86		+ 1.12	
		25 12.46		+ 3.34		
7th, 0^h	9 25 40.20		− 36.52			
		24 35.94				
12^h	9 50 16.14					

Example 1. As an example of the application of $(92)_1$, let us interpolate the moon's right ascension for 1883, July 5th, 4^h.

Since the interval of the argument w is here 12^h, we have in this case $nw = 4^h$, or $n = \frac{4}{12} = \frac{1}{3}$. Setting out from July 5th, 0^h, we have

$$
\begin{aligned}
f^{v}(T - \tfrac{1}{2}w) &= - \quad .01 \\
f^{v}(T + \tfrac{1}{2}w) &= - \quad .16 \therefore f^{v}(T) = -.085 \\
\frac{n+2}{5} f^{v} &= - \quad .040 \\
f^{iv} &= + \quad 1.940 \\
\text{Corrected, } f^{iv} &= + \quad 1.900 \\
\frac{n-2}{4} \Big\{ f^{iv} + \ldots &= - \quad .792 \\
f''' &= - \quad 1.010 \\
\text{Corrected, } f''' &= - \quad 1.802 \\
\frac{n+1}{3} \Big\{ f''' + \ldots &= - \quad .801 \\
f'' &= - \; 41.840 \\
\text{Corrected, } f'' &= - \; 42.641 \\
\frac{n-1}{2} \Big\{ f'' + \ldots &= + \; 14.214 \\
f' &= 27^m 17^s.250 \\
\text{Corrected, } f' &= 27^m 31^s.464 \\
n\{ f' + \ldots &= 9^m 10^s.488 \\
f = \alpha &= 7^h 40^m 43^s.77 \\
\text{1883, July 5th, } 4^h,\ \alpha &= 7^h 49^m 54^s.26
\end{aligned}
$$

This value agrees exactly with that found in the American Ephemeris for 1883 (see page 115).

Example 2. Let us now apply (93), to determine the moon's right ascension, July 5th, 20^h. Here we set out from July 6. As before, $n = \frac{1}{6}, f^v(T) = -.33$.

$$
\begin{array}{rl}
-\frac{n+2}{5}f^v & = + \quad .154 \\
f^{iv} & = + \quad 1.450 \\
\text{Corrected, } f^{iv} & = + \quad 1.604 \\
-\frac{n-2}{4}\left\{f^{iv} - \ldots\right. & = + \quad .668 \\
f''' & = + \quad .770 \\
\text{Corrected, } f''' & = + \quad 1.438 \\
-\frac{n+1}{3}\left\{f''' - \ldots\right. & = - \quad .639 \\
f'' & = - \; 42.080 \\
\text{Corrected, } f'' & = - \; 42.719 \\
-\frac{n-1}{2}\left\{f'' - \ldots\right. & = - \; 14.240 \\
f' & = 26^m 34^s.400 \\
\text{Corrected, } f' & = 26^m 20^s.160 \\
-n\{f' - \ldots & = \; 8^m 46^s.720 \\
f = \alpha & = 8^h 34^m 35^s.42 \\
\text{1883, July 5th, } 20^h \; \alpha & = 8^h 25^m 48^s.70
\end{array}
$$

The algebraic signs of the various corrections are determined without difficulty, as follows: If a horizontal line be drawn in the table of functions and differences (p. 75) in the vicinity of the given argument (in the first of the above examples immediately below $5^d\, 0^h$), the successive differences required will fall alternately below and above this line.

Beginning with f^v we determine the correction to f^{iv}, which is to be applied so as to bring the value nearer to that immediately below the line. In this case $f^{iv} = +1.94$; that which immediately follows is $+1.78$; therefore the correction must be subtracted from 1.94, giving the corrected $f^{iv} = 1.90$.

The value of f''' is -1.01; the value immediately above the line is -2.95. The first must be corrected so as to bring it nearer the latter, giving in this case the corrected $f''' = -1.802$, and so on for each difference in succession. That is,

When the quantity is $\left\{\begin{matrix}\text{below}\\ \text{above}\end{matrix}\right\}$ the horizontal line, apply the correction so as to bring it in the direction of the one in the same vertical column immediately $\left\{\begin{matrix}\text{above}\\ \text{below}\end{matrix}\right\}$ it.

Special Cases.

45. Whenever $(92)_1$ or $(93)_1$ can be applied, nothing more will be necessary; they require, however, a knowledge of the value of the function for several dates both before and after those between which the interpolation is made. It is sometimes necessary to interpolate between values of the function near the beginning or end of the table: as, for instance, we might require from the tabular values of the moon's right ascension, given on page 75, to determine the value between the dates July 3d, 0^h, and 3d, 12^h, or between 7th, 0^h, and 7th, 12^h. In either of these cases the series of differences terminates with f'; so the above formulæ will only give the value to first differences inclusive.

We shall consider the two cases separately.

46. *First. For arguments near the beginning of the table.*

As before, calling the arguments between which it is required to interpolate the function, T and $T + w$, we may apply formula (91), setting out from $f(T)$.

If the argument for which the value of the function is required is nearer $T+w$ than T, it will be a little simpler to set out from $T+w$ and interpolate backwards. In this case the formula requires the following modification:

Changing n into $-n$, we have

$$
\begin{aligned}
f(T-nw) = f(T) - nf'(T+\tfrac{1}{2}w) + \frac{n(n+1)}{1.2}f''(T+w) \\
- \frac{n(n+1)(n+2)}{1.2.3}f'''(T+\tfrac{3}{2}w) \\
+ \frac{n(n+1)(n+2)(n+3)}{1.2.3.4}f^{iv}(T+2w) \\
- \frac{n(n+1)(n+2)(n+3)(n+4)}{1.2.3.4.5}f^{v}(T+\tfrac{5}{2}w)\ldots
\end{aligned}
$$

From the manner of forming the successive functions, we have

$$
\begin{array}{llllll}
f'(T+\tfrac{1}{2}w) = & f'(T-\tfrac{1}{2}w)+f''(T) & & & & \\
f''(T+w) = & & f''(T)+f'''(T+\tfrac{1}{2}w) & & & \\
f'''(T+\tfrac{3}{2}w) = & & & f'''(T+\tfrac{1}{2}w)+f^{iv}(T+w) & & \\
f^{iv}(T+2w) = & & & & f^{iv}(T+w)+f^{v}(T+\tfrac{3}{2}w) & \\
f^{iv}(T+\tfrac{5}{2}w) = & & & & & f^{v}(T+\tfrac{3}{2}w)+
\end{array}
$$

Substituting these values in the above and reducing, we have

$$
\begin{aligned}
f(T-nw) = f(T) - nf'(T-\tfrac{1}{2}w) + \frac{(n-1)n}{1\cdot 2}f''(T) \\
- \frac{(n-1)n(n+1)}{1.2.3}f'''(T+\tfrac{1}{2}w) \\
+ \frac{(n-1)n(n+1)(n+2)}{1.2.3.4}f^{iv}(T+w) \\
- \frac{(n-1)n(n+1)(n+2)(n+3)}{1.2.3.4.5}f^{v}(T+\tfrac{3}{2}w). \quad (94)
\end{aligned}
$$

For greater convenience in the application, (90) and (94) may now be written as follows:

$$f(T+nw)=f(T)+n\Big\{f'(T+\tfrac{1}{2}w)+\frac{n-1}{2}\Big\{f''(T+w)$$
$$+\frac{n-2}{3}\Big\{f'''(T+\tfrac{3}{2}w)+\frac{n-3}{4}\Big\{f^{iv}(T+2w)$$
$$+\frac{n-4}{5}\Big\{f^{v}(T+\tfrac{5}{2}w)\ldots\Big\}\Big\}\Big\}\Big\}\Big\} \quad . \quad . \quad (95$$

$$f(T-nw)=f(T)+n\Big\{-f'(T-\tfrac{1}{2}w)+\frac{n-1}{2}\Big\{f''(T)$$
$$+\frac{n+1}{3}\Big\{-f'''(T+\tfrac{1}{2}w)+\frac{n+2}{4}\Big\{f^{iv}(T+w)$$
$$+\frac{n+3}{5}\Big\{-f^{v}(T+\tfrac{3}{2}w)\ldots\Big\}\Big\}\Big\}\Big\}\Big\} \quad . \quad . \quad (95)_1$$

Example 3. Required the moon's right ascension, 1883, July 3d, 4^h. Referring to the series of values (Art. 44), we have for this case $nw = 4^h$; $\therefore n = \frac{1}{3}$.

$$\begin{aligned}
f^{v} &= - \quad .06 \\
\frac{n-4}{5}f^{v} &= + \quad .044 \\
f^{iv} &= + \quad 2.010 \\
\text{Corrected, } f^{iv} &= + \quad 2.054 \\
\frac{n-3}{4}\Big\{f^{iv}\ldots &= - \quad 1.369 \\
f''' &= - \quad 6.91 \\
\text{Corrected, } f''' &= - \quad 8.279
\end{aligned}$$

$$\frac{n-2}{3}\Big\{f'''\ldots = +\ \ 4.599$$

$$f'' = -\ 27.08$$

$$\text{Corrected, } f'' = -\ 22.481$$

$$\frac{n-1}{2}\Big\{f''\ldots = +\ \ 7.494$$

$$f' = 29^m 39^s.050$$

$$\text{Corrected, } f' = 29^m 46^s.544$$

$$n\{f'\ldots = 9^m 55^s.515$$

$$f = \alpha = 5^h 45^m 15^s.680$$

$$1883, \text{ July 3d, } 4^h, \ \alpha = 5^h 55^m 11^s.195$$

Example 4. Required the moon's right ascension, 1883, July 3d, 8^h. In this case we use formula $(95)_1$, since the argument is nearer 12^h than 0^h. $n = \frac{1}{3}$.

$$-f^v = +\ \ .06$$

$$\frac{n+3}{5}\Big\{-f^v = +\ \ .04$$

$$f^{iv} = +\ 2.01$$

$$\text{Corrected, } f^{iv} = +\ 2.05$$

$$\frac{n+2}{4}\Big\{f^{iv}\ldots = +\ \ 1.172$$

$$-f''' = +\ \ 6.910$$

$$\text{Corrected, } f''' = +\ \ 8.082$$

$$\frac{n+1}{3}\Big\{f'''\ldots = +\ \ 3.592$$

$$f'' = -\ 27.080$$

$$\text{Corrected, } f'' = -\ 23.488$$

$$
\begin{aligned}
\frac{n-1}{2}\Big\{f''\ldots &= + \quad 7.829 \\
-f' &= -29^{m}39^{s}.050 \\
\text{Corrected}, f' &= -29^{m}31^{s}.221 \\
n\{-f'\ldots &= -\quad 9^{m}50.^{s}407 \\
f = \alpha &= 6^{h}14^{m}54^{s}.730 \\
\text{1883, July 3d, } 8^{h},\ \alpha &= 6^{h}\ 5^{m}\ 4^{s}.323
\end{aligned}
$$

47. *Second. Arguments near the end of the table.*

Proceeding in a manner precisely similar to that of the previous article, we readily obtain the formulæ

$$
\begin{aligned}
f(T+nw) = f(T) &+ nf'(T+\tfrac{1}{2}w) + \frac{(n-1)n}{1.2}f''(T) \\
&+ \frac{(n-1)\,n\,(n+1)}{1.2.3}f'''(T-\tfrac{1}{2}w) \\
&+ \frac{(n-1)\,n\,(n+1)\,(n+2)}{1.2.3.4}f^{iv}(T-w) \\
&+ \frac{(n-1)\,n\,(n+1)\,(n+2)\,(n+3)}{1.2.3.4.5}f^{v}(T-\tfrac{3}{2}w). \quad (97)
\end{aligned}
$$

$$
\begin{aligned}
f(T-nw) = f(T) &- nf'(T-\tfrac{1}{2}w) + \frac{n(n-1)}{1.2}f''(T-w) \\
&- \frac{n(n-1)\,(n-2)}{1.2.3}f'''(T-\tfrac{3}{2}w) \\
&+ \frac{n(n-1)\,(n-2)\,(n-3)}{1.2.3.4}f^{iv}(T-2w) \\
&- \frac{n(n-1)\,(n-2)\,(n-3)\,(n-4)}{1.2.3.4.5}f^{v}(T-\tfrac{5}{2}w). \quad (97)_1
\end{aligned}
$$

The $\left\{\begin{matrix}\text{first}\\ \text{second}\end{matrix}\right\}$ of these applies for interpolating in the

direction in which the argument $\left\{ \begin{matrix} \text{increases} \\ \text{decreases} \end{matrix} \right\}$. The above may be written as follows:

$$
\begin{aligned}
f(T + nw) = f(T) + n \Big\{ f'(T + \tfrac{1}{2}w) &+ \frac{n-1}{2} \Big\{ f''(T) \\
&+ \frac{n+1}{3} \Big\{ f'''(T - \tfrac{1}{2}w) + \frac{n+2}{4} \Big\{ f^{\text{iv}}(T - w) \\
&+ \frac{n+3}{5} \Big\{ f^{\text{v}}(T - \tfrac{3}{2}w) \ldots \Big\} \Big\} \Big\} \Big\} \Big\} \quad . \quad (98)
\end{aligned}
$$

$$
\begin{aligned}
f(T - nw) = f(T) + n \Big\{ -f'(T - \tfrac{1}{2}w) &+ \frac{n-1}{2} \Big\{ f''(T - w) \\
&+ \frac{n-2}{3} \Big\{ -f'''(T - \tfrac{3}{2}w) + \frac{n-3}{4} \Big\{ f^{\text{iv}}(T - 2w) \\
&+ \frac{n-4}{5} \Big\{ -f^{\text{v}}(T - \tfrac{5}{2}w) \ldots \Big\} \Big\} \Big\} \Big\} \Big\} \quad . \quad (98_1)
\end{aligned}
$$

Example 5. Required the moon's right ascension, 1883, July 7th, 4^{h}.

$n = \frac{1}{3}$; $\quad f^{\text{v}} = -.33$; $\quad f^{\text{iv}} = +1.12$; $\quad f''' = +3.34$;
$f'' = -36.52$; $\quad f' = 24\ 35.94$; $\quad f = 9^{\text{h}}25^{\text{m}}40^{\text{s}}.20$.

Substituting in (98) as above, we find

$$\alpha = 9^{\text{h}}\ 33^{\text{m}}\ 56^{\text{s}}.05.$$

Example 6. Required the moon's right ascension, 1883, July 7th, 8^{h}.

By substituting the numerical values in formula $(98)_1$ we find for this case

$$\alpha = 9^{\text{h}}\ 42^{\text{m}}\ 7^{\text{s}}.97.$$

It will be observed that in the application of formulæ (95), $(95)_1$, (98), and $(98)_1$ the algebraic signs of the various correc-

tions may be determined in a manner entirely similar to that explained in connection with formulæ $(92)_1$ and $(93)_1$. (See Art. 44.)

Interpolation into the Middle.

48. When the function is to be interpolated for a value of the argument half way between two consecutive dates of the table, this is called *interpolation into the middle.*

For this case either $(92)_1$ or $(93)_1$ may be used, but a more convenient formula is obtained as follows. Write $\frac{1}{2}$ in place of n in (92):

$$
\begin{aligned}
f(T+\tfrac{1}{2}w) = f(T) &+ \tfrac{1}{2}f'(T+\tfrac{1}{2}w) + \frac{\frac{1}{2}\cdot-\frac{1}{2}}{1.2}f''(T) \\
&+ \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot-\frac{1}{2}}{1.2.3}f'''(T+\tfrac{1}{2}w) + \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}}{1.2.3.4}f^{iv}(T) \\
&+ \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}}{1.2.3.4.5}f^{v}(T+\tfrac{1}{2}w) + \ldots
\end{aligned}
$$

Then in (93) let $n = \frac{1}{2}$, and set out from $(T+w)$:

$$
\begin{aligned}
f(T+\tfrac{1}{2}w) = f(T+w) &- \tfrac{1}{2}f'(T+\tfrac{1}{2}w) + \frac{\frac{1}{2}\cdot-\frac{1}{2}}{1.2}f''(T+w) \\
&- \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot-\frac{1}{2}}{1.2.3}f'''(T+\tfrac{1}{2}w) + \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}}{1.2.3.4}f^{iv}(T+w) \\
&- \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}}{1.2.3.4.5}f^{v}(T+\tfrac{1}{2}w) + \ldots
\end{aligned}
$$

Taking the mean of these equations, observing in the resulting equation that the coefficients of the odd differences, f', f''', etc., vanish, and writing

$$
\begin{aligned}
\tfrac{1}{2}\{f(T)+f(T+w)\} &= [f(T+\tfrac{1}{2}w)], \\
\tfrac{1}{2}\{f''(T)+f''(T+w)\} &= f''(T+\tfrac{1}{2}w), \\
\cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

$$f(T+\tfrac{1}{2}w) = [f(T+\tfrac{1}{2}w)] - \tfrac{1}{8}f''(T+\tfrac{1}{2}w) + \tfrac{3}{128}f^{\text{iv}}(T+\tfrac{1}{2}w) - \tfrac{5}{1024}f^{\text{vi}}(T+\tfrac{1}{2}w) + \dots \qquad (99)$$

or

$$f(T+\tfrac{1}{2}w) = [f(T+\tfrac{1}{2}w)] - \tfrac{1}{8}\{f''(T+\tfrac{1}{2}w) - \tfrac{3}{16}\{f^{\text{iv}}(T+\tfrac{1}{2}w) - \tfrac{5}{24}\{f^{\text{vi}}(T+\tfrac{1}{2}w)\dots\}\}\} \qquad (99)_1$$

Example 7. Let it be required to determine the moon's right ascension, 1883, July 5th, 6^h. We must interpolate into the middle between July 5th, 0^h, and July 5th, 12^h.

$$\begin{aligned} f^{\text{iv}} &= +\ \ 1.860 \\ -\tfrac{3}{16}f^{\text{iv}} &= -\ \ \ .349 \\ f'' &= \underline{-\ 42.345} \\ \text{Corrected, } f'' &= -\ 42.694 \\ -\tfrac{1}{8}\{f''\dots &= +\ \ 5.337 \\ [f(T+\tfrac{1}{2}w)] &= \underline{7^h 54^m 22^s.395} \end{aligned}$$

Therefore 1883, July 5th, 6^h, $\alpha = 7^h 54^m 27^s.73$

Proof of Computation.

49. The method of differences furnishes a very convenient check on the accuracy of a computation, when, for a series of values of an argument succeeding each other at regular intervals, a series of values of any function have been computed. Suppose an erroneous value of one of these quantities, $f(T) + x$, has been obtained, x being the error. The functions, with the respective differences, would then be as follows:

$$\begin{array}{lllll}
f(T-3w) & & & & \\
 & f'(T-\tfrac{5}{2}w) & & & \\
f(T-2w) & & f''(T-2w) & & \\
 & f'(T-\tfrac{3}{2}w) & & f'''(T-\tfrac{3}{2}w)+x & \\
f(T-w) & & f''(T-w)+x & & f^{\text{iv}}(T-w)-4x \\
 & f'(T-\tfrac{1}{2}w)+x & & f'''(T-\tfrac{1}{2}w)-3x & \\
f(T)+x & & f''(T)-2x & & f^{\text{iv}}(T)+6x \\
 & f'(T+\tfrac{1}{2}w)-x & & f'''(T+\tfrac{1}{2}w)+3x & \\
f(T+w) & & f''(T+w)+x & & f^{\text{iv}}(T+w)-4x \\
 & f'(T+\tfrac{3}{2}w) & & f'''(T+\tfrac{3}{2}w)-x & \\
f(T+2w) & & f''(T+2w) & & \\
 & f'(T+\tfrac{5}{2}w) & & & \\
f(T+3w) & & & &
\end{array}$$

Thus the error x in the function has increased to $6x$ in the fourth difference, the greatest deviation being in the horizontal line where the erroneous value of the function is found.

Suppose, for example, an error of 5^s had been made in computing one of the values of the moon's right ascension given in Art. 44. The scheme of differences would then be as follows:

July		$f = \alpha$	f'	f''	f'''	f^{iv}
		h. m. s.				
3d,	0^h	5 45 15.68				
			29 39.05			
	12^h	6 14 54.73		− 27.08		
			29 11.97		− 1.91	
4th,	0^h	6 44 6.70		− 28.99		− 17.99
			28 42.98		− 19.90	
	12^h	7 12 49.68		− 48.89		+ 31.95
			27 54.09		+ 12.05	
5th,	0^h	7 40 43.77		− 36.84		− 18.06
			27 17.25		− 6.01	
	12^h	8 8 1.02		− 42.85		
			26 34.40			
6th,	0^h	8 34 35.42				

We see at once without going further than second differences that the value for July 4th, 12^h, is erroneous.

Differential Coefficients.

50. When we have a series of numerical values of a function, corresponding to equidistant values of the argument, we may compute the numerical values of the differential coefficients from the tabular differences as follows: Either form of the interpolation formula is arranged according to ascending powers of n. The function $f(T + nw)$ expanded by Taylor's formula, and the differential coefficients, compared with the coefficients of the different powers of n in the above expansions, give at once values of these quantities.

The most rapid convergence, and consequently the best formulæ, will be obtained by introducing into formula (92) the arithmetical means of the odd differences situated above and below the horizontal line drawn through the function from which we set out, using the notation for the arithmetical mean given on page 71.

From the manner of forming the differences we readily see

$$f'\ (T + \tfrac{1}{2}w) = f'\ (T) + \tfrac{1}{2}f''(T);$$
$$f'''(T + \tfrac{1}{2}w) = f'''(T) + \tfrac{1}{2}f^{iv}(T).$$
$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

These values being substituted in (92), we readily derive

$$\begin{aligned} f(T + nw) = f(T) + nf'(T) \quad &+ \frac{n^2}{1.2}f''(T) \\ + \frac{(n+1)\,n\,(n-1)}{1.2.3}f'''(T) &+ \frac{(n+1)\,n^2\,(n-1)}{1.2.3.4}f^{iv}(T) \\ + \frac{(n+2)\,(n+1)\,n\,(n-1)\,(n-2)}{1.2.3.4.5}&f^{v}(T)\ \ldots \end{aligned}$$

Arranging this according to ascending powers of n, it becomes

$$\begin{aligned} f(T+nw) = f(T) &+ [f'(T) - \tfrac{1}{6}f'''(T) + \tfrac{1}{30}f^{v}(T) - \tfrac{1}{140}f^{vii}(T)\ldots]\,n \\ &+ [f''(T) - \tfrac{1}{12}f^{iv}(T) + \tfrac{1}{90}f^{vi}(T)\ldots]\,\frac{n^2}{1.2} \\ &+ [f'''(T) - \tfrac{1}{4}f^{v}(T) + \tfrac{7}{120}f^{vii}(T)\ldots]\,\frac{n^3}{1.2.3} \\ &+ [f^{iv}(T) - \tfrac{1}{6}f^{vi}(T)\ldots]\,\frac{n^4}{1.2.3.4} \\ &+ [f^{v}(T) - \tfrac{1}{3}f^{vii}(T)\ldots]\,\frac{n^5}{1.2.3.4.5} \\ &+ [f^{vi}(T)\ldots]\,\frac{n^6}{1.2.3.4.5.6}. \end{aligned}$$

Expanding the function by Taylor's formula,

$$f(T+nw)=f(T)+\frac{df}{dT}nw+\frac{d^2f}{dT^2}\frac{n^2w^2}{1.2}+\frac{d^3f}{dT^3}\frac{n^3w^3}{1.2.3}$$
$$+\frac{d^4f}{dT^4}\frac{n^4w^4}{1.2.3.4}+\frac{d^5f}{dT^5}\frac{n^5w^5}{1.2.3.4.5}+\ldots \qquad (100)$$

Comparing the coefficients of like powers of n in these two series, we have the following values for the differential coefficients:

$$\left.\begin{aligned}\frac{df(T)}{dT}&=\frac{1}{w}[f'(T)-\tfrac{1}{6}f'''(T)+\tfrac{1}{30}f^{v}(T)-\tfrac{1}{140}f^{vii}(T)\ldots];\\ \frac{d^2f(T)}{dT^2}&=\frac{1}{w^2}[f''(T)-\tfrac{1}{12}f^{iv}(T)+\tfrac{1}{90}f^{vi}(T)\ldots];\\ \frac{d^3f(T)}{dT^3}&=\frac{1}{w^3}[f'''(T)-\tfrac{1}{4}f^{v}(T)+\tfrac{7}{120}f^{vii}(T)\ldots].\end{aligned}\right\}(101)$$

. . . . . .

51. Formulæ (101) will not apply to values of the function near the beginning or end of the table. We obtain formulæ for these special cases by comparing formulæ (91) and (97), respectively—arranged according to ascending powers of n—with Taylor's formula. We thus obtain—

For arguments near the beginning of table:

$$\left.\begin{aligned}\frac{df(T)}{dT}&=\frac{1}{w}[f'(T+\tfrac{1}{2}w)-\tfrac{1}{2}f''(T+w)+\tfrac{1}{3}f'''(T+\tfrac{3}{2}w)\\&\qquad-\tfrac{1}{4}f^{iv}(T+2w)+\tfrac{1}{5}f^{v}(T+\tfrac{5}{2}w)\ldots];\\ \frac{d^2f(T)}{dT^2}&=\frac{1}{w^2}[f''(T+w)-f'''(T+\tfrac{3}{2}w)+\tfrac{11}{12}f^{iv}(T+2w)\\&\qquad-\tfrac{5}{6}f^{v}(T+\tfrac{5}{2}w)\ldots];\end{aligned}\right\}(101)_1$$

. . . . .

For arguments near end of table:

$$\left.\begin{aligned}
\frac{df(T)}{dT} &= \frac{1}{w}\big[f'(T-\tfrac{1}{2}w)+\tfrac{1}{2}f''(T-w)+\tfrac{1}{3}f'''(T-\tfrac{3}{2}w)\\
&\qquad +\tfrac{1}{4}f^{iv}(T+2w)+\tfrac{1}{5}f^{v}(T-\tfrac{5}{2}w)\ldots\big];\\
\frac{d^2f(T)}{dT^2} &= \frac{1}{w^2}\big[f''(T-w)+f'''(T-\tfrac{3}{2}w)+\tfrac{11}{12}f^{iv}(T-2w)\big];\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}\right\}\quad(101).$$

Example 8. Let it be required to compute the numerical values of the differential coefficients of the moon's right ascension with respect to the time, $\frac{d\alpha}{dT}\cdot\frac{d^2\alpha}{dT^2}\ldots$ for 1883, July 5th, 0^h.

In substituting the numerical values in (101), $w, f', f'' \ldots$ must all be expressed in the same unit. It will be convenient to express them in seconds.

From the numerical values given on page 75 we have

$$\frac{1}{w}f'(T) = \frac{1658.17}{12\times 60\times 60} = +.038\,3836;$$

$$\frac{1}{w}f''(T) = \frac{-41.84}{12\times 60\times 60} = -.000\,9685;$$

$$\frac{1}{w}f'''(T) = \frac{-1.98}{12\times 60\times 60} = -.000\,0458;$$

$$\frac{1}{w}f^{iv}(T) = \frac{1.94}{12\times 60\times 60} = +.000\,0449;$$

$$\frac{1}{w}f^{v}(T) = \frac{-.085}{12\times 60\times 60} = -.000\,0020.$$

Therefore
$$\frac{d\alpha}{dT} = +.038391;$$
$$w\frac{d^2\alpha}{dT^2} = -.000972.$$

This value of $\frac{d\alpha}{dT}$ may be regarded as the fractional part of

a second which the moon's right ascension increases in one second of time at the instant July 5th, 0^h. In the hourly ephemeris of the moon given in the Nautical Almanac there is given in connection with the moon's right ascension the "difference for one minute," which is simply the value of the differential coefficient multiplied by 60; i.e., we may suppose the α in $\frac{d\alpha}{dT}$ to be expressed in seconds, and the T in minutes. Thus we have for the example above the "difference for one minute" $= 2^s.30346$. So in connection with the solar ephemeris there is given the sun's hourly motion in right ascension, which is the value of $\frac{d\alpha}{dT}$ multiplied by 60×60. The hourly motion in declination is expressed in seconds of arc.

52. By means of these differential coefficients as given in the ephemeris, the second differences are taken into account in the interpolation in a very simple manner, for we have to second differences inclusive

$$w\frac{df(T)}{dT} = f'(T + \tfrac{1}{2}w) - \tfrac{1}{2}f''(T);$$

$$w\frac{df(T+w)}{dT} = f'(T + \tfrac{1}{2}w) + \tfrac{1}{2}f''(T).$$

The difference of these expressions is

$$f''(T) = w^2\,\frac{d^2f(T)}{dT^2},$$

and

$$f(T + nw) = f(T) + n\left(\frac{df(T)}{dT}w + \frac{n}{2}\,\frac{d^2f(T)}{dT^2}w^2\right). \quad (102)$$

Thus we have only to correct the value of the first differential coefficient by adding to it algebraically the product of

the difference of two consecutive values by one half the interval n. We then use the corrected differential coefficient, as we should do if the first differences were constant.

Example 9. Required the sun's right ascension and declination, 1883, July 4th, 4^h, Bethlehem mean time.

As the longitude of Bethlehem from Washington is $-6^m 40^s.2$, the corresponding Washington time is $3^h 53^m 19^s.8$ $=$ July 4th, $3^h.8888$ = July 4.162.

From the solar ephemeris for the meridian of Washington we then find:

Date.	α.	Hourly Motion.	δ.	Hourly Motion.
July 4.0	$6^h\,53^m\,33^s.79$	$10^s.307$	$22°\,52'\,51''.1$	$-\,13''.19$
July 5.0	$6^h\,57^m\,41^s.02$	$10^s.294$	$22°\,47'\,22''.7$	$-\,14''.18$

$$w^2 \frac{d^2\alpha}{dT^2} \cdot \frac{n}{2} = .013 \times \tfrac{1}{2}.162 = .00105$$

$$\text{Corrected hourly motion} = 10^s.306$$

$$10.306 \times 3^h.889 = 40^s.08$$

$$\text{Required } \alpha = 6^h 54^m 13^s.87.$$

$$w^2 \frac{d^2\delta}{dT^2} \cdot \frac{n}{2} = .99 \quad \times \tfrac{1}{2}.162 = .080$$

$$\text{Corrected hourly motion} - 13^s.27$$

$$13.27 \times 3^h.889 = 51''\,61$$

$$\text{Required } \delta = 22°\,51'\,59''.5.$$

53. If values of the differential coefficients are required for values of the argument between the dates of the table, we may derive the necessary formulæ by differentiating the function developed by Taylor's formula (100), viz.:

$$\left.\begin{aligned} \frac{df(T+nw)}{dT} &= \frac{df(T)}{dT} + \frac{d^2f(T)}{dT^2}nw + \frac{d^3f(T)}{dT^3}\frac{n^2w^2}{1.2}\ldots; \\ \frac{d^2f(T+nw)}{dT^2} &= \frac{d^2f(T)}{dT^2} + \frac{d^3f(T)}{dT^3}nw\ldots; \\ \cdot \quad & \cdot \quad \cdot \quad \cdot \end{aligned}\right\} \quad (103)$$

Substituting in these the values of $\frac{df(T)}{dT} \cdot \frac{d^2f(T)}{dT^2} \ldots$ (101), we have the values required.*

The Ephemeris.

54. In case the American Ephemeris and Nautical Almanac is used, most of the quantities there tabulated may be taken from the tables by the method of Art. 52, an example of the application of which has been given. The lunar distances which are given in that part of the ephemeris computed for the meridian of Greenwich form an important exception. These distances are given for three-hour intervals, together with the "*proportional logarithm of the difference.*" This proportional logarithm is simply the logarithm of 3^h—the interval of the table—divided by the difference between the two consecutive distances. It is convenient to suppose the 3^h reduced to seconds of time, and the tabular distance expressed in seconds of arc. The proportional logarithm may then be defined as *the number of seconds of time required for the distance to change one second of arc.* Thus:

1883, July 6th, 0^h, distance between centres of sun and moon $= 24° \; 2' \; 55''$

1883, July 6th, 3^h, distance between centres of sun and moon $= 25° \; 32' \; 44''$

Difference $= 1° \; 29' \; 49''$

$$\text{Proportional logarithm of difference} = \log \frac{3^h}{1° \; 29' \; 49''}$$

$$= \log \frac{10800^s}{5389''} = 0.3019$$

* A very full discussion of this subject, with elaborate tables for computing the numerical coefficients, may be found in Vol. II. of Oppolzer's "Lehrbuch zur Bahnbestimmung."

For simple interpolation, disregarding second and higher orders of differences, we proceed as follows:

Let T and $T + 3^h$ = the two consecutive dates between which the distance is to be interpolated;
$T + t$ = the time for which the distance is required;
D and $D_{\prime}$ = the distances at times T and $T + 3^h$;
D' = distance at time $T + t$;
$\varDelta = D_{\prime} - D$;
$\varDelta' = D' - D$.

Then all being expressed in seconds,

$$\varDelta' : \varDelta = t : 10800;$$
$$\log \varDelta' = \log t - PL\varDelta. \quad . \quad . \quad . \quad . \quad . \quad (104)$$

If we subtract both members of this equation from log 10800, we have

$$\log \frac{10800}{\varDelta'} = \log \frac{10800}{t} + PL\varDelta,$$

or

$$PL\varDelta' = PLt + PL\varDelta. \quad . \quad . \quad . \quad . \quad . \quad (104)_1$$

With formula (104) only the common logarithmic tables are required; with $(104)_1$ we use the tables of proportional or logistic logarithms given in works on navigation. The latter tables give at once for any angle t the logarithm of $\frac{3^h}{t^h}$ or $\frac{3^\circ}{t^\circ}$. Sometimes the tables are computed for the argument $\frac{1^h}{t}$.

The following simple example will illustrate both formulæ (104) and $(104)_1$:

Example 10. Required the distance between the centres

of the sun and moon, 1883, July 6th, $1^h\ 15^m$, Greenwich mean time.

From the ephemeris, 1883, July 6th, 0^h, $D = 24°\ 2'\ 55''$

	PL Difference $= .3019$		
$t = 1^h\ 15^m = 4500^s$	$\log t = 3.6532$		
	$\log \Delta' = 3.3513.$	Therefore $\Delta' =$	$37'\ 25''$
		$D' =$	$24°\ 40'\ 20''$

For using equation $(104)_1$ we employ the tables of proportional logarithms given in Bowditch's Navigator, Table XXII:

$$\begin{aligned} &PL \text{ Difference} = .3019 \\ &PL\ 1^h\ 15^m \quad = .3802 \\ &PL\ \Delta' \quad\quad = 6821; \qquad \Delta' = 0°\ 37'\ 25''. \end{aligned}$$

As will be seen, with the proportional logarithms the quantity Δ' is given at once in degrees, minutes, and seconds, without the necessity of reducing t in the first place from the sexagesimal to the decimal notation, and in the second place reducing Δ' from the decimal to the sexagesimal. At the end of the American Ephemeris for 1871 is given a table of "*Logarithms of small Arcs in Space or Time*," by using which this reduction is also avoided.

The foregoing process disregards second and higher orders of differences. In order to take these into account, we have in the general interpolation formula (92)

$$nw = t, \qquad w = 3^h; \quad \therefore n = \frac{t}{3^h}.$$

$$f(T + t) = D', \quad f(T) = D;$$
$$f'(T + \tfrac{1}{2}w) = \Delta, \quad f''(T) = \Delta''.$$

In which Δ'' will be the difference between two consecutive values of Δ.

Then $$\frac{n-1}{2} = \frac{\frac{t}{3^h}-1}{2} = -\frac{3^h-t}{6},$$

and formula (92), becomes $D' = D + \frac{t}{3^h}\left(\varDelta - \frac{3^h-t}{6}\varDelta''\right).$

Let $\left(\varDelta - \frac{3^h-t}{6}\varDelta''\right) = [\varDelta] =$ corrected tabular difference;

$$Q = PL\varDelta; \qquad [Q] = PL[\varDelta].$$

Then we may assume

$$\left(Q - \frac{3^h-t}{6}Q''\right) = [Q] \text{ with sufficient accuracy,} \quad (105)$$

in which Q'' is the difference between two consecutive values of Q. (Q and $\varDelta$ are inverse functions one of the other, but the algebraic sign of the correction need give no trouble.)

It will be a little more accurate if we take for Q'' the arithmetical mean of the differences between Q and both the preceding and following values found in the table.

Example 11. Required the distance between the centre of the moon and Fomalhaut, 1883, July 20th, $19^h\ 20^m\ 5^s$, Gh. M. T.

From the ephemeris,

July 20th, 15^h		$Q = .4536$	$Q'' = +211$
July 20th, 18^h	$D\ 32°\ 41'\ 20''$	$Q = .4747$	
July 20th, 21^h	$D\ 31°\ 41'\ 0''$	$Q = .4995$	$Q'' = +248$

Then $t = 1^h\ 20^m\ 5^s = 1^h.3347$	$[Q] = .4683$	$\varDelta' = 0°\ 27'\ 14''.5$
Mean $Q'' = 230$	$\log t = 3.6817$	$D' = 32°\ 14'\ 5''.5$
$-\frac{3-t}{6}Q'' = -64$	$\log \varDelta' = 3.2134$	

If we had neglected the second differences in this example we should have found $\varDelta' = 0°\ 26'\ 51''$, which can only be

considered a rough approximation. If the interpolation be extended to third differences, we find $\Delta' = 27' \, 13''.8$. This differs from the first value by a quantity which will be of very little importance in practical cases.

To Find the Greenwich Time Corresponding to a Given Lunar Distance.

55. *First.* We may interpolate the time directly from the ephemeris, neglecting the second differences; then with the time so found as a first approximation we deduce the corrected proportional logarithm $[Q]$, and repeat the computation.

t being the required quantity, either (104) or $(104)_1$ give the first approximation, viz.,

$$\log t = \log \Delta' + PL\Delta, \quad . \quad . \quad . \quad . \quad . \quad (106)$$

or

$$PLt = PL\Delta' - PL\Delta. \quad . \quad . \quad . \quad . \quad . \quad (106)_1$$

Then with this value of t we determine the corrected proportional logarithm $[Q]$ by (105), and repeat the computation.

Example 12. 1883, July 20th: determine the Gh. M. T. when the distance between the moon's centre and Fomalhaut was $32° \, 14' \, 5''.5$.

$$\begin{array}{lr}
 & .4536 \\
\text{We find from the ephemeris that on July 20th, } 18^h \; D = 32° \, 41' \, 20'' \quad PL & .4747 \\
\text{Given value of } D' = 32° \, 14' \;\; 5''.5 & .4995 \\
\log \Delta' = 3.2134 \qquad \text{Therefore } \Delta' = \quad 27' \, 14''.5 & \\
PL\Delta = \;\; .4747 & \\
\log t = 3.6881 \qquad \text{Approximate } t = 1^h \, 21^m \, 16^s & \\
\text{By (105), } -\dfrac{3^h - t}{6} Q'' = -63. \text{ Therefore } [Q] = .4684 = PL\Delta & \\
\text{Repeating computation, } PL\Delta = .4684 & \\
\log \Delta' = 3.2134 & \\
t = 1^h \, 20^m \, 06^s \qquad \log t = 3.6818 & \\
\text{Required Gh. M. T., July 20th, } 19^h \, 20^m \, 6^s. &
\end{array}$$

Table I at the end of the American Ephemeris gives the correction required on account of the second differences in the moon's motion in finding the Greenwich time corresponding to a given lunar distance. It is designed to obviate the necessity for the second computation in the case just considered. The formula for this correction is derived as follows:

Let $T + t$ = the time taken from the table when second differences are neglected;

$T + t'$ = the time taken when second differences are considered;

Q and $[Q]$ = the tabular and corrected proportional logarithms.

Then (106) $\log t = \log \Delta' + Q$;

$$\log t' = \log \Delta' + [Q];$$

$$\log t' - \log t = [Q] - Q = -\frac{3^h - t}{6} Q'', \text{ from (105).}$$

Then as $\log t' - \log t$ will never be very large, we may treat it as a differential, viz.,

$$\log t' - \log t = d \log t = M\left(\frac{t' - t}{t}\right);$$

M being the modulus $= .434294$.

Then

$$M\left(\frac{t' - t}{t}\right) = -\frac{3^h - t}{6} Q'';$$

$$t' - t = -\frac{t(180^m - t)}{2.60577} Q''. \quad . \quad . \quad . \quad (107)$$

Where t is supposed given in minutes and $t' - t$ is expressed in seconds. The correction will be applied to

t with the $\left\{\begin{matrix}\text{plus}\\ \text{minus}\end{matrix}\right\}$ sign when the proportional logarithm is $\left\{\begin{matrix}\text{diminishing}\\ \text{increasing}\end{matrix}\right\}$.

If the table is not at hand, $t' - t$ may very readily be computed from (107).

In the last example, $t = 1^h\, 21^m\, 16^s = 81^m.267;$

$$Q'' = 230.$$

Therefore $t' - t = -1^m\, 10^s.8;$

$$t = 1^h\, 20^m\, 5^s.2.$$

56. In the British Nautical Almanac the differential coefficients are not given in connection with the right ascension and declination of the sun, moon, and other bodies as in the American Ephemeris. If, therefore, it is considered necessary to carry the interpolation to second differences, it must be done by the interpolation formula.

PRACTICAL ASTRONOMY.

CHAPTER I.

THE CELESTIAL SPHERE.—TRANSFORMATION OF CO-ORDINATES.

57. When we view the heavens on a clear night, the stars and other celestial bodies appear to us to be projected on the surface of a sphere of indefinite radius, with the centre at the eye of the observer.

A few hours' observation would show us that all these bodies are apparently revolving about us from east to west, in such a manner as to make a complete revolution in about twenty-four hours. This appearance we know from other considerations is due to the diurnal revolution of the earth.

In addition to this first motion we should soon recognize a second, in consequence of which the sun appears to move among the stars from west to east, in such a manner as to complete a revolution in about one year. We know this to be due to the annual revolution of the earth about the sun. There are various other motions recognized, some of which require very long periods for completing their cycle. Of

these precession and nutation are examples. Some of these motions we shall have occasion to consider hereafter.

For our purposes it will frequently be convenient to speak of the apparent motions of the heavenly bodies as if they were the true motions. Thus we say that a star passes the meridian at a given time, when we know in fact that the meridian passes the star; or that the sun rises above the horizon, when in fact the horizon passes below the sun. The reader will never be misled by such expressions, and we are by this means often able to avoid cumbersome circumlocutions in language.

As we view the celestial sphere all the heavenly bodies appear to be at equal distances, and with few exceptions to maintain the same positions relative to each other. We can measure their directions, but at present are not concerned with their distances.

The department of astronomy with which we are now occupied deals for the most part with exact measurements—either of the co-ordinates of the stars, or of the observer's position on the earth's surface. If we know the latitude and longitude of our observatory, we can by observation determine the spherical co-ordinates of any star. If, on the other hand, the positions of the heavenly bodies are known, observation furnishes the data for determining our position in latitude and longitude. It is with problems of the latter class that this book is chiefly concerned.

Spherical Co-ordinates.

58. The position of a star on the celestial sphere is determined by means of two spherical co-ordinates, measured with reference to a fixed great circle.

Three different systems are in common use, according as the circle of reference is the horizon, the equator, or the

ecliptic. For our purposes we shall define these circles as follows:

THE HORIZON *is a great circle of the celestial sphere formed by a plane passing through the eye of the observer and perpendicular to the plumb-line.*

THE CELESTIAL EQUATOR *is a great circle of the celestial sphere formed by a plane passing through the eye of the observer and perpendicular to the earth's axis.*

THE ECLIPTIC *is a great circle of the celestial sphere formed by a plane passing through the eye of the observer and parallel to the plane of the earth's orbit.*

Either of these circles considered as the basis of a system of co-ordinates is called a *primitive circle.* The great circles formed by planes perpendicular to the primitive circle are called *secondaries.*

THE ZENITH *is the point where the plumb-line produced pierces the celestial sphere above the horizon.*

THE NADIR *is the point where the plumb-line produced below the horizon pierces the celestial sphere.*

THE ZENITH and NADIR *are the poles of the horizon.*

Vertical circles are secondaries to the horizon.

Hour-circles, or circles of declination, are secondaries to the equator.

THE MERIDIAN *is the hour-circle which passes through the zenith and nadir.*

THE MERIDIAN LINE *is the line in which the plane of the meridian intersects the plane of the horizon. The north and south points of the horizon are the points in which this line pierces the celestial sphere.*

THE PRIME VERTICAL *is the great circle whose plane is perpendicular to the plane of the meridian, and passes through the zenith.*

THE EAST AND WEST LINE *is the line in which the plane of the prime vertical intersects the plane of the horizon. The east and west points of the horizon are the points in which this line pierces the celestial sphere.*

The north and south points are the poles of the prime vertical.

The east and west points are the poles of the meridian.

The Horizon.

59. The spherical co-ordinates referred to the horizon as the primitive or fundamental plane are the *altitude* and *azimuth.*

THE ALTITUDE *of a heavenly body is its distance above the horizon, measured on a vertical circle passing through that body.*

THE AZIMUTH *of a heavenly body is the distance from the north or south point of the horizon, measured on the horizon to the foot of the vertical circle passing through the body.*

For astronomical purposes it is customary to measure the azimuth from the south point through the entire circumference in the order S., W., N., E. For geodetic purposes it is generally reckoned from the north point. Navigators and surveyors frequently use other methods, which it is not necessary to enlarge on in this place.

Instead of the altitude, the *zenith distance* of a star is frequently used; this is simply the distance from the zenith to the star, measured on a great circle. The *zenith distance* and *altitude* are complements of each other.

We shall use the following notation:

h = altitude;
a = azimuth;
z = zenith distance. $z = 90° - h.$

In consequence of the diurnal motion the altitude and azimuth of any star are constantly changing their values.

The Equator.

60. The points in which the meridian intersects the equator are the north and south points of the equator. The points in which the earth's axis pierces the celestial sphere are the poles of the equator, and are called respectively the north and south pole. This line is also the axis of the heavens.

When the equator is the fundamental plane, the position of a star may be fixed either by its declination and hour-angle or by its declination and right ascension.

THE DECLINATION *of a star is its distance north or south of the equator measured on an hour-circle passing through the star. When the star is* north *of the equator the declination is* +; *when* south, —.

THE HOUR-ANGLE *of a star is the angle at either pole between the meridian and the hour-circle passing through the star; or it is the distance measured on the plane of the equator from the south point of the equator to the foot of the hour-circle passing through the star.*

The hour-angle is reckoned from the south, in the direction S., W., N., E., from 0° to 360°, or from 0^h to 24^h. In some cases it is convenient to reckon the hour-angle towards the east, in which case it must be considered minus. The hour-angle is constantly changing, in consequence of the apparent revolution of the celestial sphere. As this revolution does not affect the position of the equator, the declination is independent of the diurnal motion.

The planes of the equator and ecliptic intersect each other

at an angle of about 23° 27′. The line in which these planes intersect is the line of the equinox, and the points where it pierces the celestial sphere are the equinoctial points. They are known respectively as the *vernal equinox* and the *autumnal equinox*. The points on the equator 90° from the equinoctial points are the *solstices*, known as the *summer solstice* and the *winter solstice*. The *equinoctial colure* is the hour-circle passing through the equinoxes. The *solstitial colure* is the hour-circle passing through the solstices.

The equinoxes are the poles of the solstitial colure, and the solstices are the poles of the equinoctial colure.

THE RIGHT ASCENSION *of a star is the arc of the equator intercepted between the vernal equinox and the foot of the hour-circle passing through the star. It is reckoned from the vernal equinox, in the order of the signs Aries, Taurus, etc., from* 0° *to* 360°, *or from* 0^h *to* 24^h.

The *right ascension* and *declination* are both independent of the diurnal motion. Instead of the *declination*, the *north-polar distance* is frequently employed. It is the distance from the north pole to the star measured on a great circle, and is the complement of the declination. We shall let

δ = Declination of a star;
α = Right ascension;
t = Hour-angle;
p = North-polar distance = $90° - \delta$.

The Ecliptic.

61. When the ecliptic is the fundamental plane, the co-ordinates are called *latitude* and *longitude*.

THE LATITUDE *of a star is its distance north or south of the ecliptic measured on a secondary to the ecliptic. When north of the ecliptic the latitude is* $+$; *when south,* $-$.

THE LONGITUDE *of a star is the distance measured on the ecliptic from the vernal equinox to the foot of the secondary passing through the star. It is reckoned in the order of the signs from* 0° *to* 360°.

Longitude will be designated by λ;
Latitude will be designated by β.

These co-ordinates must not be confounded with terrestrial latitude and longitude, with which they have no connection. The system is much used in orbit computation.

Fig. 1 will serve to illustrate the preceding definitions. It represents the sphere projected on the plane of the horizon.

Z is the zenith, CVT the ecliptic, WVE the equator, O the position of any star.

$OL =$ Declination, δ;
$LQ = LPQ =$ Hour-angle, t;
$VEQ'WL =$ Right ascension, α;
$VTCD =$ Longitude, λ;
$OD =$ Latitude, β;
$OH =$ Altitude, h;
$SH =$ Azimuth, a;
$OZ =$ Zenith distance, z;
$PO =$ N. P. distance, p.

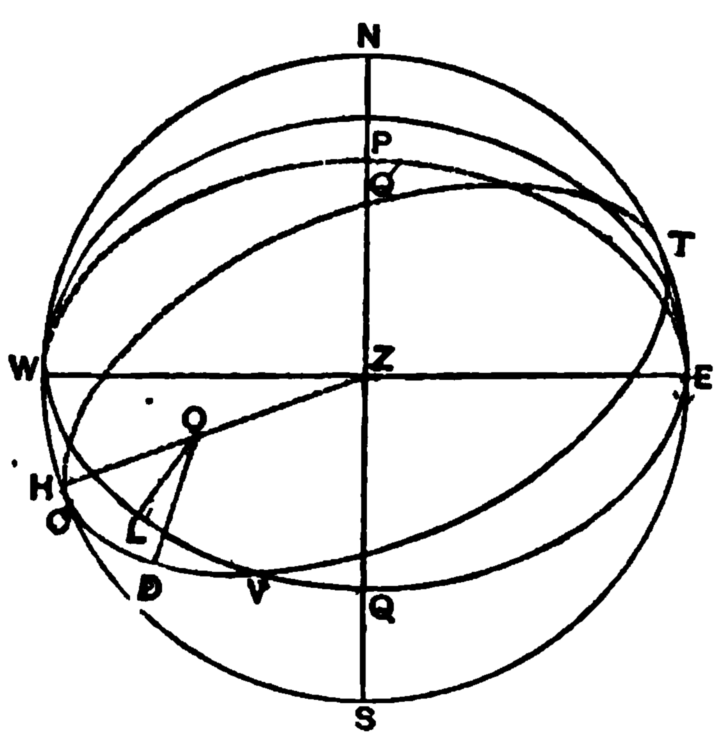

FIG. 1.

62. The following diagram will assist in giving definiteness to the symbols employed in the foregoing. The notation

should be thoroughly memorized, as the symbols will be constantly employed hereafter.

$$\text{Spherical Co-ordinates}\begin{cases}\text{Horizon}\begin{cases}\text{Azimuth} = a;\\ \text{Altitude} = h;\\ \text{Zenith distance} = z.\end{cases}\\[2ex] \text{Equator}\begin{cases}\text{Hour-angle} = t;\\ \text{Right ascension} = \alpha;\\ \text{Declination} = \delta;\\ \text{North-polar distance} = p.\end{cases}\\[2ex] \text{Ecliptic}\begin{cases}\text{Longitude} = \lambda;\\ \text{Latitude} = \beta.\end{cases}\end{cases}$$

The obliquity of the ecliptic we shall designate by ε. Its mean value for 1881.0 is $\varepsilon = 23° 27' 16''.60$. (See American Ephemeris, page 248.)

The position of the observer on the surface of the earth is given in latitude and longitude. We shall let

$$\varphi = \text{Latitude}, + \text{ when north}, - \text{ when south};$$
$$L = \text{Longitude}, + \text{ when west}, - \text{ when east}.$$

63. For astronomical purposes longitude in this country is reckoned from the meridian of Washington or Greenwich.

In Fig. 2 the large circle represents a section of the celestial sphere, and the small one a section of the earth, both formed by the intersection of the plane of the meridian. HH' is the horizon, EE' the equator, Z the zenith, Z' the nadir, P the north pole.

The latitude of the point O will be equal to the arc EZ, which by definition is the declination of the zenith of O. It is also equal to the arc PH', or the elevation of the north pole above the horizon of O.

The angle between the equator and the horizon of any place will therefore be $90° - \varphi$, φ being the latitude of the place.

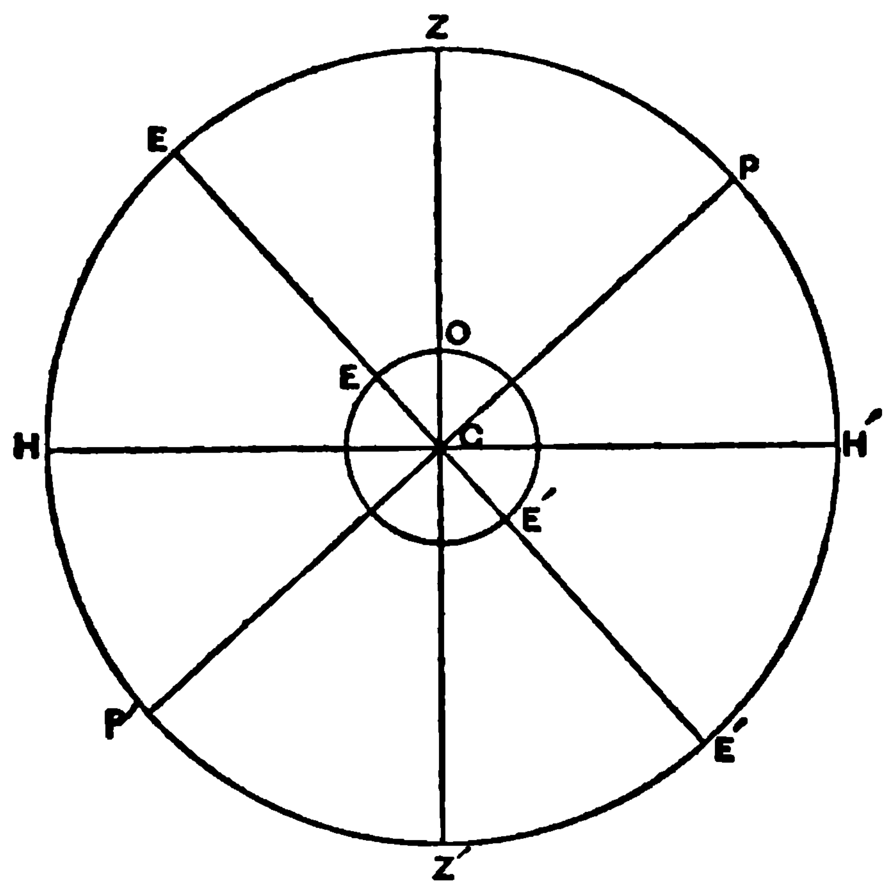

FIG. 2.

Transformation of Co-ordinates.

64. PROBLEM I. *Having given the altitude and azimuth of any star, to find the corresponding declination and hour-angle.*

Let us refer the star's position to a system of rectangular co-ordinates in which the horizon shall be the plane of XY, the positive axis of X being directed to the south point, the positive axis of Y to the west point, and the positive axis of Z to the zenith.

Then will x, y, z = the rectangular co-ordinates of the star;

Δ, h, a = the polar co-ordinates of the star;

Δ being the distance or radius vector.

We then have*

$$\left.\begin{aligned} x &= \Delta \cos h \cos a; \\ y &= \Delta \cos h \sin a; \\ z &= \Delta \sin h. \end{aligned}\right\} \quad \ldots \ldots \quad (110)$$

*See Davies' Analytical Geometry, edition of 1869, p. 302; or any other work on analytical geometry of three dimensions.

Let the star now be referred to the equator as the fundamental plane, the positive axis of X being directed to the south point of the equator, the positive axis of Y to the west point, and the positive axis of Z to the north pole.

Let now x', y', z' be the rectangular co-ordinates;
Δ, δ, t be the polar co-ordinates.

We then have

$$\left.\begin{aligned} x' &= \Delta \cos\delta \cos t; \\ y' &= \Delta \cos\delta \sin t; \\ z' &= \Delta \sin\delta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (111)$$

The problem now requires these values of x', y', and z' to be expressed in terms of x, y, and z. We observe that the axes of Y are the same in both systems; that the axes of X and Z make the angle $90° - \varphi$ with those of X' and Z'. We therefore require the formulæ for transformation of co-ordinates from one rectangular system to another having the same origin, viz.:

$$\begin{aligned} x' &= x \cos(90° - \varphi) + z \sin(90° - \varphi); \\ y' &= y; \\ z' &= -x \sin(90° - \varphi) + z \cos(90° - \varphi); \end{aligned}$$

or

$$\left.\begin{aligned} x' &= x \sin\varphi + z \cos\varphi; \\ y' &= y; \\ z' &= -x \cos\varphi + z \sin\varphi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

Substituting in (112) the values of x, y, and z from (110), and of x', y', and z' from (111), dropping at the same time the factor Δ which is common to every term, we have

$$\left.\begin{aligned} \cos\delta \cos t &= \cos h \cos a \sin\varphi + \sin h \cos\varphi; \\ \cos\delta \sin t &= \cos h \sin a; \\ \sin\delta &= -\cos h \cos a \cos\varphi + \sin h \sin\varphi. \end{aligned}\right\} \quad (113)$$

These equations express the required relation, but they are not in convenient form for logarithmic computation; besides, the required quantities δ and t are given in terms of their sines and cosines.

It is always best, when practicable, to determine an angle in terms of its tangent. The tangent varies rapidly for all angles great or small, and consequently if a small error from any cause exists in the tangent it will have but little effect on the value of the angle. On the other hand, if the value of the angle is near 90° or 270° and is given in terms of its sine, this function will vary slowly with the angle, and a small error in the sine will produce a large error in the angle. The same is true of the cosine for angles near 0° or 180°. If the angle is near 90° or 270° it may be determined with accuracy from its cosine, or if near 0° or 180° it may be accurately determined from its sine. In any case it can be determined with accuracy from its tangent.

For the purpose of effecting the required transformation in (113), let us introduce the auxiliary equations

$$\left.\begin{aligned} \sin h &= n \cos N; \\ \cos h \cos a &= n \sin N. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (114)$$

This will be possible, for we have the two arbitrary quantities n and N, and the two equations (114) for determining them. Substituting these values in (113), we have

$$\left.\begin{aligned} \cos\delta\cos t &= \quad n\sin N\sin\varphi + n\cos N\cos\varphi = n\cos(\varphi - N); \\ \cos\delta\sin t &= \cos h\sin a; \\ \sin\delta \quad &= -n\sin N\cos\varphi + n\cos N\sin\varphi = n\sin(\varphi - N). \end{aligned}\right\} (115)$$

For determining N we divide the second of (114) by the first, then we have

$$\tan N = \cot h \cos a. \quad . \quad . \quad . \quad . \quad . \quad (116)$$

For determining t we divide the second of (115) by the first, and substitute

$$n = \frac{\cos h \cos a}{\sin N}$$

from (114), viz., $\tan t = \dfrac{\sin N}{\cos(\varphi - N)} \tan a$. . . . . (117)

For determining δ, divide the third of (115) by the first:

$$\tan \delta = \tan(\varphi - N) \cos t. \quad . \quad . \quad . \quad . \quad (118)$$

We may now obtain a formula for proving the accuracy of the computation by dividing the second of (114) by the first of (115), viz.,

$$\frac{\sin N}{\cos(\varphi - N)} = \frac{\cos h \cos a}{\cos \delta \cos t}. \quad . \quad . \quad . \quad . \quad (119)$$

Formulæ (116), (117), and (118) solve the problem completely, and (119) is a proof of the accuracy of the work. The proof consists in this equation being satisfied when we substitute for δ and t the values obtained from equations (117) and (118). If the work has been correctly performed the two logarithms should not differ by more than three or four units in the last place. This proof is not always reliable, however.

Collecting together these formulæ for convenience of reference, we have

$$\left.\begin{aligned} \tan N &= \cot h \cos a; \\ \tan t &= \frac{\sin N}{\cos(\varphi - N)} \tan a; \\ \tan \delta &= \tan(\varphi - N) \cos t; \\ \frac{\sin N}{\cos(\varphi - N)} &= \frac{\cos h \cos a}{\cos \delta \cos t}. \end{aligned}\right\} \quad . \quad . \quad . \quad (I)$$

With regard to the species of these angles it is to be remarked, first, N may be taken in any quadrant which satisfies the algebraic sign of tan N; second, δ is always less than 90° and is + when tan δ is +, and − when tan is −; third, for the species of t let us examine the equation

$$\cos\delta \sin t = \cos h \sin a.$$

Cos δ and cos h will always be +, therefore the species of t will be the same as that of a.

As an example of the application of these formulæ, take the following:

$$\begin{aligned}
\text{Latitude of Sayre Observatory} &= \varphi = 40^\circ\ 36'\ 23''.9;\\
\text{Sun's altitude} &= h = 47^\circ\ 15'\ 18''.3;\\
\text{Azimuth} &= a = 80^\circ\ 23'\ \ 4''.47;
\end{aligned}$$

Required δ and t. The computation is as follows:

$$\begin{array}{rlrlrl}
\varphi = & 40^\circ\ 36'\ 23''.9 & & & & \\
h = & 47^\circ\ 15'\ 18''.3 & \cot h = & 9.9657782 & \cos h = & 9.83317007 \\
a = & 80^\circ\ 23'\ \ 4''.47 & \cos a = & \underline{9.2228053} & \cos a = & \underline{9.2228053} \\
N = & 8^\circ\ 46'\ 33''.2 & \tan N = & 9.1885835 & & 9.0545060 \\
\varphi - N = & 31^\circ\ 49'\ 50''.7 & & & & \\
t = & 46^\circ\ 40'\ \ 4''.53 & & & & \\
\delta = & 23^\circ\ \ 4'\ 24''.33 & & & &
\end{array}$$

$$\begin{array}{rlrlrl}
\tan a = & 0.7710501 & & & & \\
\sin N = & 9.1834690 & & & & \\
\sec(\varphi - N) = & \underline{.0707805} & \tan(\varphi - N) = & 9.7929304 & & \\
\tan t = & .0252996 & \cos t = & \underline{9.8364670} & \cos\ \ = & 9.8364670 \\
& & \tan\delta = & 9.6293974 & \cos\delta = & \underline{9.9637894} \\
& & & & & 9.8002564
\end{array}$$

$$\frac{\sin N}{\cos(\varphi - N)} = 9.2542495 \text{ (proof) } \frac{\cos h \cos a}{\cos\delta \cos t} = 9.2542496$$

65. PROBLEM II. *Having given the declination and hour-angle of any star, to determine the altitude and azimuth. This is the converse of the preceding problem. In this case we require the values of x, y, z in terms of the values of x', y', z'.*

Our formulæ (112) for transformation then become

$$\left.\begin{aligned} x &= x' \sin\varphi - z' \cos\varphi; \\ y &= y'; \\ z &= x' \cos\varphi + z' \sin\varphi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (120)$$

Substituting in these the values of x, y, z, x', y', z', from (110) and (111), dropping at the same time the common factor Δ, we have

$$\left.\begin{aligned} \cos h \cos a &= \cos\delta \cos t \sin\varphi - \sin\delta \cos\varphi; \\ \cos h \sin a &= \cos\delta \sin t; \\ \sin h &= \cos\delta \cos t \cos\varphi + \sin\delta \sin\varphi. \end{aligned}\right\} (121)$$

We may now adapt these equations to logarithmic computation by introducing the auxiliaries m and M, such that

$$\begin{aligned} \sin\delta &= m \sin M; \\ \cos\delta \cos t &= m \cos M; \end{aligned}$$

when, by a process like that used in solving equations (113), we find the following formulæ:

$$\left.\begin{aligned} \tan M &= \frac{\tan\delta}{\cos t}; \\ \tan a &= \frac{\cos M}{\sin(\varphi - M)} \tan t; \\ \tan h &= \frac{\cos a}{\tan(\varphi - M)}; \\ \frac{\cos M}{\sin(\varphi - M)} &= \frac{\cos\delta \cos t}{\cos h \cos a}. \end{aligned}\right\} \quad . \quad . \quad . \quad (\text{II})$$

The remarks in reference to the species of the angles in formulæ (I) will apply equally to (II).

The following example will illustrate the application of these formulæ:

Given
$$\begin{aligned} \varphi &= 40^\circ\ 36'\ 23''.9; \\ \delta &= 23^\circ\ \ 4'\ 24''.3; \\ t &= 46^\circ\ 40'\ \ 4''.5. \end{aligned}$$

Required a and h.

$$\begin{array}{rlrlrl}
\varphi = & 40^\circ\ 36'\ 23''.9 & & & & \\
\delta = & 23^\circ\ \ 4'\ 24''.3 & \tan\delta & = 9.6293972 & \cos\delta & = 9.9637894 \\
t = & 46^\circ\ 40'\ \ 4''.5 & \cos t & = \underline{9.8364670} & \cos t & = \underline{9.8364670} \\
M = & 31^\circ\ 49'\ 50''.7 & \tan M & = 9.7929302 & & 9.8002564 \\
\varphi - M = & \ \ 8^\circ\ 46'\ 33''.2 & & & & \\
a = & 80^\circ\ 23'\ \ 4''.47 & & & & \\
h = & 47^\circ\ 15'\ 18''.3 & & & &
\end{array}$$

$$\begin{array}{rlrlrl}
\tan t = & 0.0252995 & & & & \\
\cos M = & 9.9292195 & & & & \\
\operatorname{cosec}(\varphi - M) = & \underline{.8165310} & \tan(\varphi - M) = & 9.1885835 & & \\
\tan a = & 0.7710500 & \cos a = & \underline{9.2228053} & \cos a = & 9.2228053 \\
 & & \tan h = & .0342218 & \cos h = & \underline{9.8317007} \\
 & & & & & 9.0545060
\end{array}$$

$$\frac{\cos M}{\sin(\varphi - M)} = .7457505 \text{ (proof) } \frac{\cos\delta\cos t}{\cos h\cos a} = .7457504$$

66. As may readily be seen, the preceding formulæ and many more may be derived by applying the equations of Spherical Trigonometry to the triangle formed by the zenith, the pole, and the star. Thus in the figure the sides of the triangle are $90^\circ - \varphi$, $90^\circ - \delta = p$, and $90^\circ - h = z$. The angles are t, $180^\circ - a$, and q, the angle at the star, called the parallactic angle. When any three of these quantities are given, the determination of any other part is merely a question of trigonometry.

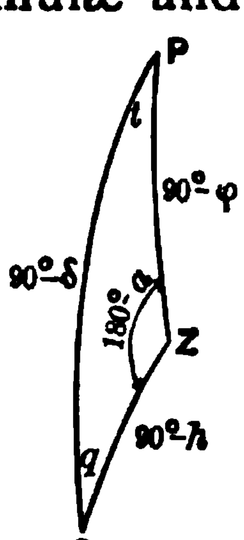

FIG. 3.

COROLLARY. *To find the hour-angle of a star when in the horizon, or at the time of rising or setting.*

When the star is in the horizon the altitude, h, is zero, and the last of equations (121) becomes

$$\cos\delta \cos t \cos\varphi + \sin\delta \sin\varphi = 0,$$

or
$$\cos t = -\frac{\sin\delta \sin\varphi}{\cos\delta \cos\varphi} = -\tan\delta \tan\varphi. \quad . \quad . \qquad (122)$$

From this equation we may determine t; but, as before remarked, it is better to determine the angle from its tangent. For this purpose first add both members of (122) to unity, then subtract both members from unity, and we have

$$1 + \cos t = \frac{\cos\delta \cos\varphi - \sin\delta \sin\varphi}{\cos\delta \cos\varphi};$$

$$1 - \cos t = \frac{\cos\delta \cos\varphi + \sin\delta \sin\varphi}{\cos\delta \cos t};$$

or
$$2\cos^2 \tfrac{1}{2}t = \frac{\cos(\varphi + \delta)}{\cos\varphi \cos\delta};$$

$$2\sin^2 \tfrac{1}{2}t = \frac{\cos(\varphi - \delta)}{\cos\varphi \cos\delta}.$$

Dividing the second of these by the first and extracting the square root,

$$\tan \tfrac{1}{2}t = \pm\sqrt{\frac{\cos(\varphi - \delta)}{\cos(\varphi + \delta)}}. \quad . \quad . \quad . \quad . \qquad (123)$$

At the time of rising the lower sign will be used; at the time of setting, the upper. This formula may be used to compute the time of sunrise and sunset at any place whose latitude is known. For example, let it be required to compute the apparent time of sunrise at Bethlehem on the morning of July 4th, 1881.

From the Nautical Almanac, page 329, we find for the sun's declination $\delta = 22° 52' 01''$.

The latitude $\varphi = 40° 36' 23''.9$.

$$\begin{aligned} \varphi - \delta &= 17° 44' 22''.9 & \cos &= 9.9788425 \\ \varphi + \delta &= 63° 28' 24''.9 & \cos &= \underline{9.6499288} \\ & & \tan^2 \tfrac{1}{2}t &= .3289137 \\ \tfrac{1}{2}t &= -\ 55° 35' 52''.5 & \tan \tfrac{1}{2}t &= .1644569_n \\ t &= -\ 111° 11' 45''.0 & & \\ t &= -\ 7^h\ 24^m\ 47^s. & & \end{aligned}$$

It being sunrise, t is minus. If we subtract this quantity from 12^h—the time when the sun is on the meridian—we have for the *apparent* time of sunrise

$$4^h\ 35^m\ 13^s.$$

This differs from the ordinary or mean time by an amount equal to the equation of time, as will be explained hereafter. (See Art. 92.)

67. PROBLEM III. *Required the distance between two stars whose right ascensions and declinations are known.*

The two stars and the pole will form the vertices of a triangle of which the sides will be $90° - \delta$, $90° - \delta'$, and d, the required distance. The angle opposite d will be $\alpha' - \alpha$.

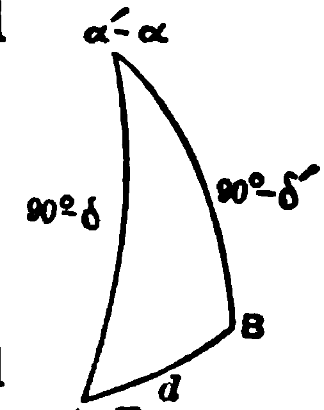

FIG. 4.

α and α' are the right ascensions of the stars.
δ and δ' are the declinations.

In the triangle two sides and the included angle are given; the third side is required.

We can apply equations (121) to this case by writing (compare Figs. 3 and 4)

$$\begin{aligned} h &= 90° - d; \\ \varphi &= \delta'; \\ t &= \alpha' - \alpha; \\ a &= 180° - B. \end{aligned}$$

Thus we have

$$\left.\begin{aligned} \sin d \cos B &= \sin \delta \cos \delta' - \cos \delta \sin \delta' \cos (\alpha' - \alpha); \\ \sin d \sin B &= \cos \delta \sin (\alpha' - \alpha); \\ \cos d &= \sin \delta \sin \delta' + \cos \delta \cos \delta' \cos (\alpha' - \alpha). \end{aligned}\right\} \quad (124)$$

If the quantity d can be determined with sufficient precision from its cosine, the last of these gives the required solution, and we may adapt it to logarithmic computation as follows:

Write
$$\begin{aligned} \sin \delta &= k \sin K; \\ \cos \delta \cos (\alpha' - \alpha) &= k \cos K. \end{aligned}$$

Then
$$\left.\begin{aligned} \tan K &= \frac{\tan \delta}{\cos (\alpha' - \alpha)}; \\ \cos d &= \frac{\sin \delta \cos (\delta' - K)}{\sin K}. \end{aligned}\right\} \quad . \ . \ . \quad (\text{IV})$$

If this does not give d with the required degree of accuracy, we may determine it in terms of the tangent in a manner precisely similar to that employed in solving equations (113) and (121). Thus, let

$$\begin{aligned} \sin \delta &= n \cos N; \\ \cos \delta \cos (\alpha' - \alpha) &= n \sin N. \end{aligned}$$

When we readily find

$$\left.\begin{aligned} \tan N &= \cot \delta \cos (\alpha' - \alpha); \\ \tan B &= \frac{\sin N}{\cos (N + \delta')} \tan (\alpha' - \alpha); \\ \tan d &= \frac{\cot (N + \delta')}{\cos B}; \\ \frac{\sin N}{\cos (N + \delta')} &= \frac{\cos \delta \cos (\alpha' - \alpha)}{\sin d \cos B}. \end{aligned}\right\} \quad . \quad \text{(IV)}.$$

Example.

Required the distance between the sun and moon, 1881, July 4th, 0^h, Bethlehem mean time.

From the Nautical Almanac for 1881, p. 114, we find, for the moon,

$$\alpha' = \quad 12^h\ 39^m\ 3^s.22;$$
$$\delta' = -\quad 9°\ 23'\ 16''.7.$$

From p. 329 of the same, for the sun,

$$\alpha = \ 6^h\ 55^m\ 32^s.73;$$
$$\delta = 22°\ 50'\ 21''.9.$$

The computation then is as follows, using equations (IV):

$\alpha' - \alpha =$	$5^h\ 43^m\ 30^s.49$		
$\alpha' - \alpha =$	$85°\ 52'\ 37''.35$	$\cos(\alpha' - \alpha) = 8.8567115$	
$\delta =$	$22°\ 50'\ 21''.9$	$\tan \delta = 9.6244585$	$\sin \delta = 9.5889992$
$K =$	$80°\ 18'\ 45''.19$	$\tan K = .7677470$	$\operatorname{cosec} K = .0062374$
$\delta' =$	$-\ 9°\ 23'\ 16''.7$		
$\delta' - K =$	$-\ 89°\ 42'\ 1''.89$		$\cos(\delta' - K) = 7.7182360$
$d =$	$89°\ 52'\ 55''.5$		$\cos d = 7.3134726$

Applying formulæ (IV)$_1$ to the solution of the same problem, we have the following:

$$
\begin{array}{rlll}
\alpha' - \alpha = & 85^\circ\ 52'\ 37''.35 & \cos = 8.8567115 & \cos = 8.8567115 \\
\delta = & 22^\circ\ 50'\ 21''.9 & \cot = \ \ .3755415 & \cos = 9.9645407 \\
\hline
N = & 9^\circ\ 41'\ 14''.8 & \tan = 9.2322530 & 8.8212522 \\
\delta' = & -\ 9^\circ\ 23'\ 16''.7 & & \\
N + \delta' = & 0^\circ\ 17'\ 58''.1 & & \\
B = & 66^\circ\ 48'\ 40''.8 & & \\
d = & 89^\circ\ 52'\ 55''.5 & &
\end{array}
$$

$$
\begin{array}{rll}
\tan(\alpha' - \alpha) = 1.1421632 & & \\
\text{sine } N = 9.2260154 & & \\
\cos(N + \delta') = 9.9999940 & \cot(N + \delta') = 2.2817621 & \\
\hline
\text{factor} = 9.2260214 & & \\
\tan B = 0.3681846 & \cos B = 9.5952317 & \cos = 9.5952317 \\
 & \tan d = 2.6865304 & \sin = 9.9999991 \\
\hline
 & & 9.5952308 \\
 & & \text{proof } 9.2260214
\end{array}
$$

$$
\frac{\sin N}{\cos(N + \delta')} = 9.2260214
$$

CHAPTER II.

PARALLAX.—REFRACTION.—DIP OF THE HORIZON.

68. The same star may be observed from points on the surface of the earth separated from each other by several thousand miles. If the distance to the star is so great that the diameter of the earth is inappreciable in comparison, it will appear in the same part of the heavens from whatever part of the earth it is seen. If, however, the diameter of the earth bears an appreciable ratio to the distance of the object, then when the observer's position changes there will be an apparent change in the place of the star. This difference in position is called parallax.

It is customary in dealing with bodies which have an appreciable parallax to reduce all positions to the earth's centre. Thus the places of the sun, moon, and planets, which we find given in the ephemeris, are the places as they would appear to an observer at the centre of the earth. This which we are considering is the *diurnal parallax*. With the subject of annual parallax, which depends upon the position of the earth in its orbit, we have at present nothing to do. It may be remarked that on account of the great distances of the fixed stars their diurnal parallax is in all cases inappreciable. It is only necessary to consider it in connection with the bodies of the solar system.

Definitions.

69. THE GEOCENTRIC POSITION *of a body is its position as seen from the earth's centre.*

THE APPARENT* *or* OBSERVED POSITION *is its place as seen from a point on the earth's surface.*

THE PARALLAX *is the difference between the geocentric and the observed place.*

It may also be defined as the angle at the body formed by two lines drawn to the centre of the earth and the place of observation respectively.

THE HORIZONTAL PARALLAX *is the parallax when the star is seen in the horizon.*

THE EQUATORIAL HORIZONTAL PARALLAX *is the parallax when seen in the horizon from a point on the earth's equator.*

It may also be defined as the angle at the body subtended by the equatorial radius of the earth.

70. PROBLEM I. *To find the equatorial horizontal parallax of a star at a given distance from the earth's centre.*

Let π = the equatorial horizontal parallax = PSC;
a = the equatorial radius of the earth = PC;
Δ = star's distance from the earth's centre = SC.

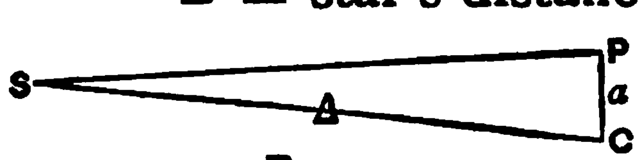

FIG. 5.

Then from the figure we have

$$\sin \pi = \frac{a}{\Delta}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

* The terms apparent place and true place are to be considered simply as relative terms. When dealing with parallax we speak of the true place as the place when corrected for parallax. So when speaking of refraction the apparent place is the place affected by refraction, and the true place is the place corrected for refraction, but it may still require corrections for parallax and a variety of other things. When dealing with the places of the fixed stars we use the term apparent place in a still different sense, as we shall see hereafter.

s being the place of the star, p a point on the surface of the earth, and c being the centre.

For astronomical purposes the mean distance of the earth from the sun is regarded as the unit of measure. Then for the sun we have

$$\Delta = 1; \qquad \sin \pi = a. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (126)$$

71. PROBLEM II. *To find the parallax of a star at any zenith distance, the earth being regarded as a sphere.*

In the figure, s represents the place of the star, z the zenith, E the centre of the earth, p a point on the surface.

Let

z' = the observed zenith distance;
z = geocentric zenith distance;
p = parallax = PSE;
a = radius of earth = PE;
Δ = distance of star = SE.

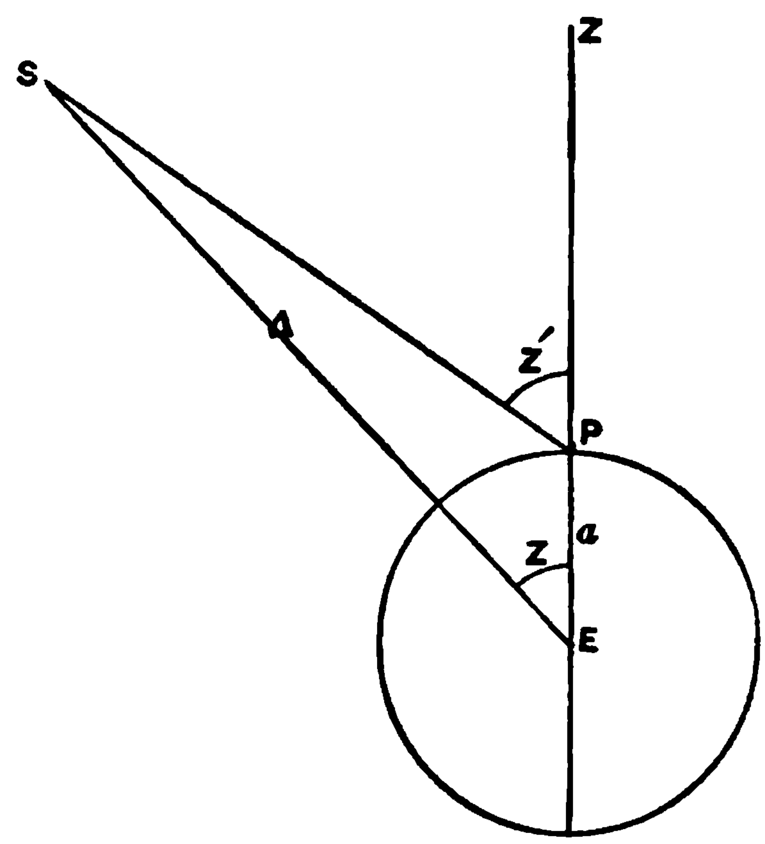

FIG. 6.

From the triangle SEP we have

$$\Delta : a = \sin z' : \sin p.$$

From which $\qquad \sin p = \dfrac{a}{\Delta} \sin z'; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (127)$

or, from (125), $\qquad \sin p = \sin \pi \sin z'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (128)$

p and π will generally be very small; hence for most purposes we may write

$$p = \pi \sin z'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (129)$$

The foregoing solution is only an approximation, the earth not being a sphere as we have there regarded it. For many purposes this is sufficiently exact, while for others, particularly where the moon is considered, it is not so. A more rigorous solution requires us to consider the true form of the earth.

Form and Dimensions of the Earth.

72. The earth is in form approximately an ellipsoid of revolution, the deviations from the exact geometrical figure being so small as to be inappreciable for our purposes.

The dimensions of the ellipsoid as given by Bessel are as follows:

$$\begin{aligned} \text{Equatorial radius } A &= 3962.8025 \text{ miles;} \\ \text{Polar radius } B &= 3949.5557 \text{ miles;} \\ \text{Eccentricity of meridian } e &= .08169683; \\ \log e &= 8.9122052. \end{aligned}$$

Many other determinations of these quantities have been made, differing more or less from the above, but these are still in more general use than any others.

Definitions.

73. THE GEOGRAPHICAL LATITUDE *of a point on the earth's surface is the angle made with the plane of the equator by a normal to the surface at this point.*

THE GEOCENTRIC LATITUDE *is the angle formed with the plane of the equator by a line joining the point with the earth's centre.*

THE ASTRONOMICAL LATITUDE *is the angle formed with the plane of the equator by a plumb-line at the given point.*

If the earth were a true ellipsoid and perfectly homogeneous, the geographical and astronomical latitude would always be the same. Practically, however, the plumb-line frequently deviates from the normal by very appreciable amounts. This deviation is always small, but in mountainous countries, as the Alps and Caucasus, deviations have been observed as great as 29″. Unless otherwise stated, when speaking of latitude the astronomical latitude is to be understood. We shall also assume for present purposes that it coincides in value with the geographical latitude.

Let the annexed figure represent a section cut from the earth's surface by a plane passing through its axis. This section will be an ellipse. Let K be any point on the surface, P and P' the north and south poles respectively. Then HH' will represent the horizon of the point K.

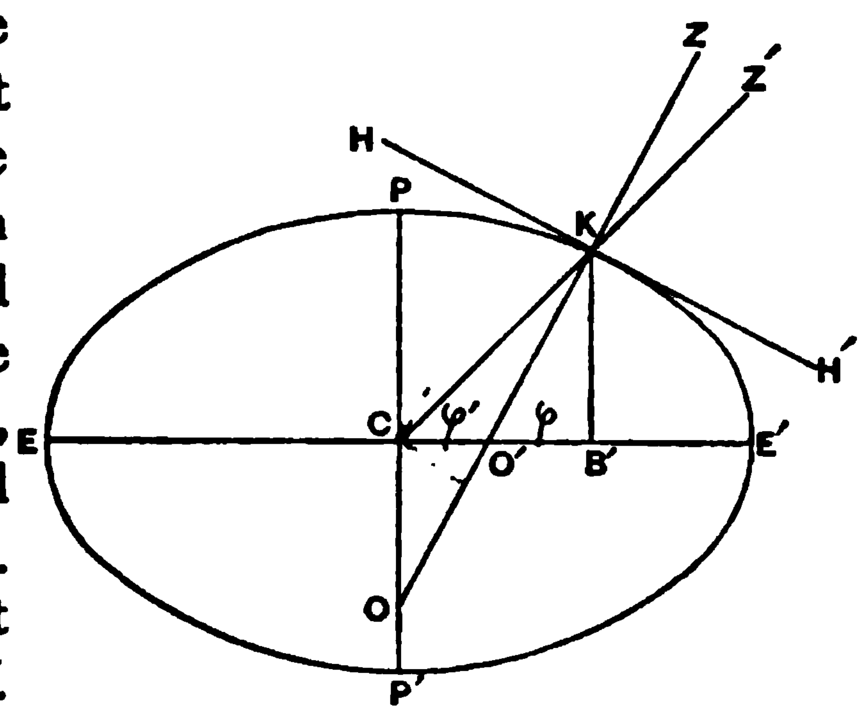

FIG. 7.

Let $\rho = CK$ = radius of the earth for latitude $KO'E'$;
$\varphi = KO'E'$ = geographical latitude of point K;
$\varphi' = KCE'$ = geocentric latitude of point K;
A = semi-major axis of ellipse = CE';
B = semi-conjugate axis of ellipse = CP.

The angle $CKO = \varphi - \varphi'$ is called the *reduction of the latitude*. For determining the parallax with precision we require $(\varphi - \varphi')$ and ρ, the determination of which for any latitude φ we shall now investigate.

To Determine $(\varphi - \varphi')$.

74. We have for the equation of the ellipse (Fig. 7)

$$A^2y^2 + B^2x^2 = A^2B^2; \quad . \quad . \quad . \quad . \quad . \quad (130)$$

$$\tan\varphi = -\frac{dx}{dy}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

φ being the angle which the normal forms with the transverse axis of the ellipse. Also,

$$\tan\varphi' = \frac{y}{x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

By differentiating (130) we find

$$-\frac{dx}{dy} = \frac{A^2y}{B^2x} = \tan\varphi. \quad . \quad . \quad . \quad . \quad . \quad (133)$$

Therefore from (132) and (133)

$$\tan\varphi' = \frac{B^2}{A^2}\tan\varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

From equation (134) φ' may be readily computed for any given value of φ. It will greatly facilitate this computation, however, to develop $(\varphi - \varphi')$ in the form of a series. For this purpose we make use of Moivre's formulæ, viz.:*

---

* As some readers may not be familiar with these very useful formulæ, we give their derivation.

Developing $n = e^x$ by Maclaurin's formula, we have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4}, \text{ etc.}; \quad . \quad . \quad . \quad (a)$$

also,

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4}, \text{ etc.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5}, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

$$\left.\begin{aligned} 2\cos x &= e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}; \\ 2\sqrt{-1}\sin x &= e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}; \\ \sqrt{-1}\tan x &= \frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1}. \end{aligned}\right\} \quad . \quad . \quad . \quad (135)$$

Writing $\tan\varphi' = p\tan\varphi$ where $p = \frac{B^2}{A^2}$, substituting for $\tan\varphi'$ and $\tan\varphi$ the value given by the last of (135), and dropping the common factor $\sqrt{-1}$, we have

$$\frac{e^{2\phi'\sqrt{-1}} - 1}{e^{2\phi'\sqrt{-1}} + 1} = p\frac{e^{2\phi\sqrt{-1}} - 1}{e^{2\phi\sqrt{-1}} + 1};$$

from which $$e^{2\phi'\sqrt{-1}} = \frac{(p+1)e^{2\phi\sqrt{-1}} - (p-1)}{(p+1) - (p-1)e^{2\phi\sqrt{-1}}}.$$

Substituting in (b) and (c) $x^2 = -z^2$, whence $x = z\sqrt{-1}$, $z = -x\sqrt{-1}$,

we have $$\cos x = 1 + \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4}\text{ etc.};$$

$$-\sqrt{-1}\sin x = z + \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5};$$

adding, $$\cos x - \sqrt{-1}\sin x = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \text{etc.}$$
$$= e^z = e^{-x\sqrt{-1}}.$$

Writing $-x$ for $+x$, we have $$\cos x - \sqrt{-1}\sin x = e^{-x\sqrt{-1}};$$
$$\cos x + \sqrt{-1}\sin x = e^{x\sqrt{-1}};$$

adding and subtracting, $$2\cos x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}};$$
$$2\sqrt{-1}\sin x = e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}.$$ Q.E.D.

Writing $q = \frac{p - 1}{p + 1}$, this becomes

$$e^{2\phi'\sqrt{-1}} = \frac{e^{2\phi\sqrt{-1}} - q}{1 - qe^{2\phi\sqrt{-1}}} = e^{2\phi\sqrt{-1}}\frac{1 - qe^{-2\phi\sqrt{-1}}}{1 - qe^{2\phi\sqrt{-1}}};$$

whence
$$e^{2\sqrt{-1}(\phi' - \phi)} = \frac{1 - qe^{-2\phi\sqrt{-1}}}{1 - qe^{2\phi\sqrt{-1}}}. \quad . \quad . \quad . \quad . \quad . \quad (136)$$

Taking the logarithms of both members of equation (136), we have

$$2\sqrt{-1}(\varphi' - \varphi) = \log(1 - qe^{-2\phi\sqrt{-1}}) - \log(1 - qe^{2\phi\sqrt{-1}}).$$

Expanding the logarithms in the second member by the formula

$$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}, \text{ etc.},$$

we have

$$2\sqrt{-1}(\varphi' - \varphi) = -qe^{-2\phi\sqrt{-1}} - \tfrac{1}{2}q^2e^{-4\phi\sqrt{-1}} - \tfrac{1}{3}q^3e^{-6\phi\sqrt{-1}}, \text{ etc.}$$
$$+ qe^{2\phi\sqrt{-1}} + \tfrac{1}{2}q^2e^{4\phi\sqrt{-1}} + \tfrac{1}{3}q^3e^{6\phi\sqrt{-1}}, \text{ etc.}$$

This becomes by the second of (135)

$$2\sqrt{-1}(\varphi' - \varphi) = 2\sqrt{-1}\, q \sin 2\varphi + 2\sqrt{-1}\,.\,\tfrac{1}{2}q^2 \sin 4\varphi$$
$$+ 2\sqrt{-1}\,\tfrac{1}{3}q^3 \sin 6\varphi, \text{ etc.},$$

or
$$\varphi' - \varphi = q \sin 2\varphi + \tfrac{1}{2}q^2 \sin 4\varphi + \tfrac{1}{3}q^3 \sin 6\varphi, \text{ etc.} \quad (137)$$

In this equation $q = \frac{p - 1}{p + 1} = \frac{B^2 - A^2}{B^2 + A^2}$.

Substituting for A and B their values given in Art. 72,

and dividing by sin 1″ in order to express the result in seconds of arc, we readily find

$$\begin{aligned} q &= -\ 690''.65; \\ \tfrac{1}{2}q^2 &= +\quad 1''.16; \\ \tfrac{1}{3}q^3 &= -\quad\ ''.003. \end{aligned}$$

Therefore we have the very convenient and practically rigorous formula

$$\varphi - \varphi' = 690''.65 \sin 2\varphi - 1''.16 \sin 4\varphi. \quad . \quad (138)$$

To Determine ρ.

75. x and y being the co-ordinates of the point K, we have

$$\rho^2 = x^2 + y^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

$$A^2y^2 + B^2x^2 = A^2B^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

$$\tan \varphi' = \frac{y}{x} = \frac{B^2}{A^2} \tan \varphi. \quad . \quad . \quad . \quad . \quad . \quad (134)$$

Combining (130) and (134), eliminating y, we have

$$x^2\left(1 + \frac{A^2}{B^2} \tan^2 \varphi'\right) = A^2,$$

or

$$x^2(1 + \tan \varphi \tan \varphi') = A^2.$$

Combining this with (139) and (134) to eliminate x, we find

$$\rho = A \frac{\sec \varphi'}{\sqrt{1 + \tan \varphi \tan \varphi'}} = A \sqrt{\frac{\cos \varphi}{\cos \varphi' \cos (\varphi' - \varphi)}}. \quad (140)$$

The computation of ρ from (140) is very simple, but it may be rendered much more so by developing ρ, or log ρ

into a series. For this purpose we shall regard A—the equatorial radius—as unity, when we have

$$\rho^2 = \frac{\sec^2 \varphi'}{1 + \tan \varphi \tan \varphi'} = \frac{1 + \dfrac{B^4}{A^4} \tan^2 \varphi}{1 + \dfrac{B^2}{A^2} \tan^2 \varphi} = \frac{\cos^2 \varphi + \dfrac{B^4}{A^4} \sin^2 \varphi}{\cos^2 \varphi + \dfrac{B^2}{A^2} \sin^2 \varphi}$$

Let us write $\dfrac{B^4}{A^4} = 1 - g^2;\qquad \dfrac{B^2}{A^2} = 1 - e^2.$

Then we have $\qquad \rho^2 = \dfrac{1 - g^2 \sin^2 \varphi}{1 - e^2 \sin^2 \varphi}.$

Taking the logarithms of both members,

$$2 \log \rho = \log (1 - g^2 \sin^2 \varphi) - \log (1 - e^2 \sin^2 \varphi).$$

Developing the second member by the logarithmic formula,

$$2 \log \rho = M \begin{bmatrix} - g^2 \sin^2 \varphi - \frac{1}{2} g^4 \sin^4 \varphi - \frac{1}{3} g^6 \sin^6 \varphi - \text{etc.} \\ + e^2 \sin^2 \varphi + \frac{1}{2} e^4 \sin^4 \varphi + \frac{1}{3} e^6 \sin^6 \varphi + \text{etc.} \end{bmatrix};$$

or $$\log \rho = \tfrac{1}{2} M (e^2 - g^2) \sin^2 \varphi + \tfrac{1}{4} M (e^4 - g^4) \sin^4 \varphi + \tfrac{1}{6} M (e^6 - g^6) \sin^6 \varphi, \text{ etc.}$$

Substituting for e, g, and M their values,—M being the modulus of the common system of logarithms $= .43429448$,—we readily find

$$\log \rho = - .00143968 \sin^2 \varphi - .00001438 \sin^4 \varphi - .00000015 \sin^6 \varphi. \quad (141)$$

76. From this the computation of $\log \rho$ is very simple. A better series is, however, obtained by expressing it in terms of functions of the multiple angles, instead of powers of the sine as here.

For effecting the required transformation, let us write (141)

$$\log \rho = \alpha \sin^2\varphi + \beta \sin^4\varphi + \gamma \sin^6\varphi;$$

also
$$\sin \varphi = \frac{1}{2\sqrt{-1}}(e^{\phi\sqrt{-1}} - e^{-\phi\sqrt{-1}});$$

and for convenience write $e^{\phi\sqrt{-1}} = x; \qquad e^{-\phi\sqrt{-1}} = \frac{1}{x}.$

Then
$$\alpha \sin^2\varphi = -\frac{\alpha}{4}\left[x^2 - 2 + \frac{1}{x^2}\right];$$

$$\beta \sin^4\varphi = +\frac{\beta}{16}\left[x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}\right];$$

$$\gamma \sin^6\varphi = -\frac{\gamma}{64}\left[x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}\right].$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdot$$

Therefore
$$\begin{aligned}\log \rho = &-\left[\frac{\alpha}{2} + \frac{3}{8}\beta + \frac{5}{16}\gamma + \text{etc.}\right];\\ &-\left[\frac{\alpha}{4} + \frac{\beta}{4} + \frac{15}{16}\gamma + \text{etc.}\right]\left[x^2 + \frac{1}{x^2}\right];\\ &+\left[\frac{1}{16}\beta + \frac{3}{32}\gamma + \text{etc.}\right]\left[x^4 + \frac{1}{x^4}\right];\\ &-\left[\frac{1}{64}\gamma + \text{etc.}\right]\left[x^6 + \frac{1}{x^6}\right].\end{aligned}$$

But
$$x^2 + \frac{1}{x^2} = e^{2\phi\sqrt{-1}} + e^{-2\phi\sqrt{-1}} = 2 \cos 2\varphi;$$

$$x^4 + \frac{1}{x^4} = e^{4\phi\sqrt{-1}} + e^{-4\phi\sqrt{-1}} = 2 \cos 4\varphi;$$

$$x^6 + \frac{1}{x^6} = e^{6\phi\sqrt{-1}} + e^{-6\phi\sqrt{-1}} = 2 \cos 6\varphi.$$

Substituting these values with the numerical values of α, β, and γ as given in (141), and we find

$$\log \rho = 9.9992747 + .0007271 \cos 2\varphi - .0000018 \cos 4\varphi. \quad (142)$$

77. We therefore have for computing $(\varphi - \varphi')$ and $\log \rho$,

$$\left.\begin{aligned} \varphi - \varphi' &= [2.839258] \sin 2\varphi + [0.06446_n] \sin 4\varphi; \\ \log \rho &= 9.999\,2747 + [6.861594] \cos 2\varphi + [4.25527_n] \cancel{\sin 4\varphi}\ \cos 4\varphi. \end{aligned}\right\} \quad \text{(V)}$$

In which the quantities in brackets are logarithms of the coefficients.

Let us apply formulæ (V) to the determination of $\varphi - \varphi'$ and $\log \rho$ for latitude $40° \ 36' \ 23''.9$.

$$\begin{aligned} \text{We have} \qquad 2\varphi &= \ \ 81° \ 12' \ 48''; \\ 4\varphi &= 162° \ 25' \ 36''. \end{aligned}$$

$$\begin{array}{lll} & & 9.9992747 \\ [2.839258] \sin 2\varphi = + 682''.54 & [6.861594] \cos 2\varphi = & 1110.6 \\ [0.06446_n] \sin 4\varphi = - \qquad ''.35 & [4.25527_n] \cos 4\varphi = & 17.2 \\ \hline \text{Therefore} \quad \varphi - \varphi' = 11' \ 22''.19 & \log \rho = & 9.9993875 \end{array}$$

78. We are now prepared for the complete solution of the problem of parallax. The following method is that of Olbers (see Bode's Jahrbuch, 1811, p. 95).

We shall consider four cases, viz.:

First—*To determine the parallax in zenith distance and azimuth, having given the geocentric zenith distance and azimuth.*

Second—*Parallax in zenith distance and azimuth, having given the observed zenith distance and azimuth.*

Third—*Parallax in declination and right ascension, having given the geocentric declination and right ascension.*

Fourth—*Parallax in declination and right ascension, having given the observed declination and right ascension.*

Case First.

79. Let the star be referred to a system of rectangular axes, the horizon of the observer being the plane of XY, the positive axis of X being directed to the south point, the positive axis of Y to the west point, and the positive axis of Z to the zenith.

Let ξ', η', ζ' = the rectangular co-ordinates;
Δ', z', a' = the polar co-ordinates.

Then
$$\left.\begin{aligned} \xi' &= \Delta' \sin z' \cos a'; \\ \eta' &= \Delta' \sin z' \sin a'; \\ \zeta' &= \Delta' \cos z'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (143)$$

Next let the star be referred to a system of co-ordinate axes parallel to the first, the origin being at the centre of the earth.

Let ξ, η, ζ = the rectangular co-ordinates;
Δ, a, z = the polar co-ordinates;

and we have
$$\left.\begin{aligned} \xi &= \Delta \sin z \cos a; \\ \eta &= \Delta \sin z \sin a; \\ \zeta &= \Delta \cos z. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (144)$$

Let the co-ordinates of the first origin referred to the second be

$$\xi_0, \eta_0, \zeta_0 = \text{rectangular co-ordinates};$$
$$\rho, (\varphi - \varphi'), a_0 = \text{polar co-ordinates}.$$

With the co-ordinate planes situated as in the present case, a_0 will be zero. We shall write $a_0 = a - a$, as this form will be found convenient in a future transformation.

We then have

$$\left.\begin{aligned} \xi_0 &= \rho \sin (\varphi - \varphi') \cos (a - a); \\ \eta_0 &= \rho \sin (\varphi - \varphi') \sin (a - a); \\ \zeta_0 &= \rho \cos (\varphi - \varphi'). \end{aligned}\right\} \quad . \quad . \quad . \quad (145)$$

The formulæ for passing from the first system (143) to the second (144) will be

$$\xi' = \xi - \xi_0; \qquad \eta' = \eta - \eta_0; \qquad \zeta' = \zeta - \zeta_0. \quad (146)$$

Substituting for these quantities their values (143), (144), and (145), we have

$$\left.\begin{aligned} \Delta' \sin z' \cos a' &= \Delta \sin z \cos a - \rho \sin (\varphi - \varphi') \cos (a - a); \\ \Delta' \sin z' \sin a' &= \Delta \sin z \sin a - \rho \sin (\varphi - \varphi') \sin (a - a); \\ \Delta' \cos z' &= \Delta \cos z - \rho \cos (\varphi - \varphi'). \end{aligned}\right\} (147)$$

These equations express the required relation between the quantities given, viz., a and z, and those required, a' and z'. It remains to transform them so as to render their application convenient.

Let us divide the equations through by Δ and write from (125)

$$\sin \pi = \frac{1}{\Delta}, \qquad (a \text{ being unity in this case.})$$

also $\qquad f^* = \dfrac{\Delta'}{\Delta}$; viz.:

$$\left.\begin{aligned} f \sin z' \cos a' &= \sin z \cos a - \rho \sin \pi \sin (\varphi - \varphi') \cos (a - a); \\ f \sin z' \sin a' &= \sin z \sin a - \rho \sin \pi \sin (\varphi - \varphi') \sin (a - a); \\ f \cos z' &= \cos z - \rho \sin \pi \cos (\varphi - \varphi'). \end{aligned}\right\} (148)$$

In these equations let all horizontal angles be diminished

* As f is eliminated from our formulæ, we are not concerned with its value.

by a; the resulting equations will be what we should have obtained if our original axes of ξ, ξ', and ξ_0 had been directed to a point whose azimuth was a, instead of zero as in the present case. We thus obtain

$$\left.\begin{aligned} f \sin z' \cos (a' - a) &= \sin z - \rho \sin \pi \sin (\varphi - \varphi') \cos a; \\ f \sin z' \sin (a' - a) &= \rho \sin \pi \sin (\varphi - \varphi') \sin a. \end{aligned}\right\} \quad (149)$$

Let us write $$m = \frac{\rho \sin \pi \sin (\varphi - \varphi')}{\sin z}. \quad . \quad . \quad . \quad . \quad (150)$$

Then (149) become

$$f \sin z' \cos (a' - a) = \sin z (1 - m \cos a);$$
$$f \sin z' \sin (a' - a) = m \sin z \sin a;$$

and by division,

$$\tan (a' - a) = \frac{m \sin a}{1 - m \cos a}. \quad . \quad . \quad . \quad . \quad (151)$$

(150) and (151) determine the parallax in azimuth.

To determine $(z' - z)$ we proceed as follows:

Multiply the first of (149) by $\cos \frac{1}{2}(a' - a)$, and the second by $\sin \frac{1}{2}(a' - a)$; add, and divide the result by $\cos \frac{1}{2}(a' - a)$. A simple reduction then gives

$$f \sin z' = \sin z - \rho \sin \pi \sin (\varphi - \varphi') \frac{\cos \frac{1}{2}(a' + a)}{\cos \frac{1}{2}(a' - a)}. \quad (152)$$

Let us write

$$\sin (\varphi - \varphi') \frac{\cos \frac{1}{2}(a' + a)}{\cos \frac{1}{2}(a' - a)} = \cos (\varphi - \varphi') \tan \gamma,$$

or $$\tan \gamma = \tan (\varphi - \varphi') \frac{\cos \frac{1}{2}(a' + a)}{\cos \frac{1}{2}(a' - a)}. \quad . \quad . \quad (153)$$

(152) then becomes

$$\left.\begin{array}{l} f \sin z' = \sin z - \rho \sin \pi \cos (\varphi - \varphi') \tan \gamma ; \\ \text{and the last of (148),} \\ f \cos z' = \cos z - \rho \sin \pi \cos (\varphi - \varphi'). \end{array}\right\} \quad (154)$$

Multiplying the first of (154) by $\cos z$ and the second by $\sin z$, and subtracting, then multiplying the first by $\sin z$ and the second by $\cos z$, and adding, we find

$$f \sin (z' - z) = \rho \sin \pi \cos (\varphi - \varphi') \frac{\sin (z - \gamma)}{\cos \gamma};$$

$$f \cos (z' - z) = 1 - \rho \sin \pi \cos (\varphi - \varphi') \frac{\cos (z - \gamma)}{\cos \gamma}.$$

Writing now $$n = \rho \frac{\sin \pi \cos (\varphi - \varphi')}{\cos \gamma}, \quad . \quad . \quad . \quad . \quad (155)$$

and we have

$$f \sin (z' - z) = n \sin (z - \gamma);$$
$$f \cos (z' - z) = 1 - n \cos (z - \gamma);$$
$$\tan (z' - z) = \frac{n \sin (z - \gamma)}{1 - n \cos (z - \gamma)}. \quad . \quad . \quad . \quad . \quad . \quad (156)$$

(155) and (156) now determine the parallax in zenith distance, and the problem is completely solved.

80. Formulæ (150), (151), (155), and (156) may be placed in a form more convenient for logarithmic computation, as follows: Write

$$\sin \vartheta = m \cos a = \frac{\rho \sin \pi \sin (\varphi - \varphi') \cos a}{\sin z}. \quad (157)$$

Then

$$\tan(a' - a) = \frac{\sin \vartheta \tan a}{1 - \sin \vartheta}$$

$$= \tan a \frac{\sin \vartheta}{\cos^2 \frac{1}{2}\vartheta - 2 \sin \frac{1}{2}\vartheta \cos \frac{1}{2}\vartheta + \sin^2 \frac{1}{2}\vartheta}$$

$$= \tan a \frac{\sin \vartheta}{(\cos \frac{1}{2}\vartheta - \sin \frac{1}{2}\vartheta)^2}$$

$$= \tan a \frac{\sin \vartheta}{\cos^2 \frac{1}{2}\vartheta - \sin^2 \frac{1}{2}\vartheta} \cdot \frac{\cos \frac{1}{2}\vartheta + \sin \frac{1}{2}\vartheta}{\cos \frac{1}{2}\vartheta - \sin \frac{1}{2}\vartheta}$$

$$= \tan a \tan \vartheta \frac{1 + \tan \frac{1}{2}\vartheta}{1 - \tan \frac{1}{2}\vartheta}.$$

But

$$\frac{1 + \tan \frac{1}{2}\vartheta}{1 - \tan \frac{1}{2}\vartheta} = \tan (45^\circ + \tfrac{1}{2}\vartheta);$$

therefore

$$\tan(a' - a) = \tan a \tan \vartheta \tan (45^\circ + \tfrac{1}{2}\vartheta). \quad . \quad (158)$$

In a similar manner writing

$$\sin \vartheta' = n \cos (z - \gamma) = \frac{\rho \sin \pi \cos (\varphi - \varphi') \cos (z - \gamma)}{\cos \gamma}, \quad (159)$$

we find

$$\tan (z' - z) = \frac{\sin \vartheta' \tan (z - \gamma)}{1 - \sin \vartheta'}$$

$$= \tan \vartheta' \tan (45^\circ + \tfrac{1}{2}\vartheta') \tan (z - \gamma). \quad (160)$$

For computing γ we have

$$\tan \gamma = \tan(\varphi - \varphi') \frac{\cos \frac{1}{2}(a' + a)}{\cos \frac{1}{2}(a' - a)} = \tan(\varphi - \varphi') \frac{\cos [a + \frac{1}{2}(a' - a)]}{\cos \frac{1}{2}(a' - a)}.$$

Therefore

$$\tan \gamma = \tan (\varphi - \varphi') [\cos a - \sin a \tan \tfrac{1}{2}(a' - a)].$$

By Maclaurin's formula we have

$$\tan x = x + \tfrac{1}{3}x^3, \text{ etc.}$$

Therefore if we neglect terms of the third and higher orders in γ, $(\varphi - \varphi')$, and $(a' - a)$, all of which are small quantities, we have

$$\gamma = (\varphi - \varphi') [\cos a - \sin a \tfrac{1}{2}(a' - a)]. \quad . \quad (161)$$

From $$\tan (a' - a) = \frac{m \sin a}{1 - m \cos a}$$

we have, by neglecting terms of the higher orders,

$$(a' - a) = m \sin a = \frac{\rho \sin \pi(\varphi - \varphi') \sin a}{\sin z},$$

Substituting this in (161), we have

$$\gamma = (\varphi - \varphi') \cos a - \frac{\rho \sin \pi \sin^2 a(\varphi - \varphi')^2 \sin 1''}{2 \sin z}. \quad (162)$$

This is accurate to terms of the second order of $(\varphi - \varphi_1)$ inclusive.

It will readily appear that for any value of z not less than $(\varphi - \varphi')$ the second term will always be inappreciable. When z is very near zero the formula is apparently inapplicable. As we shall not have occasion to apply it to such cases, it will not be necessary for our purposes to discuss it further. We may therefore compute γ from the practically rigorous formula

$$\gamma = (\varphi - \varphi') \cos a. \quad . \quad . \quad . \quad . \quad . \quad (163)$$

81. We have therefore the following complete formulæ for computing the parallax in zenith distance and azimuth, having given the geocentric zenith distance and azimuth.

$$\left.\begin{aligned}\sin \vartheta &= \frac{\rho \sin \pi \cos a \sin (\varphi - \varphi')}{\sin z};\\ \tan (a' - a) &= \tan a \tan \vartheta \tan (45^\circ + \tfrac{1}{2}\vartheta);\\ \gamma &= (\varphi - \varphi') \cos a;\\ \sin \vartheta' &= \frac{\rho \sin \pi \cos (z - \gamma) \cos (\varphi - \varphi')}{\cos \gamma};\\ \tan (z' - z) &= \tan (z - \gamma) \tan \vartheta' \tan (45^\circ + \tfrac{1}{2}\vartheta').\end{aligned}\right\}(\text{VI})$$

In the meridian, $a = a' = 0$. Therefore for this case (VI) become

$$\left.\begin{aligned}\gamma &= \varphi - \varphi';\\ \sin \vartheta' &= \rho \sin \pi \cos [z - (\varphi - \varphi')]\\ \tan (z' - z) &= \tan [z - (\varphi - \varphi')] \tan \vartheta' \tan (45^\circ + \tfrac{1}{2}\vartheta').\end{aligned}\right\}(\text{VI})_1$$

As an example of the application of (VI) let us take the following:

1881, July 4th, 9^h, mean Bethlehem time, the geocentric position of the moon was as follows:

$$\begin{aligned}\text{Zenith distance} = z &= 65^\circ\ 40'\ 46''.5;\\ \text{Azimuth} = a &= 48^\circ\ 19'\ 49''.8.\end{aligned}$$

Required the parallax in azimuth and zenith distance for Bethlehem.

We have found for the latitude of Bethlehem (Art. 77)

$$\begin{aligned}\varphi - \varphi' &= 11'\ 22''.19;\\ \log \rho &= 9.9993875.\end{aligned}$$

From the Nautical Almanac, page 113,

$$\pi = 56'\ 20''.4.$$

Our computation is now as follows:

$a = 48^\circ\ 19'\ 49''.8$	$\cos a = 9.8227125$
$\varphi - \varphi' = 11'\ 22''.19$	$\sin = 7.5194794$
$z = 65^\circ\ 40'\ 46''.5$	$\text{cosec} = .0403593$
$\gamma = 7'\ 33''.54$	
	$\log \rho = 9.9993875$
$\pi = 56'\ 20''.4$	$\sin \pi = 8.2145238$
$z - \gamma = 65^\circ\ 33'\ 12''.96$	$\cos = 9.6168344$
	$\cos(\varphi - \varphi') = 9.9999976$
	$\sec \gamma = .0000009$
$\vartheta = 8''.145$	$\sin \vartheta = 5.5964625$
$\vartheta' = 23'\ 16''.92$	$\sin \vartheta' = 7.8307442$
$45^\circ + \frac{1}{2}\vartheta = 45^\circ\ 00'\ 4''.07$	
$45^\circ + \frac{1}{2}\vartheta' = 45^\circ\ 11'\ 38''.46$	
$\cos a = 9.8227125$	$\tan a = 0.0506037$
$\log(\varphi - \varphi') = 2.8339053$	$\tan \vartheta = 5.5964625$
	$\tan(45^\circ + \frac{1}{2}\vartheta) = .0000171$
$\log \gamma = 2.6566178$	$\tan(a' - a) = 5.6470833$
	$(a' - a) = 9''.152$
	$\tan \vartheta' = 7.8307540$
	$\tan(45^\circ + \frac{1}{2}\vartheta') = 0029412$
	$\tan(z - \gamma) = .3423734$
	$\tan(z' - z) = 8.1760686$
	$(z' - z) = 51'\ 33''.58$
We thus have for the apparent place	$a' = 48^\circ\ 19'\ 59''.0$
	$z' = 66^\circ\ 32'\ 20''.1$

Take the following example of application of $(\text{VI})_1$:

		$\log \rho = 9.9993875$
		$\sin \pi = 8.2138035$
Zenith distance of moon at culmination,	$z = 51^\circ\ 06'\ 45''.5$	$\cos[z - (\varphi - \varphi')] = 9.7995903$
Equatorialhorizontalparallax,	$\pi = 56'\ 14''.8$	$\sin \vartheta' = 8.0127813$
	$\varphi - \varphi' = 11'\ 22''.19$	$\tan[z - (\varphi - \varphi')] = 0.0904399$
	$\vartheta' = 35'\ 24''.29$	$\tan \vartheta' = 8.0128043$
	$45^\circ + \frac{1}{2}\vartheta' = 45^\circ\ 17'\ 42''.15$	$\tan(45^\circ + \frac{1}{2}\vartheta') = .0044727$
	$z' - z = 44'\ 3''.13$	$\tan(z' - z) = 8.1077169$

Case Second.

82. To compute the parallax in azimuth and zenith distance, having given the observed azimuth and zenith distance.

To obtain the expression for $(z' - z)$ we multiply the first of (154) by $\cos z'$ and the second by $\sin z'$, and subtract. We thus have

$$\sin (z' - z) = \frac{\rho \sin \pi \cos (\varphi - \varphi') \sin (z' - \gamma)}{\cos \gamma}. \quad . \quad (164)$$

For $(a' - a)$ we multiply the first of (148) by $\sin a'$, the second by $\cos a'$ and subtract. recollecting that $\cos (a - a) = 1$, $\sin (a - a) = 0$. We thus find

$$\sin (a' - a) = \frac{\rho \sin \pi \sin (\varphi - \varphi') \sin a'}{\sin z}. \quad . \quad . \quad . \quad (165)$$

We thus have for the parallax in zenith distance and azimuth, having given the apparent zenith distance and azimuth,

$$\left.\begin{aligned} \gamma &= (\varphi - \varphi') \cos a; \\ \sin (z' - z) &= \frac{\rho \sin \pi \cos (\varphi - \varphi') \sin (z' - \gamma)}{\cos \gamma}; \\ \sin (a' - a) &= \frac{\rho \sin \pi \sin (\varphi - \varphi') \sin a'}{\sin z}. \end{aligned}\right\} \quad \text{(VII)}$$

To compute we may substitute a' for a without appreciable error.

To compute $(a' - a)$ we must first obtain z by applying the correction $(z' - z)$ to the observed zenith distance.

In the meridian, $a = a' = 0$, whence $\gamma = \varphi - \varphi'$, $a' - a = 0$, and (VII) become

$$\sin (z' - z) = \rho \sin \pi \sin [z' - (\varphi - \varphi')] \, . \quad \text{(VII)}_1$$

For all bodies except the moon (VII) may be greatly simplified, as follows:

$(z' - z)$, $(a' - a)$, and π being very small, we may write the arcs in place of their sines. $(\varphi - \varphi')$ and γ being small, we may write for their cosines unity. We then have

$$\left.\begin{aligned} \gamma &= (\varphi - \varphi') \cos a; \\ z' - z &= \pi\rho \sin (z' - \gamma); \\ a' - a &= \pi\rho \sin (\varphi - \varphi') \sin a' \operatorname{cosec} z. \end{aligned}\right\} \quad \text{(VIII)}$$

In computing these we may use a and z or a' and z' indifferently in the second terms. It will often be sufficiently accurate to use

$$\left.\begin{aligned} z' - z &= \pi \sin z'; \\ a' - a &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(VIII)}_1$$

These last are what we obtained when we treated the earth as a sphere.

Application of Formulæ (VII).

Latitude of Bethlehem $= \varphi = 40° \, 36' \, 23''.9$
Apparent azimuth of moon $= a' = 48° \, 19' \, 59''.0$
Apparent zenith distance of moon $= z' = 66° \, 32' \, 20''.1$
Equatorial horizontal parallax $= \pi = 56' \, 20''.4$
$\varphi - \varphi' = 11' \, 22''.19$

$\log (\varphi - \varphi') = 2.8339053$	$\cos (\varphi - \varphi') = 9.9999976$
$\cos a' = 9.8226904$	$\sin (z' - \gamma) = 9.9621103$
	$\sec \gamma = .0000009$
$\log \gamma = 2.6565957$	
$\gamma = 453''.52$	$\log \rho = 9.9993875$
$z' - \gamma = 66° \, 24' \, 46''.58$	$\sin \pi = 8.2145238$
	$\sin (\varphi - \varphi') = 7.5194794$
	$\sin a' = 9.8733333$
	$\operatorname{cosec} z = .0403593$
$z' - z = 51' \, 33''.58$	$\sin (z' - z) = 8.1760201$
$z = 65° \, 40' \, 46''.52$	
$a' - a = 9''.152$	$\sin (a' - a) = 5.6470833$
$a = 48° \, 19' \, 49''.85$	

Application of (VII)$_1$.

$$\left.\begin{array}{l}\text{Apparent zenith distance of the moon}\\ \text{at meridian passage}\end{array}\right\} = z' = 51^\circ\ 50^\circ\ 48''.6$$

$$\begin{aligned}
\text{Equatorial horizontal parallax} \quad = \pi &= \quad 56'\ 14''.8 \\
\varphi - \varphi' &= \quad 11'\ 22''.19 \\
\log \rho &= 9.9993875 \\
\sin \pi &= 8.2138035 \\
\sin [z' - (\varphi - \varphi')] &= 9.8944903 \\
\hline
\sin (z' - z) &= 8.1076813 \\
z' - z &= 44'\ 3''.13
\end{aligned}$$

Application of (VIII).

Find the parallax in azimuth and zenith distance of Venus as seen from Bethlehem, having given the following:

$a = 271^\circ\ 56'\ 21''$	$\log (\varphi - \varphi') = 2.83390$	$\sin (z - \gamma) = 9.96312$
$z = 66^\circ\ 43'\ 35''$	$\cos a = 8.52941$	
$\pi = 13''.61$		$\log \rho = 9.99939$
	$\log \gamma = 1.36331$	$\log \pi = 1.13386$
$\gamma = 23''.1$		
$z - \gamma = 66^\circ\ 43'\ 12''$		$\sin (\varphi - \varphi') = 7.51947$
		$\sin a = 9.99975_n$
		$\operatorname{cosec} z = .03686$
	$z' - z = +\ 12''.48$	$\log (z' - z) = 1.09637$
	$a' - a = -\ ''.05$	$\log (a' - a) = 8.68933_n$

For this case formula (VIII$_1$) gives

$$\begin{aligned}
\log \pi &= 1.13386 \\
\sin z &= 9.96314 \\
\hline
\log (z' - z) &= 1.09700 \qquad z' - z = +\ 12''.50
\end{aligned}$$

Application of (VII)$_1$.

Zenith distance of Venus at time of transit $= z = 24° \; 15' \; 35''$
Equatorial horizontal parallax $= \pi = 13''.57$

$$\varphi - \varphi' = 11' \; 22''$$

$$\log \pi = 1.13258$$
$$\log \rho = 9.99939$$
$$\sin [z - (\varphi - \varphi')] = 9.61051$$
$$\log (z' - z) = .74248 \qquad z' - z = 5''.53$$

Case Third.

83. Required the parallax in right ascension and declination, having given the geocentric right ascension and declination.

Let the equator of the observer be taken as the plane of x, y, the positive axis of x being directed to the vernal equinox, the positive axis of y to the summer solstice, and the positive axis of z to the north pole of the heavens.

Let x', y', z' = the rectangular co-ordinates;
$\Delta', \alpha', \delta'$ = the polar co-ordinates.

We then have

$$\left.\begin{aligned} x' &= \Delta' \cos \delta' \cos \alpha'; \\ y' &= \Delta' \cos \delta' \sin \alpha'; \\ z' &= \Delta' \sin \delta'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (166)$$

In the second system let the origin be at the centre of the earth, the axes being respectively parallel to those of the first system.

Let x, y, z be the rectangular co-ordinates;
Δ, α, δ be the polar co-ordinates.

Then
$$\left.\begin{aligned} x &= \Delta \cos\delta \cos\alpha; \\ y &= \Delta \cos\delta \sin\alpha; \\ z &= \Delta \sin\delta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (167)$$

Let now

x_0, y_0, z_0 = rectangular co-ordinates } of the observer's position referred to the earth's centre.
ρ, φ', θ = polar co-ordinates }

Here ρ is, as before, the line joining the observer's position with the centre of the earth, and φ' and θ are respectively the declination and right ascension of the point where this line produced pierces the celestial sphere; or in other words, of the geocentric zenith. The declination of the zenith, as we have seen (Art. 63), is equal to the latitude $= \varphi'$ in this case.

The right ascension of the zenith, θ, equals the right ascension of the observer's meridian—all points on the same meridian having the same right ascension. This we shall see hereafter is equal to the observer's sidereal time.

We have then
$$\left.\begin{aligned} x_0 &= \rho \cos\varphi' \cos\theta; \\ y_0 &= \rho \cos\varphi' \sin\theta; \\ z_0 &= \rho \sin\varphi'; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (168)$$

and for passing from system (166) to (167),

$$x' = x - x_0; \qquad y' = y - y_0; \qquad z' = z - z_0. \qquad (169)$$

Therefore

$$\left.\begin{aligned} \Delta' \cos\delta' \cos\alpha' &= \Delta \cos\delta \cos\alpha - \rho \cos\varphi' \cos\theta; \\ \Delta' \cos\delta' \sin\alpha' &= \Delta \cos\delta \sin\alpha - \rho \cos\varphi' \sin\theta; \\ \Delta' \sin\delta' &= \Delta \sin\delta - \rho \sin\varphi'. \end{aligned}\right\} (170)$$

As before, let us divide through by Δ, and write

$$f = \frac{\Delta'}{\Delta}; \qquad \sin \pi = \frac{1}{\Delta}.$$

Then

$$\left.\begin{aligned} f \cos \delta' \cos \alpha' &= \cos \delta \cos \alpha - \rho \sin \pi \cos \varphi' \cos \theta; \\ f \cos \delta' \sin \alpha' &= \cos \delta \sin \alpha - \rho \sin \pi \cos \varphi' \sin \theta; \\ f \sin \delta' &= \sin \delta \qquad\quad - \rho \sin \pi \sin \varphi'. \end{aligned}\right\} \quad (171)$$

Let us diminish all horizontal angles by α, which will be equivalent to transforming our rectilinear systems to others in which the axes of x and x' make respectively the angle α with the original axes. We thus derive

$$\left.\begin{aligned} f \cos \delta' \cos (\alpha' - \alpha) &= \cos \delta - \rho \sin \pi \cos \varphi' \cos (\theta - \alpha); \\ f \cos \delta' \sin (\alpha' - \alpha) &= \qquad\quad - \rho \sin \pi \cos \varphi' \sin (\theta - \alpha). \end{aligned}\right\} \quad (172)$$

Let us write $\qquad m' = \dfrac{\rho \sin \pi \cos \varphi'}{\cos \delta}, \quad . \quad . \quad . \quad . \quad . \quad (173)$

which substituted in (172) and the second divided by the first, we find

$$\tan (\alpha' - \alpha) = \frac{m' \sin (\alpha - \theta)}{1 - m' \cos (\alpha - \theta)}. \quad . \quad . \quad (174)$$

84. As in case first, we may give this a form better adapted to logarithmic computation, as follows: Write

$$\sin \vartheta = m' \cos (\alpha - \theta) = \frac{\rho \sin \pi \cos \varphi' \cos (\alpha - \theta)}{\cos \delta}. \quad (175)$$

Then (174) becomes

$$\tan (\alpha' - \alpha) = \tan (\alpha - \theta) \frac{\sin \vartheta}{1 - \sin \vartheta}.$$

But

$$\begin{aligned}
\frac{\sin \vartheta}{1-\sin \vartheta} &= \frac{\sin \vartheta}{\cos^2 \frac{1}{2}\vartheta - 2 \sin \frac{1}{2}\vartheta \cos \frac{1}{2}\vartheta + \sin^2 \frac{1}{2}\vartheta} \\
&= \frac{\sin \vartheta}{(\cos \frac{1}{2}\vartheta - \sin \frac{1}{2}\vartheta)^2} \\
&= \frac{\sin \vartheta(\cos \frac{1}{2}\vartheta + \sin \frac{1}{2}\vartheta)}{(\cos \frac{1}{2}\vartheta + \sin \frac{1}{2}\vartheta)(\cos \frac{1}{2}\vartheta - \sin \frac{1}{2}\vartheta)(\cos \frac{1}{2}\vartheta - \sin \frac{1}{2}\vartheta)} \\
&= \frac{\sin \vartheta}{\cos^2 \frac{1}{2}\vartheta - \sin^2 \frac{1}{2}\vartheta)} \cdot \frac{\cos \frac{1}{2}\vartheta + \sin \frac{1}{2}\vartheta}{\cos \frac{1}{2}\vartheta - \sin \frac{1}{2}\vartheta} \\
&= \tan \vartheta \tan (45^\circ + \tfrac{1}{2}\vartheta).
\end{aligned}$$

Therefore

$$\tan (\alpha' - \alpha) \doteq \tan (\alpha - \theta) \tan \vartheta \tan (45^\circ + \tfrac{1}{2}\vartheta), \quad (176)$$

which determines $(\alpha' - \alpha)$. For determining $(\delta' - \delta)$ we multiply the first of (172) by $\cos \frac{1}{2}(\alpha' - \alpha)$, the second by $\sin \frac{1}{2}(\alpha' - \alpha)$; add the products, and divide the result by $\cos \frac{1}{2}(\alpha' - \alpha)$. By this process we obtain

$$f \cos \delta' = \cos \delta - \rho \sin \pi \cos \varphi' \frac{\cos [\frac{1}{2}(\alpha' + \alpha) - \theta]}{\cos \frac{1}{2}(\alpha' - \alpha)}.$$

The last of (171) is

$$f \sin \delta' = \sin \delta - \rho \sin \pi \sin \varphi'. \quad (177)$$

Let us write

$$\tan \gamma = \frac{\tan \varphi' \cos \frac{1}{2}(\alpha' - \alpha)}{\cos [\frac{1}{2}(\alpha' + \alpha) - \theta]}. \quad . \quad . \quad . \quad (178)$$

Then (177) become

$$\left.\begin{aligned} f \sin \delta' &= \sin \delta - \rho \sin \pi \sin \varphi'; \\ f \cos \delta' &= \cos \delta - \rho \sin \pi \sin \varphi' \cot \gamma. \end{aligned}\right\} \quad . \quad . \quad (179)$$

Multiply the first of these by cos δ, the second by sin δ, and subtract; then multiply the first by sin δ, the second by cos δ, and add. We thus obtain

$$\left.\begin{aligned} f \sin(\delta' - \delta) &= \rho \sin \pi \sin \varphi' \frac{\sin(\delta - \gamma)}{\sin \gamma}; \\ f \cos(\delta' - \delta) &= 1 - \rho \sin \pi \sin \varphi' \frac{\cos(\delta - \gamma)}{\sin \gamma}. \end{aligned}\right\} \quad (180)$$

Let us write $$n' = \frac{\rho \sin \pi \sin \varphi'}{\sin \gamma}. \quad \ldots \ldots \quad (181)$$

Introducing this value and dividing the first equation by the second, we find

$$\tan(\delta' - \delta) = \frac{n' \sin(\delta - \gamma)}{1 - n' \cos(\delta - \gamma)}.$$

Then writing

$$\sin \vartheta' = n' \cos(\delta - \gamma) = \frac{\rho \sin \pi \sin \varphi' \cos(\delta - \gamma)}{\sin \gamma}, \quad (182)$$

this equation becomes

$$\tan(\delta' - \delta) = \frac{\sin \vartheta' \tan(\delta - \gamma)}{1 - \sin \vartheta'} = \tan(\delta - \gamma) \tan \vartheta' \tan(45° + \tfrac{1}{2}\vartheta'). \quad (183)$$

Equations (175), (176), (178), (182), and (183) give the complete solution of the problem.

We thus have for computing the parallax in right ascension and declination, having given the *geocentric* right ascension and declination, the following formulæ:

$$\left.\begin{aligned}
\sin \vartheta &= \frac{\rho \sin \pi \cos \varphi' \cos (\theta - \alpha)}{\cos \delta};\\
\tan (\alpha - \alpha') &= \tan (\theta - \alpha) \tan \vartheta \tan (45^\circ + \tfrac{1}{2}\vartheta);\\
\tan \gamma &= \frac{\tan \varphi' \cos \frac{1}{2}(\alpha - \alpha')}{\cos [\frac{1}{2}(\alpha + \alpha') - \theta]};\\
\sin \vartheta' &= \frac{\rho \sin \pi \sin \varphi' \cos (\gamma - \delta)}{\sin \gamma};\\
\tan (\delta - \delta') &= \tan (\gamma - \delta) \tan \vartheta' \tan (45^\circ + \tfrac{1}{2}\vartheta').
\end{aligned}\right\} \text{(IX)}$$

In the meridian, $\alpha = \alpha' = \theta$. Therefore $\gamma = \varphi'$, and the above become

$$\left.\begin{aligned}
\sin \vartheta' &= \rho \sin \pi \cos (\varphi' - \delta);\\
\tan (\delta - \delta') &= \tan (\varphi' - \delta) \tan \vartheta' \tan (45^\circ + \tfrac{1}{2}\vartheta').
\end{aligned}\right\} \text{(IX)}_1$$

Application of Formulæ (IX).

Required the parallax of the moon in right ascension and declination, 1881, July 4th, 9^h, Bethlehem mean time, as seen from Bethlehem.

Converting 9^h mean time into sidereal time by the method to be explained hereafter (p. 170), we have

Bethlehem sidereal time $= \theta =$	$15^h\ 52^m\ 50^s.2$
From the Nautical Almanac, p. 114, we find $\alpha =$	$12^h\ 57^m\ 10^s.56$
$\delta =$	$-11^\circ\ 3'\ 48''.4$
Astronomical latitude of Bethlehem $= \varphi =$	$40^\circ\ 36'\ 23''.9$
$\varphi - \varphi' =$	$11'\ 22''.2$
Geocentric latitude of Bethlehem $= \varphi' =$	$40^\circ\ 25'\ 1''.7$
Nautical Almanac, p. 113, equatorial horizontal parallax $= \pi =$	$56'\ 20''.4$
$\theta - \alpha =$	$2^h\ 55^m\ 39^s.64$
$=$	$43^\circ\ 54'\ 54''.6$

$$
\begin{array}{rl}
\cos(\theta - \alpha) = & 9.8575542 \\
\sec \delta = & .0081471 \\
\cos \varphi' = & 9.8815812 \\
\hline
\log \rho = & 9.9993875 \\
\sin \pi = & 8.2145238 \\
\hline
\sin \varphi' = & 9.8118080 \\
\cos(\gamma - \delta) = & 9.6861710 \\
\operatorname{cosec} \gamma = & .1164301 \\
\hline
\sin \vartheta = & 7.9611938 \\
\vartheta = & 31' \; 26''.36 \\
45^\circ + \frac{1}{2}\vartheta = & 45^\circ \; 15' \; 43''.2 \\
\hline
\sin \vartheta' = & 7.8283204 \\
\vartheta' = & 0^\circ \; 23' \; 9''.15 \\
45^\circ + \frac{1}{2}\vartheta' = & 45^\circ \; 11' \; 34''.6
\end{array}
$$

$$
\begin{array}{rl}
\tan \varphi' = & 9.9302268 \\
\cos \frac{1}{2}(\alpha - \alpha') = & 9.9999957 \\
\sec\left[\frac{1}{2}(\alpha + \alpha') - \theta\right] = & .1443121 \\
\hline
\tan \gamma = & .0745346 \\
\gamma = & 49^\circ \; 53' \; 33''.56 \\
\gamma - \delta = & 60^\circ \; 57' \; 21''.96 \\
\tan(\gamma - \delta) = & 0.2554636 \\
\tan \vartheta' = & 7.8283302 \\
\tan(45^\circ + \frac{1}{2}\vartheta') = & .0029250 \\
\hline
\tan(\delta - \delta') = & 8.0867188 \\
\delta - \delta' = & 41' \; 58''.39 \\
\tan(\theta - \alpha) = & 9.9835502 \\
\tan \vartheta = & 7.9612118 \\
\tan(45^\circ + \frac{1}{2}\vartheta) = & .0039719 \\
\hline
\tan(\alpha - \alpha') = & 7.9487339 \\
\alpha - \alpha' = & 0^\circ \; 30' \; 32''.94 \\
\hline
\alpha = & 194^\circ \; 17' \; 38''.4 \\
\text{therefore } \alpha' = & 193^\circ \; 47' \; 5''.5 \\
\frac{1}{2}(\alpha + \alpha') = & 194^\circ \; 2' \; 21''.9 \\
\theta = & 238^\circ \; 12' \; 33''.0 \\
\frac{1}{2}(\alpha' + \alpha) - \theta = & 315^\circ \; 49' \; 48''.9
\end{array}
$$

We therefore have for the position of the moon as seen from Bethlehem, 1881, July 4th, 9^h, mean time,

$$
\begin{aligned}
\alpha' &= \quad 12^h \; 55^m \; 8^s.36; \\
\delta' &= -\,11^\circ \; 45' \; 46''.79.
\end{aligned}
$$

Application of (IX,).

At the time of meridian passage at Bethlehem, 1881, July

4th, the moon's declination and equatorial horizontal parallax were as follows:

$$\delta = -10^\circ\ 30'\ 21''.6 \qquad \text{Required } (\delta - \delta').$$
$$\pi = 56'\ 14''.8$$
$$\varphi' = 40^\circ\ 25'\ 1''.7$$
$$\varphi' - \delta = 50^\circ\ 55'\ 23''.3$$

$\log \rho = 9.9993875$
$\sin \pi = 8.2138035$
$\cos(\varphi' - \delta) = 9.7995903$ $\qquad \tan(\varphi' - \delta) = .0904399$

$\sin \vartheta' = 8.0127813$ $\qquad \tan \vartheta' = 8.0128043$
$\vartheta' = 35'\ 24''.29$ $\qquad \tan(45^\circ + \frac{1}{2}\vartheta) = .0044726$
$45^\circ + \frac{1}{2}\vartheta' = 45^\circ\ 17'\ 42''.1$

$\tan(\delta - \delta') = 8.1077168$
$\delta - \delta' = 44'\ 3''.13$

Case Fourth.

85. Required the parallax in right ascension and declination, having given the apparent right ascension and declination.

Multiply the first of (171) by $\sin \alpha'$, the second by $\cos \alpha'$; subtract and reduce. The result is

$$\sin(\alpha - \alpha') = \frac{\rho \sin \pi \cos \varphi' \sin(\theta - \alpha')}{\cos \delta}. \quad . \quad (184)$$

To obtain $\delta - \delta'$ we make use of (179). Multiply the first by $\cos \delta'$, the second by $\sin \delta'$; subtract and reduce. We thus have

$$\sin(\delta - \delta') = \frac{\rho \sin \pi \sin \varphi' \sin(\gamma - \delta')}{\sin \gamma}. \quad . \quad (185)$$

We have therefore the following formulæ for the parallax in right ascension and declination, having given the apparent co-ordinates:

$$\left.\begin{aligned}\sin(\alpha - \alpha') &= \frac{\rho \sin \pi \cos \varphi' \sin(\theta - \alpha')}{\cos \delta};\\ \tan \gamma &= \frac{\tan \varphi' \cos \frac{1}{2}(\alpha - \alpha')}{\cos [\frac{1}{2}(\alpha + \alpha') - \theta']};\\ \sin(\delta - \delta') &= \frac{\rho \sin \pi \sin \varphi' \sin(\gamma - \delta')}{\sin \gamma}.\end{aligned}\right\} \quad . \quad (\mathrm{X})$$

To compute the first of these we require δ, which will be unknown until after we have computed the last, which in turn requires a knowledge of α obtained from the first. We must therefore proceed indirectly as follows: Compute $(\alpha-\alpha')$, using in the denominator δ' instead of δ. With the approximate value of α so obtained compute $(\delta - \delta')$; this gives us δ, with which we recompute $(\alpha - \alpha')$. It will never be necessary to repeat the computation of $\delta - \delta'$ with this new value of α.

In the meridian, $\alpha = \alpha' = \theta$. Therefore $\gamma = \varphi'$, and formulæ (X) become

$$\sin(\delta - \delta') = \rho \sin \pi \sin(\varphi' - \delta') \quad . \quad . \quad . \quad (\mathrm{X})_1$$

For all bodies except the moon we may write, without appreciable error,

$$\sin(\alpha-\alpha')=(\alpha-\alpha'); \quad \sin(\delta-\delta')=(\delta-\delta'); \quad \cos\tfrac{1}{2}(\alpha-\alpha')= 1;$$
$$\sin \pi = \pi; \quad \cos \delta' = \cos \delta; \quad \tfrac{1}{2}(\alpha+\alpha')= \alpha';$$

giving the following approximate formulæ:

$$\left.\begin{aligned}\alpha - \alpha' &= \frac{\pi\rho \cos \varphi' \sin(\theta - \alpha')}{\cos \delta'};\\ \tan \gamma &= \frac{\tan \varphi'}{\cos(\theta - \alpha')};\\ \delta - \delta' &= \frac{\pi\rho \sin \varphi' \sin(\gamma - \delta')}{\sin \gamma}.\end{aligned}\right\} \quad . \quad . \quad . \quad (\mathrm{XI})$$

In these formulæ we may use either the geocentric co-ordinates (α and δ) or the observed (α' and δ') indifferently.

In the meridian, where $\theta = \alpha = \alpha'$, $\gamma = \varphi'$ and (XI) become

$$\delta - \delta' = \pi\rho \sin(\varphi' - \delta'). \quad . \quad . \quad . \quad . \qquad \text{(XI)}_1$$

Application of (X).

Required the geocentric place of the moon, having given the apparent place as seen from Bethlehem, 1881, July 4th, 9^h, Bethlehem mean time, as follows:

$$\begin{aligned}
\text{Apparent right ascension} &= \alpha' = 12^h\ 55^m\ 8^s.36; \\
\text{Apparent declination} &= \delta' = -\ 11^\circ\ 45'\ 46''.79.
\end{aligned}$$

$$\begin{aligned}
\text{From Nautical Almanac, p. 113,} \quad \pi &= 56'\ 20''.4 \\
\text{Geocentric latitude,} \quad \varphi' &= 40^\circ\ 25'\ 1''.7 \\
\text{Sidereal time,} \quad \theta &= 15^h\ 52^m\ 50^s.2 \\
\theta - \alpha' &= 44^\circ\ 25'\ 27''.6
\end{aligned}$$

$$\begin{aligned}
\sec\delta' &= .009\ 2176 \\
{}^*\sec\delta &= .008\ 1471 \\
\cos\varphi' &= 9.881\ 5812 \\
\sin(\theta - \alpha') &= 9.845\ 0774 \\
\hline
\log\rho &= 9.999\ 3875 \\
\sin\pi &= 8.214\ 5238 \\
\hline
\sin\varphi' &= 9.811\ 8080 \\
\sin(\gamma - \delta') &= 9.944\ 5358 \\
\operatorname{cosec}\gamma &= .116\ 4320 \\
\hline
\sin(\delta - \delta') &= 8.086\ 6871 \\
\delta - \delta' &= 41'\ 58''.39 \\
\delta &= -\ 11^\circ\ 3'\ 48''.4
\end{aligned}$$

$$\begin{aligned}
\text{Approx. } \sin(\alpha - \alpha') &= 7.9497875 \\
\text{Approx. } (\alpha - \alpha') &= 30'\ 37''.5 \\
\alpha' &= 193^\circ\ 47'\ 5''.4 \\
\text{Approx. } \alpha &= 194^\circ\ 17'\ 43 \\
\tfrac{1}{2}(\alpha + \alpha') &= 194^\circ\ 2'\ 24''.2 \\
[\tfrac{1}{2}(\alpha + \alpha') - \theta] &= 315^\circ\ 49'\ 51''.2 \\
\tfrac{1}{2}(\alpha - \alpha') &= 15'\ 18''.8 \\
\hline
\tan\varphi' &= 9.9302268 \\
\cos\tfrac{1}{2}(\alpha - \alpha') &= 9.9999957 \\
\sec[\tfrac{1}{2}(\alpha + \alpha') - \theta] &= .1443074 \\
\hline
\tan\gamma &= .0745299 \\
\gamma &= 49^\circ\ 53'\ 32''.5 \\
\gamma - \delta' &= 61^\circ\ 39'\ 19''.3 \\
\text{Corrected } \sin(\alpha - \alpha') &= 7.9487170 \\
\text{True } (\alpha - \alpha') &= 30'\ 32''.94 \\
\alpha &= 194^\circ\ 17'\ 38''.34 \\
&= 12^h\ 57^m\ 10^s.55
\end{aligned}$$

---

* This value is inserted after the computation of the parallax in declination.

Application of $(X)_1$.

1881, July 4th, at meridian passage, Bethlehem, the moon's apparent declination and equatorial horizontal parallax were as follows:

$\delta' = -\ 11^\circ\ 14'\ 24''.7$ Required the parallax in declination.
$\pi = \quad 56'\ 14''.8$

$$\varphi' = \quad 40^\circ\ 25'\ \ 1''.7$$
$$\varphi' - \delta' = \quad 51^\circ\ 39'\ 26''.4$$

$$\log \rho = 9.9993875$$
$$\sin \pi = 8.2138035$$
$$\sin (\varphi' - \delta') = 9.8944903$$
$$\sin (\delta - \delta') = 8.1076813 \qquad \delta - \delta' = 44'\ 3''.13$$

Application of (XI).

1881, July 4th, 16^h, Bethlehem mean time, the right ascension, declination, and equatorial horizontal parallax of Venus were as follows:

From Nautical Almanac, p. 355, $\alpha = \ 3^h\ 46^m\ 12^s.25$
$\delta = 16^\circ\ 18'\ 23''.3$
From Nautical Almanac, p. 388, $\pi = \quad 13''.61$
Sidereal time,* $\theta = 22^h\ 53^m\ 59^s.2$

* See p. 170.

The computation is then as follows:

$\theta - \alpha = 19^h\ 7^m\ 47^s$		
$= 286°\ 56'\ 45''$	$\cos(\theta - \alpha') = 9.46459$	$\cos\varphi' = 9.88156$
$\varphi' = 40°\ 25'\ 1''.7$	$\tan\varphi' = 9.93027$	$\sin(\theta - \alpha) = 9.98072_n$
$\gamma = 71°\ 6'\ 27''$		$\sec\delta = .01783$
$\gamma - \delta = 54°\ 48'\ 4''$	$\tan\gamma = .46568$	
		$\log\rho = 9.99939$
		$\log\pi = 1.13386$
		$\sin(\gamma - \delta) = 9.91231$
		$\sin\varphi' = 9.81183$
		$\operatorname{cosec}\gamma = .02405$
	$\alpha - \alpha' = -10''.31$	$\log(\alpha - \alpha') = 1.01336_n$
	$= -{}^s.69$	$\log(\delta - \delta') = .88144$
	$\delta - \delta' = +7''.61$	

Application of (XI)$_1$.

To compute the parallax of Venus in declination at the time of meridian passage, Bethlehem, 1881, July 4th.

The data are as follows:

$\delta = 16°\ 20'\ 48''.5$		$\log\pi = 1.13258$
$\pi = 13''.57$		$\log\rho = 9.99939$
$\varphi' = 40°\ 25'\ 1''.7$		$\sin(\varphi' - \delta) = 9.61051$
$\delta - \delta' =$	$5''.53$	$\log(\delta - \delta') = .74248$

Refraction.

86. When a ray of light passes obliquely from a rarer into a denser medium, it is bent or refracted out of its original course towards the normal drawn to the surface separating the two media, at the point where the ray pierces this surface. The angle which the original direction of the ray makes with this normal is the *angle of incidence*, and the angle formed with the normal by the bent or refracted ray is the *angle of refraction.*

According to Descartes, refraction takes place in accordance with the following laws:

I. *Whatever the obliquity of the incident ray, the ratio which the sine of the angle of incidence bears to the sine of the angle of refraction is always constant for the same two media, but varies with different media.*

II. *The incident and refracted ray are in the same plane, which is perpendicular to the surface separating the two media.*

If the density of the air were uniform and constant, the determination of the effect of refraction would be a comparatively easy matter in accordance with these laws. Neither condition is realized, however.

The density of the air is a maximum at the surface of the earth, and it continually decreases as we ascend above the surface, until it practically disappears at an altitude of 45 or 50 miles. It is also continually varying in density, as shown by the readings of the barometer and thermometer.

In consequence of the decrease in density of the air as we ascend above the surface of the earth, it follows that the path of a ray of light through the atmosphere is not a straight line, but a curve, as shown in the figure. We see a star in the direction of a tangent drawn to the curve at the point where it enters the eye. In consequence, the altitudes of all celestial bodies appear to us greater than they really are; but in accordance with Descartes' second law, the azimuths are not affected at all.

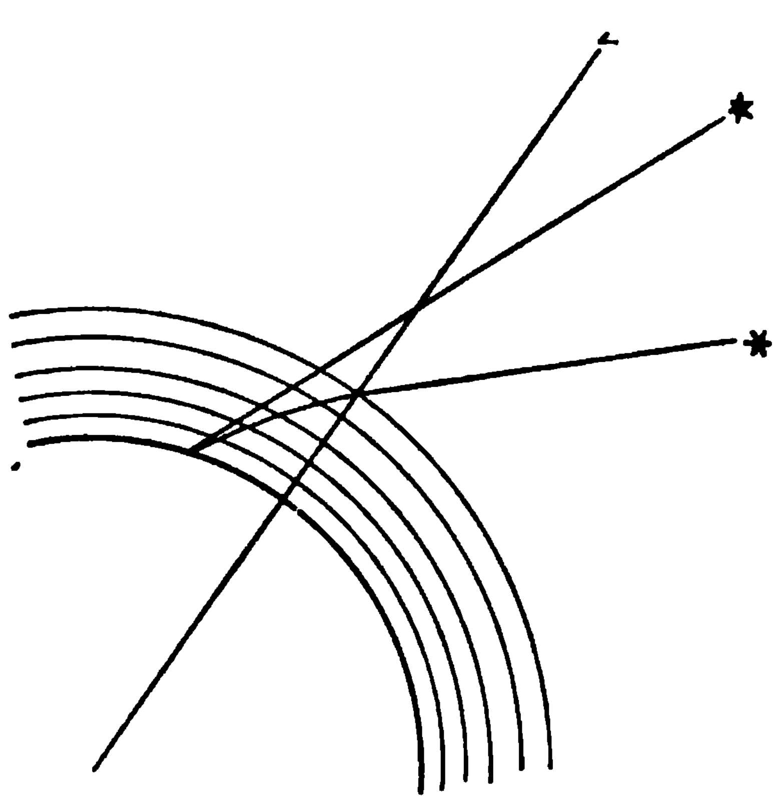

FIG. 8.

It sometimes happens that there are lateral deviations of an anomalous character, but these are beyond the scope of

theory, and when they exist are generally to be counted among the accidental errors to which observations are liable.

The complete investigation of the laws of astronomical refraction is a very complex and difficult problem, and one which has never been solved with entire satisfaction. We shall not enter into the theory here, but confine ourselves to the explanation of the use of our refraction tables based on those of Bessel.

Bessel's formula for the amount of refraction at any zenith distance z is

$$r = \alpha \,.\, \beta^A \gamma^\lambda \tan z. \quad . \quad . \quad . \quad . \quad . \quad (186)$$

In which r is the refraction; α varies slowly with the zenith distance; A and λ also vary with the zenith distance, and differ but little from unity. This difference is never appreciable except for large zenith distances: for our purposes it will generally be sufficiently accurate to regard them as unity. β is a factor depending on the barometer reading. As this reading depends on the pressure of the air and the temperature of the mercury, it is tabulated in the form

$$\beta = t \times B.$$

In which B depends on the reading of the barometer, and t upon the *attached* thermometer.

γ depends upon the temperature of the air as shown by the *detached* thermometer.

We may therefore use the formula

$$r = R \times B \times t \times T. \quad . \quad . \quad . \quad . \quad . \quad (187)$$

In which $R = \alpha \tan z$ is given in table II A;

B depends upon the barometer and is given in table II B;

t depends upon the attached thermometer and is given in table II C;

T depends upon the detached thermometer and is given in table II D.

As an example take the following:

Apparent altitude $= h' =$	$31° 49' 48''$
Barometer reading	29.51 inches
Attached thermometer	$78°.2$
Detached thermometer	$82°.1$

Table II A,	$R = 93''.6$	$\log = 1.9713$
II B,	$B = .983$	$\log = 9.9928$
II C,	$t = .997$	$\log = 9.9990$
II D,	$T = .941$	$\log = 9.9736$
	$r = 1' 26''.4$	$\log r = 1.9367$

For many purposes, especially for small zenith distances, it will be sufficiently accurate to use the mean refraction R without correcting for barometer and thermometer.

An approximate value may be obtained by the formula

$$r = 57''.7 \tan z. \qquad (188)$$

This will be accurate enough for many purposes, and may be of service in cases where tables are not available. This would give for our example above

$$r = 1' 32''.95.$$

When the greatest precision is demanded, table III must be employed. For the above example we have

Table III A,	$\log \alpha =$ 1.76021	$A =$ 1.00	$\lambda = 1.004$
III B, III C,	$A \cdot \log \beta = -$ .00306	$\log B = -$ .00127; $\log t = -$ .00179	
III D,	$\lambda \cdot \log \gamma = -$ .02757	$\log \gamma = -$ .02746	
	$\tan z =$ .20709		
$r = 1' 26''.43$	$\log r =$ 1.93667		

In the volume of astronomical observations of the Washington Observatory for 1845 may be found refraction tables carried out much farther than those given here. They are convenient when many computations are to made with great precision.

Refraction in Right Ascension and Declination.

87. As our tables give the refraction in zenith distance or altitude, if we require the effect in right ascension and declination it will be necessary to express the increments of these quantities in terms of the increment of the zenith distance. Differential formulæ will be accurate enough for any case which is likely to arise. Such formulæ are given in works on Trigonometry. Those required for this particular purpose are derived as follows:

Let us assume the general formulæ of spherical trigonometry, viz.:

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A; \\ \sin a \cos B &= \cos b \sin c - \cos c \sin b \cos A; \\ \sin a \sin B &= \sin b \sin A. \end{aligned}\right\}. \quad (189)$$

Applying these formulæ to the triangle formed by the zenith, the pole, and the star, we have

$$\left.\begin{aligned} \sin \delta &= \sin \varphi \cos z - \cos \varphi \sin z \cos a; \\ \cos \delta \cos q &= \sin \varphi \sin z + \cos \varphi \cos z \cos a; \\ \cos \delta \sin q &= \cos \varphi \sin a. \end{aligned}\right\} (190)$$

Also,

$$\left.\begin{aligned} \cos z &= \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t; \\ \sin z \cos q &= \sin \varphi \cos \delta - \cos \varphi \sin \delta \cos t; \\ \sin z \sin q &= \cos \varphi \sin t. \end{aligned}\right\} (191)$$

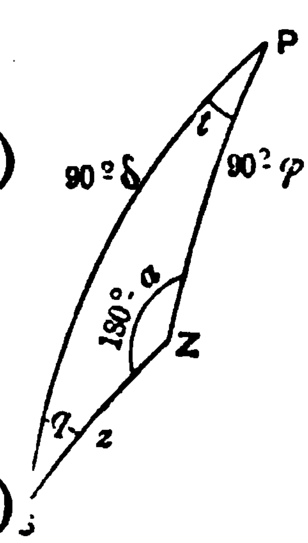

FIG. 9.

Now differentiating the first of (190), regarding δ and z only as variables,

$$\cos \delta d\delta = -(\sin \varphi \sin z + \cos \varphi \cos z \cos a)dz.$$

Combining this with the second of (190), we have

$$d\delta = -\cos q dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad (192)$$

Differentiating the first of (191), regarding z, δ, and t as variables,

$$-\sin z dz = (\sin \varphi \cos \delta - \cos \varphi \sin \delta \cos t)d\delta - \cos \varphi \cos \delta \sin t dt.$$

Combining this with the second and third of (191) and with (192), we readily derive

$$\cos \delta dt = +\sin q dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad (193)$$

In (192) and (193),

$$dz = \text{the refraction in zenith distance} = r;$$
$$t = \Theta - \alpha; \qquad \text{therefore} \qquad dt = -d\alpha.$$

Our formulæ then become

$$\left.\begin{aligned} d\delta &= -r \cos q; \\ \cos \delta d\alpha &= -r \sin q. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (194)$$

For applying these formulæ we must compute q, and we require z for taking from the table the refraction in zenith distance.

Equations (191) give these quantities, the solution of which is as follows:

Let $$n \sin N = \cos \varphi \cos t;$$
$$n \cos N = \sin \varphi.$$

Then
$$\cos z = n \sin(\delta + N);$$
$$\sin z \cos q = n \cos(\delta + N);$$
$$\sin z \sin q = \cos \varphi \sin t;$$

and finally,
$$\left.\begin{aligned} \tan N &= \cot \varphi \cos t; \\ \tan q &= \frac{\sin N}{\cos(\delta + N)} \tan t; \\ \tan z &= \frac{\cot(\delta + N)}{\cos q}; \\ \frac{\sin N}{\cos(\delta + N)} &= \frac{\cos \varphi \cos t}{\sin z \cos q}. \end{aligned}\right\} \quad . \; . \; . \quad \text{(XII)}$$

As an example of the application of formulæ (194), take the following:

Given the sun's right ascension $\alpha =$	$21^h\ 47^m\ 59^s.92$
Declination $\delta =$	$-\ 13°\ 17'\ 38''.7$
Latitude $\varphi =$	$40°\ 36'\ 24''$
Sidereal time $\Theta =$	$0^h\ 0^m\ 0^s$
Barometer reading	29.5 inches
Attached thermometer	65°.1
Detached thermometer	70°.0

From (XII) we find

$z = 61°\ 58'.0$; $\cos q = 9.94620$; $\sin q = 9.67068$.

From table II A,	$R = 1'\ 49''.0$	log =	2.0374
II B,	.983		9.9927
II C,	.998		9.9994
II D,	.962		9.9834
		$\log r =$	2.0129
		$\cos q =$	9.9462
		$\sin q =$	9.6707
	$d\delta = -\ 91''.0$	log =	1.9591
	$\cos \delta d\alpha = -\ 48''.3$	log =	1.6836

Dip of the Horizon.

88. At sea, altitudes of the heavenly bodies are measured from the visible horizon, which is generally a clearly defined line. As the eye of the observer is elevated above the surface of the water, this visible horizon, owing to the curvature of the earth, will be below the true horizon.

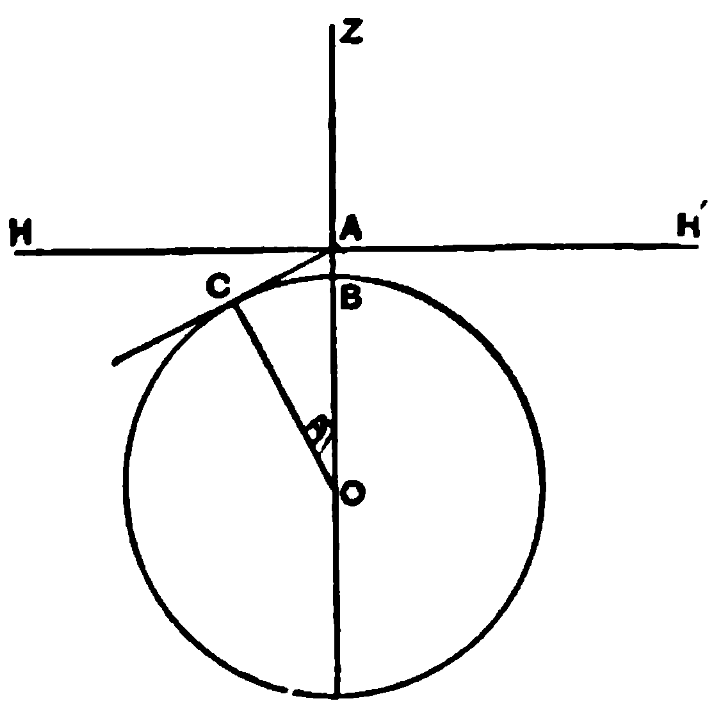

FIG. 10.

Thus, in the figure, let the circle represent a section of the earth made by a vertical plane passing through the eye of the observer at A. Then AH will be a section of the true horizon; AC will be a section of the visible horizon; the dip will be the angle $HAC = AOC$.

Let D = the dip;
a = the radius of the earth in feet;
$x = AB$, the height of the eye above the water in feet.

Then from the triangle ACO,

$$AC = a \tan D = \sqrt{\overline{AO}^2 - \overline{CO}^2} = \sqrt{(a+x)^2 - a^2} = \sqrt{2ax + x^2},$$

or $$\tan D = \frac{\sqrt{2ax + x^2}}{a}.$$

As x^2 will be very small in comparison with $2ax$, we may neglect it without appreciable error. Also, D being a small angle, we may write

$$\tan D = D \tan 1''.$$

Therefore we have $D = \dfrac{1}{\tan 1''}\sqrt{\dfrac{2}{a}}\sqrt{x}$,

or $$D = 63''.82\sqrt{x \text{ in feet.}} \quad . \quad . \quad . \quad . \quad (195)$$

This formula would give us the true value of the correction if there were no refraction, the effect of which is to diminish D. The refraction very near the horizon is always a somewhat uncertain quantity, but for a mean state of the air the dip corrected for refraction will be found by multiplying the value given by (81) by the factor .9216,

or $$D' = 58''.82\sqrt{x \text{ in feet.}} \quad . \quad . \quad . \quad . \quad (196)$$

An approximate value sometimes used by navigators is obtained by taking the square root of the number of feet above the water and calling the result minutes. Thus if the eye is 25 feet above the water, this process would give for the dip 5′; formula (196) gives 4′ 54″.

The dip must be subtracted from the observed altitude to obtain the true altitude.

CHAPTER III.

TIME.

89. For astronomical purposes the day is considered as beginning at noon instead of at midnight; the hours are reckoned from zero to twenty-four, instead of from zero to twelve as in civil time. Thus, July 4th, 9^h A.M., civil reckoning, would be July 3, 21^h, astronomically.*

In all operations of practical astronomy the time when an observation is made is a very important element. There are various methods of reckoning time, of which three are in common use, viz., *mean solar*, *apparent solar*, and *sidereal* time. Before entering upon the relations between these different kinds of time, some preliminary considerations are necessary.

90. The *transit*, *culmination*, or *meridian passage* of a heavenly body at any place is its passage across the meridian of that place.

Every meridian is bisected at the poles; and as a star in the course of its apparent diurnal revolution crosses both branches, it is necessary to distinguish between the *upper* culmination and *lower* culmination.

The Upper Culmination of a heavenly body is its passage over that branch of the meridian which contains the observer's zenith.

The Lower Culmination is the passage over that branch which contains the observer's nadir.

Any star whose north-polar distance does not exceed the

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* The prime meridian conference which assembled at Washington October 1st, 1884, recommended the adoption of a universal day for astronomical and other scientific purposes, to begin at Greenwich mean midnight and to be reckoned from 0^h to 24^h. The Astronomer Royal of England has adopted the suggestion for the Greenwich Observatory. Whether it will be generally adopted remains to be seen.

north latitude of the place of observation is constantly above the horizon, and may be observed at both upper and lower culmination. Any star whose south-polar distance does not exceed the north latitude of the place of observation is always below the horizon, and therefore cannot be observed at all.* Stars between these limits can be observed at upper culmination only.

The rotation of the earth on its axis being uniform, it follows that the intervals of time between the successive transits of a point on the equator over either branch of the meridian will be of equal length. Such an interval is a sidereal day, and the point with the transit of which the sidereal day is regarded as beginning is the vernal equinox.

A SIDEREAL DAY *is the interval between two successive transits of the vernal equinox over the upper branch of the meridian.*

THE SIDEREAL TIME *at any meridian is the hour-angle of the vernal equinox at that meridian.*

The right ascensions being reckoned from the vernal equinox, it follows that a star whose right ascension is α will culminate at α hours, sidereal time.

Therefore the sidereal time at any meridian is equal to the right ascension of that meridian.

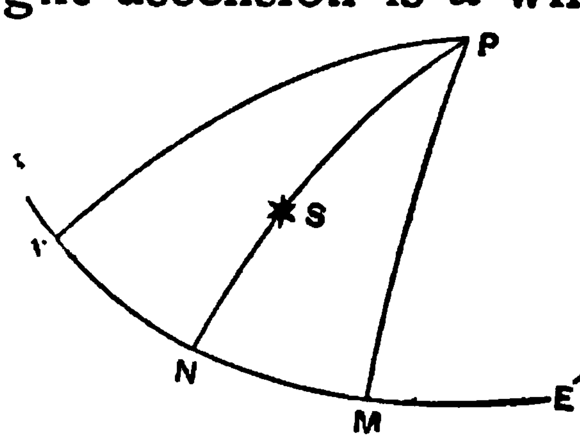

FIG. 11.

In the figure let EE' be the equator, P the pole, PM the meridian of any place, PN the hour-circle of any star S, ♈ the vernal equinox. Then from our definitions,

MPN = hour-angle of star $S = t$;
$NP♈$ = right ascension of star $S = \alpha$;
$MP♈$ = the sidereal time at the meridian $PM = \Theta$.

* If the latitude of the place of the observer is south, obviously these conditions will be reversed.

Therefore $$\Theta = \alpha + t. \qquad \ldots\ldots \qquad (197)$$

Thus, if we have by any method determined the hour-angle of a star, this equation gives the sidereal time; α, the right ascension, being taken from the ephemeris, or from a star catalogue.

The interval between two successive transits of the sun over the upper branch of the meridian is an APPARENT SOLAR DAY.

The hour-angle of the sun at any meridian is the APPARENT TIME *at that meridian.*

Owing to the annual revolution of the earth, the sun's right ascension is constantly increasing; therefore it follows that the solar day will be longer than the sidereal day. Thus in one year the sun moves through 24 hours of right ascension. In one year there are, according to Bessel, 365.24222 mean solar days; therefore in one day the sun's right ascension increases $\frac{24^h}{365.24222} = 3^m\,56^s.555$. In one hour one twenty-fourth of this amount $= 9^s.8565$.

These figures represent the mean or average rate of change. The actual change, however, is not uniform, and in consequence the apparent solar days are not of equal length. This want of uniformity results from two causes, which will now be explained.

First Inequality of the Solar Day.

91. The apparent orbit of the sun about the earth is an ellipse with the earth in one of the foci. Let the ellipse, Fig. 12, represent this apparent orbit. When the sun is at A the right ascension is increasing more rapidly than when it is at A'; therefore in the first case it will have a larger arc to pass over between two successive meridian passages than in the second. This inequality alone being considered, the

length of the solar day will be a maximum when the sun is in perigee, and a minimum when it is in apogee. We may imagine a fictitious sun to move in the ecliptic in such a way that the angular distances AEP, PEP_1, $P'EP_1'$, etc., described in equal times, shall be equal. Let both start together from A on January 1st, moving in the direction of the arrow. On January 2d the true sun will be in advance of the fictitious sun, and will continue so until June 30th, when they will be together at A'. Therefore from January 1st to June 30th the fictitious sun, having the smaller right ascension, will always pass the meridian in advance of the true sun. From A' to A the fictitious sun will be in advance of the true sun, and will consequently pass the meridian later, until they both reach A, when they will again be together, January 1st.

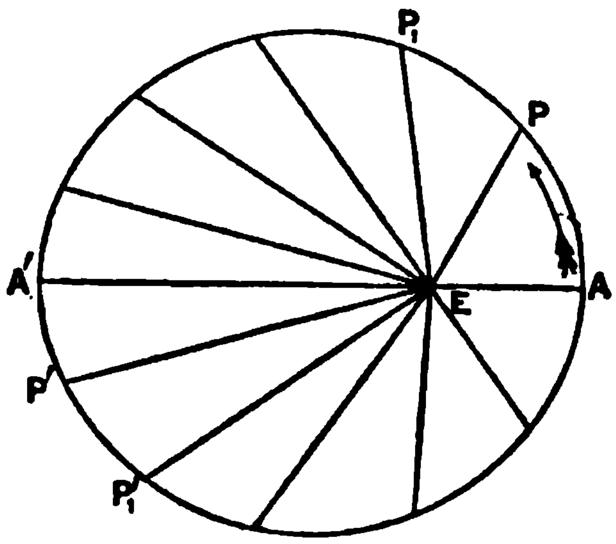

FIG. 12.

Second Inequality of the Solar Day.

92. The figure represents a projection of the sphere on the plane of the equinoctial colure. P is the north pole, P' the south pole, ♈ O ♎ the equator, ♈ ♋ ♎ ♑ the ecliptic. Now the fictitious sun before considered moves in the ecliptic describing the equal arcs ♈A, AB, BC, etc., in equal times. Let the hour-circles PAP', PBP', etc., be drawn; then the distances ♈a, ab, bc, etc., intercepted on the equator, will not be equal, but the distance ♈♋ = ♈O, both being quadrants.

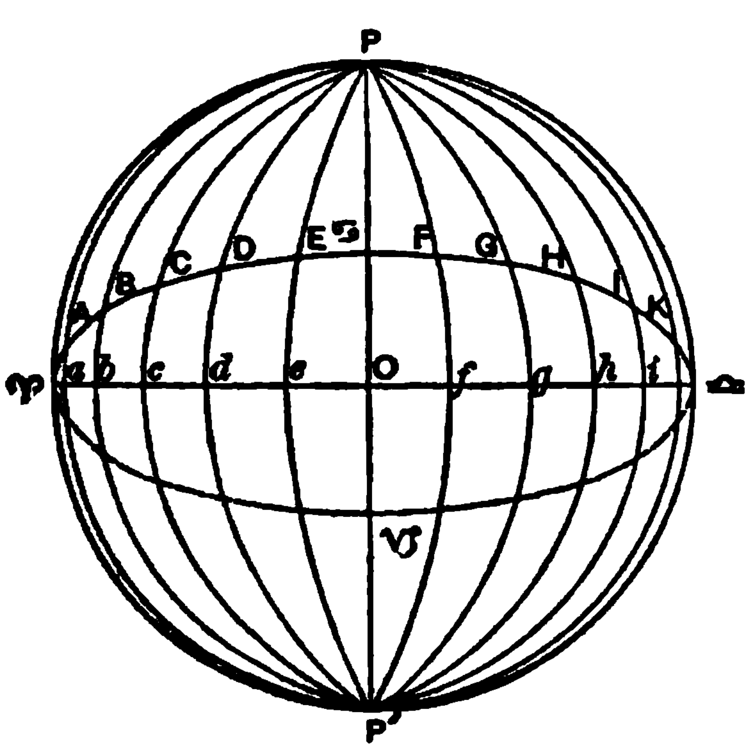

FIG. 13.

We may now suppose a second fictitious sun to move in the equator in such a way as to complete the circuit of the equator in the same time that the first completes the circuit of the ecliptic.

Let both start from the vernal equinox ♈ together on March 20th; on March 21st the second fictitious sun will be in advance of the first, and will continue so until June 20th, when they will both have completed a quadrant and will be on the solstitial colure at the same instant, the first at ♋ and the second at o. Therefore from March 21st until June 20th the right ascension of the first fictitious sun will be less than that of the second, and it will always pass the meridian first.

From June 20th to September 22d the first fictitious sun will be in advance of the second, at which time they will both be together at ♎. From September 22d until December 21st the second will be in advance of the first, at which time they will both again be on the solstitial colure at the same instant, the first at ♑ and the second at o. From this until March 20th the first will again be in advance of the second, when finally they will again be together at ♈, having completed an entire revolution.

As the second fictitious sun describes equal arcs of the equator in equal times, it follows that the intervals of time between each two successive transits over the same branch of the meridian will be equal.

A MEAN SOLAR DAY *is the interval between two successive transits of the second fictitious sun, or the mean sun over the upper branch of the meridian.*

THE MEAN SOLAR TIME *at any meridian is the hour-angle of the second fictitious sun or the mean sun at that meridian.*

THE EQUATION OF TIME *is the quantity which must be added algebraically to the apparent time to produce the mean time.*

The equation of time is given in the Nautical Almanac, p. 326 and following, for Washington apparent noon of each day in the year. If we require its value for any other time, we must interpolate between the values there given. It is the algebraic sum of the two inequalities explained above. From the foregoing we readily see that the equation of time will be zero four times in the course of the year; also that there will be two maxima and two minima values.

By referring to the ephemeris for 1881, we find the value to be zero on April 14th, June 13th, August 31st, and December 23d. The maxima values $+ 14^m\ 28^s$ and $+ 6^m\ 15^s$ occur February 10th and July 25th respectively; the minima values $- 3^m\ 51^s$ and $- 16^m\ 18^s$ on May 14th and November 2d.

We have the following simple precepts:

To convert a given instant apparent time at any meridian into the corresponding mean time, add algebraically to the apparent time the equation of time taken from the ephemeris.

To convert the mean time at any meridian into the corresponding apparent time, subtract the value of the equation of time taken from the ephemeris.

Example 1. 1881, July 4th, $5^h\ 7^m\ 16^s$, Bethlehem apparent time; find the corresponding mean time.

Longitude of Bethlehem	$- 6^m\ 40^s.3$
Bethlehem apparent time	$5^h\ 7^m\ 16^s.$
Washington apparent time	$5^h\ 0^m\ 35^s.7 =$ July 4.21

From the Nautical Almanac (p. 329) we find

$$\begin{array}{rl} \text{Eq. of time July } 4 = & + 4^m\ 11^s.30 \\ \text{July } 5 = & + 4^m\ 21^s.69 \\ \hline \text{Difference} & 10^s.39 \end{array}$$

$$.21 \times 10^s.39 = 2^s.18$$

$$\begin{array}{rcl} \text{Eq. of time July 4} & = & 4^m\ 11^s.30 \\ \hline \text{July 4.21} & = & 4^m\ 13^s.48 \\ \text{Apparent time} & = & 5^h\ 7^m\ 16^s. \\ \hline \text{Mean time} & = & 5^h\ 11^m\ 29^s.48 \end{array}$$

Example 2. 1881, November 12th, $10^h\ 15^m\ 7^s$, Bethlehem mean time; find the apparent time.

From the Nautical Almanac we find

$$\begin{array}{rcl} \text{Equation of time} & = & -\ 15^m\ 34^s.71 \\ \text{Mean time} & = & 10^h\ 15^m\ 7^s.00 \\ \hline \text{Apparent time} & & 10^h\ 30^m\ 41^s.71 \end{array}$$

Comparative Length of the Sidereal and Mean Solar Unit.

93. Owing to the annual revolution of the earth about the sun, the number of sidereal days in a year will be greater by one than the number of mean solar days. According to Bessel the year contains

365.24222 mean solar days;*
366.24222 sidereal days.

Therefore

$$\begin{aligned} \text{One mean solar day} &= \frac{366.24222}{365.24222} \text{ sidereal days} \\ &= 1.00273791 \text{ sidereal days;} \\ \text{One sidereal day} &= \frac{365.24222}{366.24222} \text{ mean solar days} \\ &= 0.99726957 \text{ mean solar days.} \end{aligned}$$

* These values given for 1800 are not absolutely constant; the length of the year is diminishing at the rate of $0^s.595$ in 100 years.

Let $I_{\odot} =$ mean solar interval;
$I_{*} =$ sidereal interval;
$\mu = 1.00273791$.

Then

$$\left.\begin{aligned} I_* &= I_{\odot}\mu = I_{\odot} + I_{\odot}(\mu - 1) = I_{\odot} + .00273791 I_{\odot}; \\ I_{\odot} &= \frac{I_*}{\mu} = I_* - I_*\left(1 - \frac{1}{\mu}\right) = I_* - .00273043 I_*. \end{aligned}\right\} \qquad (198)$$

By the use of these formulæ the process is very simple. It is rendered still more so by the use of tables II and III of the appendix to the Nautical Almanac. Table II gives the quantity $\left(1 - \frac{1}{\mu}\right)I_*$, with the argument I_*, and table III gives $(\mu - 1)I_{\odot}$, with the argument $I_{\odot}$.

One or two examples will illustrate their use.

Example 1. Given the mean solar interval $I_{\odot} = 4^h\ 40^m\ 30^s$. Find the corresponding sidereal interval.

	$I_{\odot} = 4^h\ 40^m\ 30^s.000$
Table III gives for $4^h\ 40^m$	$+\ 45^s.997$
Table III gives for 30^s	$+\ .082$
	$I_* = 4^h\ 41^m\ 16^s.079$

Example 2. Given the sidereal interval $I_* = 4^h\ 41^m\ 16^s.079$. Find the corresponding mean solar interval.

	$I_* = 4^h\ 41^m\ 16^s.079$
Table II gives for $4^h\ 41^m$	$-\ 46^s.035$
Table II gives for $16^s.079$	$-\ .044$
	$I_{\odot} = 4^h\ 40^m\ 30^s.000$

To Convert the Mean Solar Time at any Meridian into the Corresponding Sidereal Time.

94. Referring to Fig 11 and formula (197), we see that if S represents the mean sun, then

$MPN =$ the mean time $= T$;
$NP\Upsilon =$ the right ascension of the mean sun $= \alpha_{\odot}$.

Then we have $$\Theta = \alpha_{\odot} + T. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (199)$$

The right ascension of the mean sun, $\alpha_{\odot}$, is given in the solar ephemeris of the Nautical Almanac, for Washington mean noon of each day. It is there called the *sidereal time of mean noon*, which it is readily seen is the right ascension of the mean sun at noon, since at mean noon the mean sun is on the meridian when its right ascension is equal to the sidereal time.

If $L =$ the longitude of the meridian from which T is reckoned, then $(T + L) =$ the time past Washington mean noon.

Let $V_{\circ} =$ sidereal time of mean noon at Washington.

Then $$\alpha_{\odot} = V_{\circ} + (T + L)(\mu - 1),$$
and $$\Theta = T + V_{\circ} + (T + L)(\mu - 1). \quad . \quad . \quad . \qquad (200)$$

The last term may be taken from table III before used, or we may compute it by the method given in Art. 90. We there found the hourly change in right ascension of the mean sun to be $9^s.8565$. If we express $(T + L)$ in hours, we have

$$\alpha_{\odot} = V_{\circ} + (T + L)\, 9^s.8565.$$

When this operation has frequently to be performed at any meridian other than Washington, it is a little more convenient to use the sidereal time of mean noon at the meridian itself.

Let $V =$ the sidereal time of mean noon at meridian whose

longitude is L. Then if we consider L as reckoned towards the west, the Washington time of mean noon at the given meridian will be L, and we shall have

$$V = V_{\odot} + L(\mu - 1),$$

or $$V = V_{\odot} + 9^s.8565 L; \qquad L \text{ being expressed in hours.}$$

Formula (200) then becomes

$$\Theta = V + T + T(\mu - 1). \quad . \quad . \quad . \quad . \quad (201)$$

Example 1. Longitude of Bethlehem $= -6^m\,40^s.3 = -^h.1112$; Mean solar time, 1881, July 4th, $9^h\,00^m\,00^s$.

Required the corresponding sidereal time.

From the Nautical Almanac, p. 329, we find

	$V_{\odot} =$	$6^h\,51^m\,22^s.610$
$-.1112 \times 9^s.8565$, or from table III, N. A., $(\mu - 1)L$	$=$	$-\,1^s.096$
	$V =$	$6^h\,51^m\,21^s.514$
Mean solar time	$T =$	$9^h\,00^m\,00^s.000$
Table III, $(\mu - 1)T$		$+\,1^m\,28^s.708$
Sidereal time	$\Theta =$	$15^h\,52^m\,50^s.222$

Example 2. $T =$ 1881, July 4th, $21^h\,7^m\,3^s.2$, Ann Arbor mean time. Required Θ.

Longitude of Ann Arbor $= +26^m\,43^s.1 \;= {}^h.4453$

	$V_{\odot} =$	$6^h\,51^m\,22^s.610$
$.4453 \times 9^s.8565$, or table III, $(\mu - 1)L$	$=$	$+\,4^s.389$
	$V =$	$6^h\,51^m\,26^s.999$
	$T =$	$21^h\,7^m\,3^s.200$
Table III, $(\mu - 1)T$	$=$	$+\,3^m\,28^s.145$
Sidereal time	$\Theta =$	$4^h\,01^m\,58^s.344$

To Convert Sidereal into Mean Solar Time.

95. This process, the converse of the preceding, may be briefly stated as follows:

First. Subtract from the given sidereal time the sidereal time of mean noon; we then have the sidereal interval past noon, viz., $\Theta - V$.

Second. Convert the sidereal interval $(\Theta - V)$ into the corresponding mean time interval, by subtracting the quantity $(\Theta - V)\left(1 - \frac{1}{\mu}\right)$ found in table II, N. A.

The formula is as follows:

$$T = (\Theta - V) - (\Theta - V)\left(1 - \frac{1}{\mu}\right). \quad . \quad . \quad (202)$$

Example 1. Given 1881, July 4th, $15^h\ 52^m\ 50^s.222$ Bethlehem sidereal time.

Required the corresponding mean solar time.

	$\Theta =$	$15^h\ 52^m\ 50^s.222$
As before,	$V =$	$6^h\ 51^m\ 21^s.514$
	$\Theta - V =$	$9^h\ 01^m\ 28^s.708$
Table II, $(\Theta - V)\left(1 - \frac{1}{\mu}\right)$	$=$	$1^m\ 28^s.708$
Mean time	$T =$	$9^h\ 00^m\ 00^s.$

Example 2. Given 1881, July 4th, $4^h\ 1^m\ 58^s.344$ Ann Arbor sidereal time.

Required the mean solar time.

	$\Theta =$	$4^h\ 1^m\ 58^s.344$
As before,	$V =$	$6^h\ 51^m\ 26^s.999$
	$\Theta - V =$	$21^h\ 10^m\ 31^s.345$
Table II, $(\Theta - V)\left(1 - \frac{1}{\mu}\right)$	$=$	$3^m\ 28^s.145$
Mean time	$T =$	$21^h\ 7^m\ 03^s.2$

It is sometimes necessary to convert mean solar time into sidereal, or vice versa, in reducing old observations made before the publication of the solar ephemeris in the form now employed. Bessel's *Tabulæ Regiomontanæ* furnish the data necessary for solving the problem for any date between 1750 and 1850. The method of using these tables for this purpose is fully explained in Art. 362 of this work.

CHAPTER IV.

ANGULAR MEASUREMENTS.—THE SEXTANT.—THE CHRONOMETER AND CLOCK.

96. The circles of astronomical instruments are graduated continuously from zero to 360°. With ordinary field-instruments the smallest division is commonly 10′, though sometimes less. The large circles of fixed observatories are graduated much finer. Fractional parts of a division are read by means of the vernier, or reading microscope.

The edge of the circle on which the division is marked is called the *limb*. The circle or arm which carries the index is called the *alidade*.

The *vernier*, also called the *nonius*, is an arc carried by the alidade, and graduated in the manner described below, for measuring fractional parts of a division.

Let AB (Fig. 14) be a portion of the limb of a circle. Each division is supposed to be one degree of the circle. The arc CD, carried by the alidade and graduated as shown, forms a vernier.

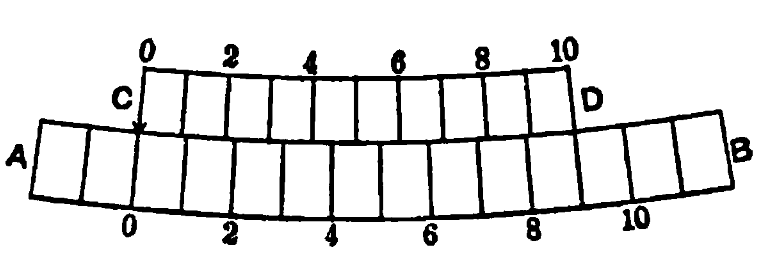

FIG 14.

In this case there are ten divisions on the vernier, covering a space equal to nine divisions of the limb. Each space on the vernier is therefore shorter by $\frac{1}{10}$ of one degree (equals 6′) than a space on the limb. In the figure the index coincides with the zero-point of the limb; division one of the vernier falls behind division one of the limb, 6′; division two of

the vernier falls behind division two of the limb, $2 \times 6' = 12'$, etc., etc.

The method of using the vernier will now be clear by referring to Fig. 15. In this case the index falls between 42° and 43° on the limb. The reading of the circle is therefore 42° plus a fractional part of a degree. This fraction is given by the vernier as follows: Looking along the scale until we find a line of the vernier which coincides with a line of the limb, we find this to be the case with the one marked 4. Therefore, following down the vernier scale towards the zero-point, it is evident that

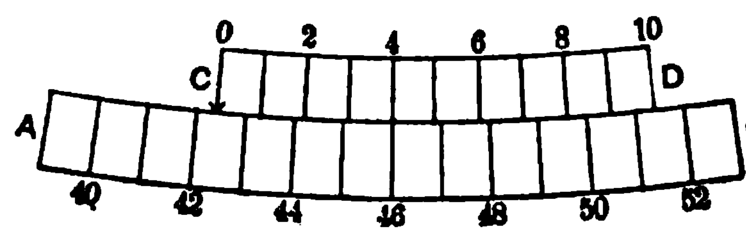

FIG 15.

Line 3 of the vernier is $6'$ to the right of 45° of the limb;
Line 2 of the vernier is $2\times6'=12'$ to the right of 44° of the limb;
Line 1 of the vernier is $3\times6'=18'$ to the right of 43° of the limb;
Line 0 of the vernier is $4\times6'=24'$ to the right of 42° of the limb.

The reading is therefore $42^\circ\ 24'$ or $42^\circ.4$, the number on the vernier where the line of the latter coincides with a line of the limb, giving the tenths of a degree at once.

In general let

d = the value of one division of the limb;
d' = the value of one division of the vernier;
n = the number of divisions of the vernier corresponding to $n - 1$ of the limb.

Then
$$(n-1)d = nd',$$
and
$$d - d' = \frac{1}{n}d. \quad \quad (203)$$

$d - d'$ is the least reading of the vernier. We have therefore the following very simple rule:

To find the least reading of a vernier: Divide the length of one division of the limb by the number of spaces of the vernier.

For example, suppose the limb graduated to 10′, and the number of divisions of the vernier-scale to be 60. Then the least reading of the vernier will be

$$\frac{10'}{60} = \frac{600''}{60} = 10''.$$

This is a very common arrangement.

In the vernier just described n divisions of the vernier were equal to $n - 1$ of the limb. Verniers are sometimes made in which n divisions are equal to $n + 1$ of the limb.

Then $\qquad (n + 1)d = nd'$ and $d' - d = \frac{1}{n}d$, as before.

It is to be observed that in this case the reading of the vernier proceeds in a direction opposite to that of the limb.

Many different forms of division and arrangement are found in verniers, but they all follow the same general principle, a practical familiarity with which makes the reading of any form of vernier very simple.

The Reading Microscope.

97. Instead of the vernier, in very fine instruments the alidade carries a microscope the optical axis of which is perpendicular to the plane of the circle. This is a compound microscope with a positive eye-piece. In the common focus of the object-lens and eye-piece are the micrometer-threads for reading the circle. The micrometer (Fig. 16*a*) consists of a frame of brass, across which are stretched two spider-lines. Sometimes these lines make an acute angle with each other, as shown in the figure; sometimes they are made parallel and quite close together. The plane of the frame is parallel to

the plane of the circle *MN*, and it is moved parallel to a tangent to the circle by the screw *G*. Attached to the screw and revolving with it is the cylinder *FE*, graduated, as shown in the figure, for recording the fractional parts of a revolution of the screw. The cylinder is generally graduated into either 60 or 100 parts. Suppose now the distance between two divisions of the circle to be 5′, and that five revolutions of the screw are just sufficient to move the cross-threads over this distance: then evidently one revolution moves the threads over 1′. If the head is divided into 60 parts, then each division of the head corresponds to a motion of the cross-threads over 1″. By making the screw sufficiently fine and increasing the number of divisions of the head, at the same time increasing the power of the microscope, this division of space may be carried to an almost unlimited extent. For the purpose under consideration, however, we should soon reach a limit beyond which nothing would be gained by increasing the delicacy of the microscope.

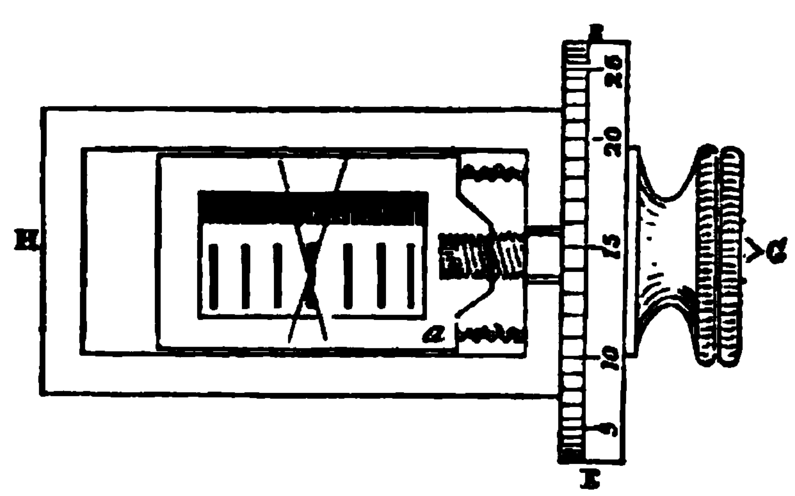

FIG. 16*a*.—THE MICROMETER.

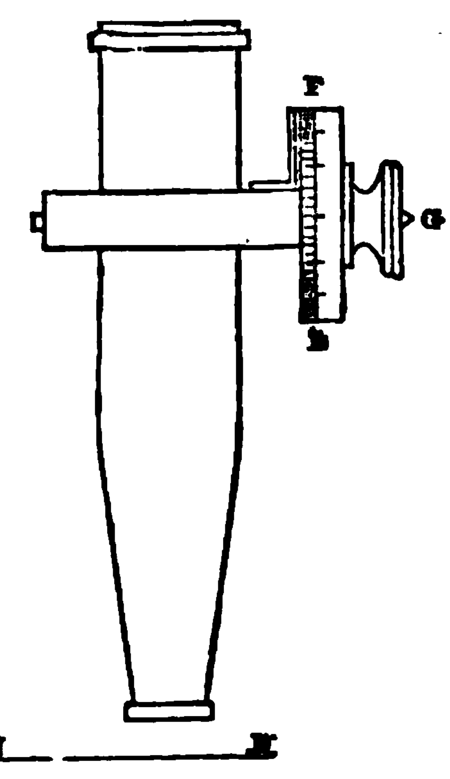

FIG. 16.—THE READING MICROSCOPE.

For reading the entire number of revolutions of the screw there is sometimes a scale attached to the outside of the box in which the slide moves. More frequently the scale is inside the box, placed at one side of the field of view. When so placed it consists of a strip of metal in the edge of which

notches are cut; the distance between two consecutive notches being equal to one revolution of the screw. Every fifth notch is made deeper than the others for facility in counting.

Suppose now the cross-threads to stand opposite the centre notch (which is generally distinguished in some manner), and the zero point of the head to be exactly at the index-mark. The point in the field now occupied by the cross-threads is the fixed point to which all angular measurements are referred; it corresponds exactly to the zero-point of the vernier. Suppose, further, the zero-point of the circle to be exactly under the intersection of the threads. Now let the instrument be revolved on its axis through any angle: the number of divisions of the circle which pass by this point of reference will then be the measure of the angle.

For the purpose of fixing the idea, let the arrangement be that described above, viz., the circle graduated to 5′, and the micrometer reading to single seconds. If now the revolution of the instrument has brought the scale into the position shown in Fig. 17, we see from the position of the threads that the entire angle passed over is between 45° 15′ and 45° 20′. By means of the screw let the cross-threads be moved so as to coincide with division 15′. Then the entire number of revolutions of the screw will give the number of minutes to be added to 45° 15′, and the fractional part of a revolution given by the head will be expressed in seconds. Thus if the whole number of revolutions were two, and the reading of the head 53, the angle would be 45° 17′ 53″. In making the bisection, the screw should always be turned in the same direction, to guard against the effect of slip or lost motion in the screw. If the thread is to be moved in a negative direction it should be moved back beyond the line, and the final bisection made by bringing it up from the other side.

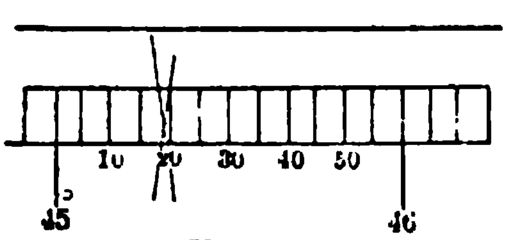

FIG. 17.

98. When everything is in perfect order a whole number

of revolutions of the screw is exactly equal to the distance between two consecutive lines on the circle. This is provided for by an arrangement for changing the focal length of the microscope, and for moving the object-lens nearer to or farther from the plane of the circle. This adjustment is subject to small disturbances, on account of changes of temperature and other causes. The error caused by an imperfect adjustment is called the error of runs. The correction for runs is found by reading the microscope on two consecutive divisions of the circle. If this does not correspond to the exact number of revolutions of the screw, the excess or deficiency is to be distributed in the proper proportion to measurements made with the screw.

For determining the correction a number of readings should be made in different parts of the circle in order to eliminate from the result the accidental errors of graduation. Some observers in certain kinds of work always read the micrometer on both divisions of the limb between which the zero-point falls. For example, in Fig. 15 the micrometer-thread would be set on both division 15′ and 20′, thus eliminating from the resulting reading the effect of runs, and to some extent the accidental errors of graduation and of bisection.

For insuring greater accuracy two or more microscopes or verniers are used. When there are two they are placed opposite each other, or 180° apart. When there are three or more they are placed at uniform distances around the circle. If the probable error of the reading of one microscope be $1''$, that of the mean of two will be* $\frac{1''}{\sqrt{2}} = ''.71$; that of four will be $\frac{1''}{\sqrt{4}} = ''.5$.

The principal value of two or more microscopes, however, is for eliminating the error of eccentricity.

* See Introduction, Art. 14, Eq. (25).

Eccentricity of Graduated Circles.

99. The centre of the alidade seldom coincides exactly with the centre of the graduated circle. This deviation from exact coincidence is called *eccentricity*.

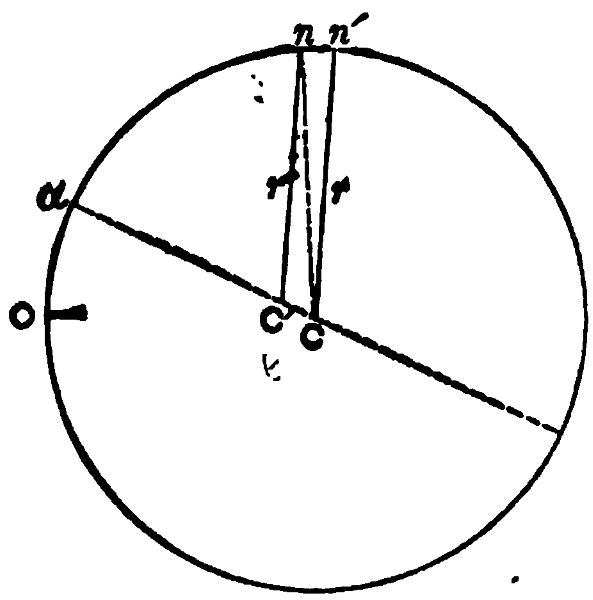

FIG. 18.

In order to understand the effect of eccentricity, let

C be the centre of the circle;
C', the centre of the alidade;
O, the zero-point of the limb;
α, the point on the limb where it is intersected by a line joining C and C';
$C'n$, the direction of the line drawn from the centre of the alidade to the zero-point of the vernier when the telescope is directed to any object.

The true position of the object is given by the direction of the line $C'n$, while the reading of the circle gives the direction Cn, differing from the former by the small angle $n'Cn = CnC'$.

Let now $CnC' = p$; $CC' = e$; $Cn = r$; $C'n = r'$; Angle $OCn = n$; $OC\alpha = \alpha$. Then $C'Cn = n - \alpha$.

From the triangle $C'Cn$ we have

$$r' \sin p = e \sin (n - \alpha);$$
$$r' \cos p = r - e \cos (n - \alpha);$$

from which
$$\tan p = \frac{\frac{e}{r} \sin (n - \alpha)}{1 - \frac{e}{r} \cos (n - \alpha)}. \quad . \quad . \quad . \quad (204)$$

The angle p will always be small, and the denominator of (204) differs but little from unity. We may therefore write, without appreciable error,

$$p = \frac{e}{r} \sin (n - \alpha). \quad . \quad . \quad . \quad . \quad . \quad (205)$$

100. It is more elegant to expand the above expression into a series in terms of ascending powers of $\frac{e}{r}$. Equation (204) is of the form

$$\frac{\sin p}{\cos p} = \frac{a \sin x}{1 - a \cos x};$$

from which we readily find

$$\sin p = a \sin (p + x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (206)$$

Now add $\sin (p + x)$ to both members of (206); then subtract $\sin (p + x)$ from both members; finally, divide the first expression by the second:

$$\frac{\sin p + \sin (p + x)}{\sin p - \sin (p + x)} = \frac{(a + 1) \sin (p + x)}{(a - 1) \sin (p + x)};$$

from which $$\tan (p + \tfrac{1}{2}x) = \frac{1 + a}{1 - a} \tan \tfrac{1}{2}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (207)$$

Applying to this the process of development made use of in Art. 74, Eq. (137), we find

$$p = a \sin x + \tfrac{1}{2}a^2 \sin 2x + \tfrac{1}{3}a^3 \sin 3x, \text{ etc.}$$

Writing for a and x their values and dividing by $\sin 1''$, in order to express p in seconds of arc, we find

$$p = \frac{e}{r \sin 1''} \sin (n - \alpha) + \frac{e^2}{2r^2 \sin 1''} \sin 2(n - \alpha) + \frac{e^3}{3r^3 \sin 1''} \sin 3(n - \alpha). \quad (208)$$

The first term is identical with (205), and will always give the necessary accuracy without using the following terms.

101. Besides the eccentricity above considered there is a similar effect due to the play of the axis of the instrument in

its socket. This is not a determinate quantity like that we have been considering, but when two verniers or microscopes 180° apart are used, the effect of both will be eliminated, as appears from the following:

Let n' and n'' be the readings of the two microscopes;
n, the true value of the angle.

Then from the first microscope

$$n = n' + e'' \sin (n' - \alpha).$$

Similarly, $$n = n'' + e'' \sin (n'' - \alpha).$$

In which e'' has been written for $\dfrac{e}{r \sin 1''}$.

Now n'' differs very little from $180° + n'$, so that no appreciable error will be introduced by writing the second of the above equations

$$n = n'' + e'' \sin [180° + (n' - \alpha)] = n'' - e'' \sin (n' - \alpha).$$

Therefore $n = \frac{1}{2}(n' + n'')$, from which the correction for eccentricity is eliminated. In a similar manner it may be shown that the mean of three microscopes will be free from the effect of eccentricity. In case of four, as the mean of each pair 180° apart is free from this error, it follows that the mean of the four will be.

The constants e'' and α may be determined very readily by taking readings in different parts of the circle; but with a complete circle they will not be required. It is only in the case of the sextant, where we have a limited arc of the circle read by a single vernier, that this becomes a matter of importance. The application to this case will be considered in the proper place.

The Sextant.

102. In the determination of time and latitude when extreme accuracy is not required, the sextant is one of the most convenient and useful of astronomical instruments. It is light and easy of transportation; in observing it is simply held in the hand, and consequently entails no loss of time in

FIG. 19.—THE SEXTANT.

mounting and adjusting; it is therefore especially adapted to the requirements of navigation and exploration. For use on land the sextant is sometimes mounted on a tripod, which adds something to its accuracy. When the instrument is used by a skilful observer, however, the advantage is not great. In most cases where such an arrangement could be made use of the sextant will not be employed at all, but will give place to an instrument of greater precision.

The principal features of the sextant may be seen from Fig. 19. The graduated arc is about 60° in extent, hence the name, sextant. This arc of 60° is divided into 120 parts, called degrees for reasons which will soon appear. The arc commonly reads directly to 10′, and by means of the vernier to 10″. A mirror, *C*, called the *index-glass*, is attached to the arm carrying the vernier, and revolves with it about a pivot at the centre. A second mirror, *N*, is attached to the frame of the instrument, and is called the *horizon-glass*. Only half of this glass is silvered, viz., that next the plane of the instrument—an arrangement which makes it possible to see an object directly through the unsilvered part by means of the telescope, and at the same time the image of the same object, or of a second one, reflected from the silvered part of the mirror. In order to make these images equally distinct an adjusting-screw is provided (not shown in the figure), by which the telescope can be moved nearer to the plane of the instrument or farther from it. Attached to the frame are several colored glasses, *E* and *F*, which may be brought into a position to protect the eye when observing the sun. These are sometimes attached to an axis so that they can be at once reversed, the object being to eliminate any error due to want of parallelism of the surfaces by taking half of a series of measurements in each position. There is also a revolving disk attached to the eye-piece of some instruments containing a number of colored glasses of different shades. Other minor features can best be learned by the inspection of the instrument itself.

103. The principle which lies at the foundation of the sextant and instruments of like character is the following: If a ray of light suffers two successive reflections in the same plane by two plane mirrors, then the angle between the first and last direction of the ray is double the angle of the mirrors. In Fig. 20 let *M* and *m* be the two mirrors supposed

perpendicular to the plane of the paper; let AM be the first direction of a ray of light falling on the mirror M; it will be reflected in the direction Mm, and finally from m in the direction mE. Draw MB parallel to mE, MP perpendicular to M, Mp perpendicular to m. The angle between the first and

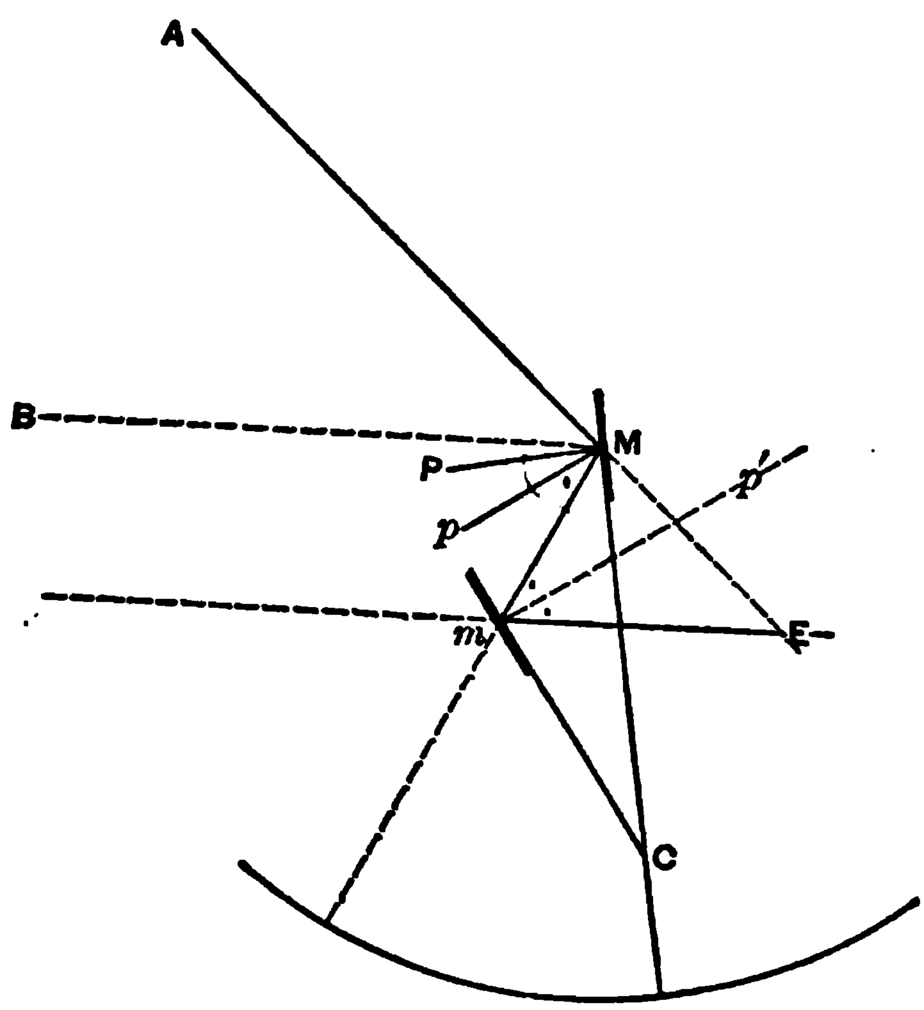

FIG. 20.

last direction of the ray is equal to the angle AMB. The angle between the mirrors is equal to PMp. We have now to show that $AMB = 2PMp$.

Consider first the mirror m. The incident ray Mm makes with the normal the angle

$$Mmp' = mMp = pMB = pMP + PMB. \quad . \quad . \quad (a)$$

Consider now M. The angle

$$mMP = PMA = AMB + PMB. \quad . \quad . \quad . \quad . \quad (b)$$

Subtracting (*a*) from (*b*),

$$mMP - mMp = AMB - pMP,$$

from which $\qquad 2pMP = AMB. \qquad$ Q. E. D.

If now the angle between two objects is to be measured, the instrument is held so that the plane of the graduated arc passes through both. The telescope is then directed to one of the objects, which is seen through the unsilvered part of the horizon-glass, and the index-arm is revolved until the reflected image of the second object is brought in contact with the direct image of the first. The reading of the limb will then be the required angle; the graduation before explained, viz., each degree being divided into two, gives the angle between the objects, which is twice that of the mirrors.

104. In the prismatic sextant of Pistor & Martins (Fig. 21) the horizon-glass is replaced by a totally reflecting prism. The arrangement has this advantage, viz., that by its use angles of all sizes from 0° to 180°, and even larger, can be measured, while the common form of sextant is not adapted to the measurement of angles much greater than 120°.

In using the instrument the prism *B* interferes with the rays of light which should reach the index-glass, *A*, when the angle is about 140°; but angles of this magnitude may be measured by turning the instrument over and holding it in the reverse position. If, for instance, the double altitude of the sun is being measured, the instrument will ordinarily be held in the right hand, with the arc below and the telescope above. If, however, the double altitude is about 140°, the instrument must be held in the left hand, with the telescope below and the arc above. In case the head of the observer interferes, as will be the case when the angle is near 180°, the difficulty is overcome by means of the prism *E*

placed back of the eye-piece so as to reflect the rays of light coming through the telescope in a direction at right angles to its axis.

105. The arc of the sextant may be extended to an entire circumference, and the index-arm produced so as to carry a

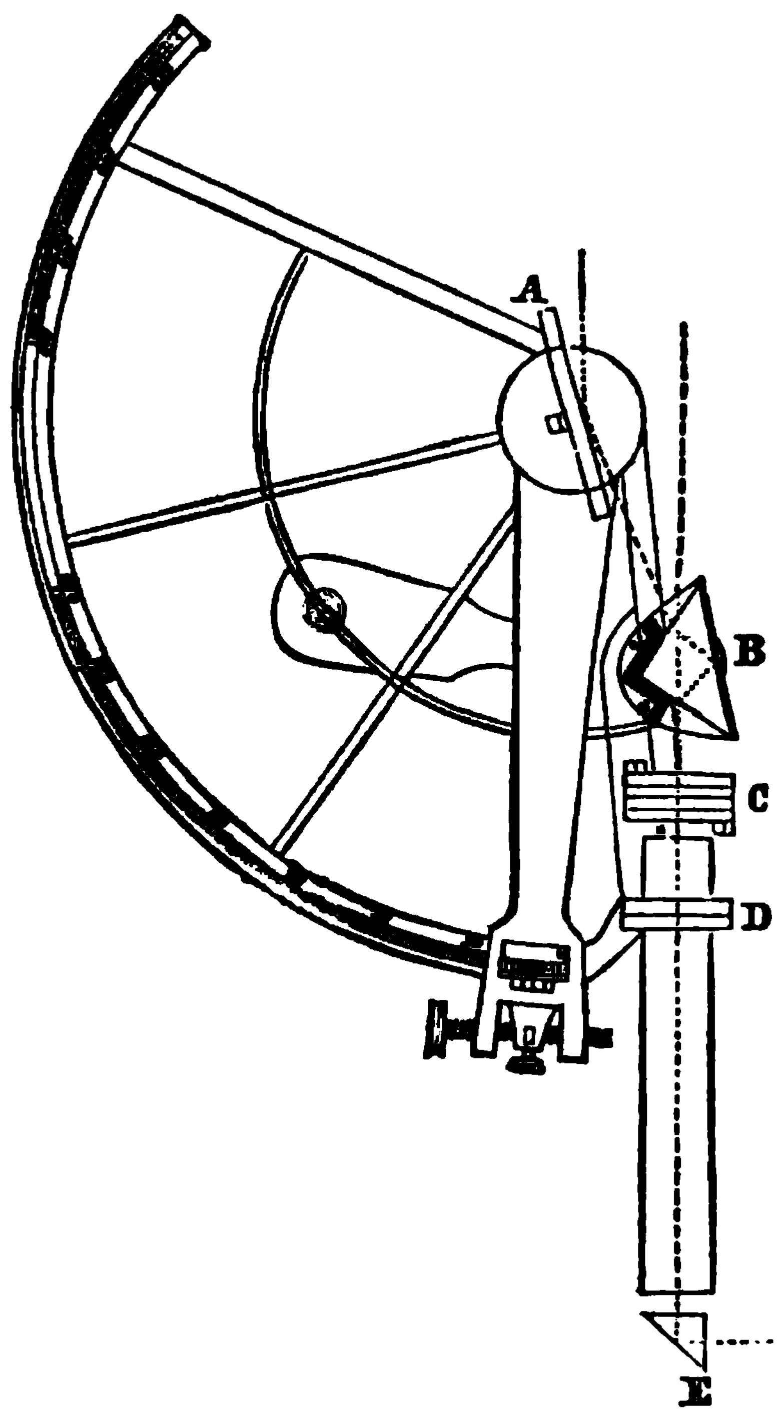

FIG. 21.—THE PRISMATIC SEXTANT.

vernier at each extremity. The instrument then becomes the simple *reflecting circle*. As previously shown, this arrangement possesses the advantage of eliminating the eccentricity, and to some extent the errors of graduation. This instrument is used precisely like the sextant.

Other forms of reflecting circles have been made possessing advantages in certain directions, but they do not seem to have met with great favor, although they are theoretically much more perfect instruments than the sextant; practically, however, this superiority is not so great. This is no doubt due in part to the fact that, except in the hands of an observer of more than usual skill, the errors of observation are so great as practically to neutralize their greater theoretical advantages.

Adjustments of the Sextant.

106. *First Adjustment.* THE INDEX-GLASS. *The plane of the reflecting surface must be perpendicular to the plane of the sextant.*

To ascertain whether this is the case, place the index near the middle of the arc, then look into the glass so as to see the image of the arc reflected. If the adjustment is perfect, the arc seen directly will be continuous with its reflected image.

This adjustment is attended to by the maker and is not liable to derangement; for this reason no provision is commonly made for correcting a want of perpendicularity. It may be corrected when necessary by removing the glass from its frame and filing down one of the points against which it rests, or by loosening the screws holding the frame to the index-arm and inserting a piece of paper or other thin substance under one side.

107. *Second Adjustment.* THE HORIZON-GLASS. *The plane of this mirror must also be perpendicular to the plane of the sextant.*

The index-glass must first be in adjustment; if then it is possible to place it in a position parallel to the horizon-glass by moving the index-arm, then the latter will also be perpendicular to the plane of the sextant. To test this adjust-

ment proceed as follows: Bring the index near the zero-point and direct the telescope to a well-defined point—a star is best. If then the index-arm be moved slightly one way and then the other—the plane of the instrument being vertical—the reflected image of the object will move up and down through the field. If the adjustment of the two glasses is perfect, the two images may be made to coincide exactly, otherwise the reflected image, instead of passing over the direct, will pass to one side or the other of it. Two small capstan-headed screws are provided for making this adjustment when necessary. A pair of adjusting-screws is also provided for correcting the position of the glass in the opposite direction, viz., to make it parallel to the index-glass when the vernier is at zero. If the direct and reflected image of the star are brought into exact coincidence by means of the tangent-screw, the reading of the vernier, if not zero, is called the index error. The screws just mentioned are for correcting this error. It will be found better in practice not to attempt this adjustment, but to determine the error and apply the necessary correction to the angles measured, as will be explained hereafter.

108. *Third Adjustment. The axis of the telescope must be parallel to the plane of the instrument.*

Two parallel threads are placed in the eye-piece to mark approximately the middle of the field: they should be made parallel to the plane of the instrument by revolving the eye-piece. The axis of the telescope will now be the line drawn through the optical centre of the object-glass and a point midway between these lines. To determine whether this line is parallel to the plane of the instrument, select two well-defined objects 100° or more apart, and bring the reflected image of one in contact with the direct image of the other, making the contact on one of the threads; then move the instrument so as to bring the images on the other thread.

If the contact still remains perfect, the line is in adjustment; if any correction is required, there will be found a pair of screws for the purpose on opposite sides of the ring which holds the telescope.

The above test will be found difficult to apply, especially if the observer has not a considerable amount of experience in the use of the instrument. One less difficult is the following: Place the instrument face upward on a table, then lay on the arc two strips of metal or wood, the width of which must be the same and equal to the distance of the axis of the telescope from the plane of the instrument. Now sight across the upper edges of these strips, and have an assistant mark with a pencil on the wall of the room (which should be 15 or 20 feet distant) the place where the sight-line intersects it; then, without disturbing anything, look through the telescope, which has been previously directed to this part of the wall and properly focused, and see whether this mark is found in the middle of the field; if so, then the adjustment is satisfactory.

Method of Observing with the Sextant.

109. *To Measure the Distance between Two Stars.* Direct the telescope to one of the stars, then revolve the instrument about the axis of the telescope until its plane passes through the other (taking care to have the index-glass on the right side), then move the index-arm until the image of the second star is brought into the field, clamp the instrument and bring the two images into perfect contact by means of the tangent-screw. The reading of the vernier corrected for index error will be the required distance. Unless the two stars are quite near each other it will be expedient to compute the distance approximately before attempting the observation. The index may then be set at the approximate distance, which will

greatly facilitate finding the two images. A common observation of this character is that of obsèrving the distance of the moon from the sun or a star for determining longitude. In the Nautical Almanac will be found given for every day throughout the year the distance of the moon from the sun, and certain stars and planets, which may be used for this purpose. The index may at once be set at the approximate angle without any preliminary computation. If the distance of the moon from a star is measured, the image of the star is brought into contact with the bright limb of the moon, the contact being made at the point where the great circle joining the star with the centre of the moon intersects the limb. To ascertain this point the instrument must be revolved through a small arc back and forth about the axis of the telescope (supposed to be directed to the star); the image of the moon's limb will then pass back and forth across the field, and should appear to pass exactly through the centre of the star's image, which will in general not be reduced to a simple point by the feeble telescope of the sextant.

This distance is to be corrected for the moon's semidiameter in order to give the distance between the star and the centre of the moon.

In measuring the distance between the moon and sun, the bright limb of the moon is brought in contact with the nearest limb of the sun. The measured distance must then be corrected for the semidiameters of both moon and sun.

110. *Measurement of Altitudes.* At sea altitudes are measured by bringing the reflected image of the body in contact with the line of the horizon as seen directly through the telescope. In order that the result may be correct the plane of the instrument must be held exactly vertical. To accomplish this the instrument is revolved or vibrated slightly about the axis of the telescope, at the same time moving it so as to keep the image in the centre of the field.

The image will appear to describe an arc of a circle, the lowest point of which must be made tangent to the horizon by moving the index-arm. If the sun is observed, the lower limb must be made tangent to the horizon. As the altitude of the sun's centre is required, the reading of the vernier must be corrected for index error, refraction, parallax, and semidiameter. If a star is observed, there will be no correction for semidiameter or parallax.

111. For observing altitudes on land the artificial horizon must be used. This is a shallow basin, about 3 inches by 5, for holding mercury. It is provided with a roof formed of two pieces of plate glass set at right angles to each other in a metal frame, for protecting the mercury from agitation by the wind. The surface of the mercury forms a mirror from which the image of the sun or star is reflected; and as it is perfectly horizontal the reflected image will appear at an angular distance below the horizon equal to the altitude of the body itself above the horizon. If now the image of a star reflected from the mirrors of the sextant is brought into contact with the image reflected from the mercury, the angle which will be measured is evidently twice the altitude of the star.

The opposite sides of the glass plates forming the roof to the horizon should be exactly parallel, otherwise the prismatic form introduces an error into the measured angle. It is possible to derive a formula for the correction necessary to free an observation from this source of error, but it will be better in practice to observe half of a series of altitudes with one side of the roof next the observer and then reverse it, taking the remaining half in the opposite position.

The mercury must be freed from the particles of dust and impurities which will generally be found floating on its surface. It may be strained through a piece of chamois-skin or through a funnel of paper brought down to a fine point

at the end. Another method is to add a small amount of tin-foil to the mercury, when the amalgam which will be formed will rise to the top and may be drawn to one side with a card, leaving the surface entirely free from specks of any kind.

112. In measuring altitudes for any purpose, a number of measures should be made in quick succession and the mean taken. In this way the accidental errors of contact and reading will be greatly diminished. Thus, in taking the altitude of the sun for determining the time, a series of not less than three altitudes should be measured on each limb. Suppose the observations made when the sun is east of the meridian, and the altitudes therefore to be increasing; the readings on the upper limb will be made first, as follows: Set the index on an even division of the limb at a reading 10′ or 15′ greater than the double altitude of the upper limb. When the two images are then brought into the field they will appear separated, but will be approaching each other. The observer watches until they become tangent, when the time is carefully noted by the chronometer. The index is then moved ahead 10′, 15′, or 20′, and the same process repeated. A little practice will enable the observer to take the altitudes in this manner at intervals of 10′ without difficulty, in which case five readings may be taken which will correspond to an increase of 40′ in the double altitude or 20′ in the actual altitude. As the sun's diameter is about 32′ of arc, the index may now be moved back to the first reading, and five readings on the lower limb taken at the same altitudes as before. In this case the images will overlap and will gradually separate, the time to be noted being that when the two disks are tangent.

If the sun is observed west of the meridian, the readings on the lower limb will be made first. The altitudes will of course be decreasing.

113. The beginner will sometimes find difficulty in bringing the two images into the field together. A convenient way of accomplishing this is as follows: Bring the index near the zero-point and direct the telescope to the sun, when two images will be seen; then bring the instrument down towards the mercury horizon, at the same time moving the arm so as to keep the reflected image in the field until the image reflected from the mercury is found, when both will be in the field together. A little practice will make this process very easy.

In observing stars care must be taken to avoid bringing the direct image of one star in contact with the reflected image of another. Sometimes a small level is attached to the index-arm to facilitate finding the reflected image, and at the same time for preventing mistakes of the kind just mentioned. It may be shown geometrically that when the two images of any star are brought in contact in the manner we have been describing, the angle formed with the

horizon by the index-glass will be equal to that formed with the horizon-glass by the axis of the telescope. As both telescope and horizon-glass are fixed to the frame of the instrument, it is therefore a constant angle. If then the level above mentioned is adjusted so that the bubble will play (the plane of the instrument being vertical) when the index-glass makes this constant angle with the horizon, it may be used for the purpose mentioned. The method of finding the reflected image will then be as follows: Look through the telescope at the image reflected from the mercury; then, holding the instrument in the same position, move the index-arm until the bubble plays. If the reflected image is not then in the field also, the reason will be that the plane of the instrument is not vertical. It will be brought into the field by revolving the instrument back and forth about the axis of the telescope.

To adjust this level, bring the two images of the sun or a known star into the centre of the field and move the tube until the bubble plays.

Errors of the Sextant.

114. Among the various theoretical errors to which sextant observations are liable there are two which call for a detailed investigation, viz., *index error* and *ecentricity.*

To Determine the Index Error. The arc is graduated a short distance backward from the zero-point; when the reading falls on this side of the zero-point the reading is said to be *off arc;* a direct reading being *on arc.*

First Method of Determining the Index Error. By a Star. Direct the telescope to a star, and by means of the tangent-screw bring the direct and reflected images into exact coincidence. The reading of the vernier will then be the index error, and it must be applied as a correction to all angles measured with the instrument.

The correction will be + when the reading is off arc;
The correction will be − when the reading is on arc.

The mean of several readings should always be taken so as to diminish the effect of errors of observation.

Example. The following readings were made with a Pistor & Martins sextant for determining the index correction:

On arc.
45″
60″
70″
70″
75″
60″
30″
75″
70″
65″

Mean of ten readings, $1'\ 2''.0$.

The index correction being I, we have therefore

$$I = -\ 1'\ 2''.0.$$

115. *Second Method. By the Sun.* Measure the apparent diameter of the sun by bringing the direct and reflected images tangent to each other and read the vernier; then bring the opposite limbs into the position of tangency and again read the vernier. If the first reading is on arc, the second will be off arc, and vice versa.

Let $r =$ the reading on arc;
$r' =$ the reading off arc;
$I =$ the index correction;
$S =$ the true diameter of the sun.

Then
$$S = r + I;$$
$$S = r' - I;$$
from which
$$I = \tfrac{1}{2}(r' - r). \qquad (209)$$

When observations are made on the sun for any purpose, the gradual heating up of the instrument sometimes changes the value of the index correction. For this reason some observers determine its value both at the beginning and end of such a series of observations. The following example taken from the *Astronomische Nachrichten*, Band 23, No. 548, will illustrate this, and at the same time the application of formula (209):

First Determination.		Second Determination.	
On arc.	Off arc.	On arc.	Off arc.
32′ 20″	30′ 60″	32′ 5″	31′ 15″
20″	60″	0″	10″
25″	50″	0″	20″
20″	50″	0″	10″
$r = 32' 21''.2$	$r' = 30' 55''$	$r = 32' 1''.2$	$r' = 31' 13''.8$
$I = -43'.1$		$I = -23''.7$	

Eccentricity of the Sextant.

116. As the arc of the sextant is limited and is read by a single vernier, the effect of eccentricity is not eliminated; it should therefore be investigated. This can only be done by comparing the values of angles measured by it with their known values determined in some other way. The angles between terrestrial objects may be measured with a good theodolite, and the same angles measured with the sextant, or, what is better, stars may be used.

In using stars for the purpose we may proceed in either of two ways.

First, by measuring the distances between known stars. The right ascensions and declinations of the stars will be taken from the Nautical Almanac (it will be best to use none except Nautical Almanac stars for the purpose). The posi-

tions of the stars as they seem to us will differ from those given in the Nautical Almanac by the amount of refraction in α and δ. The necessary corrections must be computed by (194), and the apparent distances of the stars by (IV) or $(\mathrm{IV})_1$, Art. 67.

Second, by measuring the altitudes of known stars. The latitude of the place of observation must be known and the true time. Then from (II), Art. 65, the true altitude of the star may be computed, or, if it is very near, the meridian formula (244) may be used. This altitude must be corrected for refraction to make it comparable with that measured by the sextant. Whatever plan is adopted, the angles chosen should be such that the measurements will be distributed with some approach to uniformity over the entire arc of the sextant.

Let n' = the value of the angle given by the instrument;
n = the true value of the same angle;
z = the correction of zero-point for eccentricity.

Then since in the sextant the reading of the arc is double the actual angle passed over by the index-arm, we shall have, from formula ~~208~~, 205.

$$p = [n - (n' - z)] = 2e'' \sin(\tfrac{1}{2}n - \alpha);$$

and for the zero-point, $\qquad z = -2e'' \sin \alpha.$

Subtracting, $\quad n - n' = 2e''[\sin(\tfrac{1}{2}n - \alpha) + \sin \alpha];$
from which $\quad n - n' = 4e'' \sin \tfrac{1}{2}n \cos(\tfrac{1}{2}n - \alpha). \quad . \quad . \quad (210)$

When the constants e'' and α are to be determined from observation, equation (210) must be transformed as follows:

Expanding $\cos(\tfrac{1}{2}n - \alpha)$, the equation becomes

$$(4e'' \cos \alpha) \sin \tfrac{1}{2}n \cos \tfrac{1}{2}n + (4e'' \sin \alpha) \sin^2 \tfrac{1}{2}n = n - n'.$$

Let $\left.\begin{aligned} 4e'' \cos \alpha &= x; \\ 4e'' \sin \alpha &= y; \end{aligned}\right\}$ . . . . (211)

$z =$ the sum of any outstanding constant errors;
$\sin \frac{1}{2}n \cos \frac{1}{2}n = A$, a known coefficient;
$\sin^2 \frac{1}{2}n = B$, a known coefficient;
$n - n' = N$, the quantity given by observation.

Then each measured angle gives us one equation for determining the unknown quantities x, y, and z, viz.,

$$Ax + By + z = N. \quad . \quad . \quad . \quad . \quad . \quad (212)$$

If everything were perfect, three such equations would completely solve the problem. In order to obtain a result of practical value, however, a considerable number of angles must be measured and the resulting equations combined by the method of least squares.

Having determined x and y, we have e'' and α from (211). With these values a table of corrections is then to be computed by (210).

These corrections may be computed for intervals of 10°, from zero up to the largest angles ever measured with the instrument. The correction for any intermediate point may then be taken out by interpolation.

*Example.**

We give as an example the investigation of the eccentricity of sextant "Stackpole 4152," made by Prof. Boss of the U. S. Northern Boundary Survey. The observations were made 1873, August 20, at the U. S. Astronomical Station No. 8.

Latitude $= \varphi = 49° \ 1' \ 2''.4$; determined by zenith telescope.†
Longitude $= L = 1^h \ 41^m \ 18^s$ west of Washington.

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* For a full understanding of the details of this example a knowledge is required of some principles which are explained later. It will be advisable to read Chapter V. before attempting it.

† See Chapter VIII.

Eleven angles were carefully measured, each measurement consisting of ten readings. All except two were measurements of double altitudes of stars. All were north stars except one, viz., α *Aquilæ*, observed on the meridian. The north stars were in most cases observed both before and after meridian passage; by this arrangement any small undetermined error of the time is practically eliminated.

The chronometer correction was determined by measuring the altitudes of α *Bootis* west of the meridian and α *Andromedæ* east, both being observed at exactly the same altitude.*

The two angles which form the exception above referred to were measurements of the distances between α *Andromedæ* and α *Pegasi*, and α *Ursæ Minoris* and γ *Cephei* respectively.

The index correction, determined both at the beginning and end of the series, was as follows:

$$\text{Beginning,} \quad I = -3' \, 43''.$$
$$\text{End,} \quad I = -3' \, 42''.5.$$

The following will serve as a specimen of the form of record and method of reduction. The series of ten readings is divided into two parts so that one may serve as a check on the other.

Double Altitude of α Ursæ Majoris.

	Sextant.	Chronometer.	Sextant.	Chronometer.
	1. 63° 25′ 50″	$19^h\ 12^m\ 21^s$	6. 62° 39′ 45″	$19^h\ 17^m\ 18^s$
	2. 15 50	13 23	7. 29 50	18 22
	3. 63 6 45	14 23	8. 21 10	19 16
	4. 62 57 10	15 21	9. 13 5	20 12
	5. 48 10	16 20	10. 3 55	21 9
Means	63° 6′ 45″	$19^h\ 14^m\ 21^s.6$	62° 21′ 33″	$19^h\ 19^m\ 15^s.4$
Chron. correction		$\Delta T - 22\ \ 50\ .0$		$\Delta T - 22\ \ 50\ .0$
True time $= \theta =$		$18^h\ 51^m\ 31^s.6$		$18^h\ 56^m\ 25^s.4$
From ephemeris, $\alpha =$		10 55 52 .0		10 55 52 .0
Hour-angle $t =$		$7^h\ 55^m\ 39^s.6$		$8^h\ 0^m\ 33^s.4$
" $t =$		118° 54′ 54″		120° 8′ 21″

The true altitude of the star at the instant of observation is then computed by formulæ (II), Art. 65:

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* See Articles 125, 126, and 127.

$\varphi = 49°\ 01'\ 2''.4$		
$^*\ \delta = 62°\ 26'\ 11''.3$	$\tan\delta = 0.282349$	
$t = 118°\ 54'\ 54''.0$	$\cos t = 9.684407_n$	$\tan t = 0.257769_n$
$M = 75°\ 50'\ 7''.4$	$\tan M = 0.597942_n$	$\cos M = 9.388649$
$\varphi - M = 124°\ 51'\ 9''.8$		$\operatorname{cosec}(\varphi - M) = 0.085856$
$a = 151°\ 38'\ 15''.2$		$\tan a = 9.732274_n$
$h = 31°\ 29'\ 58''.3$		
Refraction $r = 1'\ 30''.4$		Proof 9.474505
$h' = 31°\ 31'\ 28''.7$		
$2h' = 63°\ 2'\ 57''.4$		$\cos\delta = 9.665329$
Index Cor. $I = 3'\ 43''.0$		$\cos t = 9.684407_n$
Computed $n = 63°\ 6'\ 40''.4$		
Measured $n' = 63°\ 6'\ 45''.0$		9.349736_n
	$\tan(\varphi - M) = 0.157152_n$	
$n - n' = -4''.6$	$\cos a = 9.944463_n$	$\cos a = 9.944463_n$
	$\tan h = 9.787311$	$\cos h = 9.930768$
		9.875231_n
		Proof 9.474505
$\varphi = 49°\ 01'\ 02''.$		
$^*\delta = 62°\ 26'\ 11''.3$	$\tan\delta = 0.282349$	
$t = 120°\ 8'\ 21''.0$	$\cos t = 9.700792_n$	$\tan t = 0.236128_n$
$M = 75°\ 18'\ 50''.1$	$\tan M = 0.581557_n$	$\cos M = 9.404017$
$\varphi - M = 124°\ 19'\ 52''.5$		$\operatorname{cosec}(\varphi - M) = 0.083130$
$a = 152°\ 7'\ 51''.6$		$\tan a = 9.723275_n$
$h = 31°\ 7'\ 16''.5$		
$r = 1'\ 31''.7$		Proof 9.487147
$h' = 31°\ 8'\ 48''.2$		
$2h' = 62°\ 17'\ 36''.4$		$\cos\delta = 9.665329$
Index Cor. $I = 3'\ 43''.0$		$\cos t = 9.700792_n$
Computed $n = 62°\ 21'\ 19''.4$		
Measured $n' = 62°\ 21'\ 33''.0$		9.366121_n
	$\tan = 0.165609_n$	
$n - n' = -13''.6$	$\cos a = 9.946462$	$\cos a = 9.946462_n$
	$\tan h = 9.780853$	$\cos h = 9.932512$
		9.878974_n
		Proof 9.487147

Mean $= N = -9''.1$.

The computation for determining the true angular distance between

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* The declination, δ, is taken from the ephemeris.

α Andromedæ and *α Pegasi* is also given in full. We take from the ephemeris for 1873, August 20—

$$\alpha\ Andromedæ:\ \alpha = 0^h\ 1^m\ 51^s.78 \qquad \alpha\ Pegasi:\ \alpha = 22^h\ 58^m\ 28^s.50$$
$$\delta = 28°\ 23'\ 30''.8 \qquad \delta = 14°\ 31'\ 33''.2$$

The observed distance was 20° 15′ 20″.5

Chronometer time $20^h\ 26^m\ 3^s.6$.

Refraction factor $B \times t \times T = .960$. [See Eq. (187).]

We first determine q and s by equations (XII); then the refraction in right ascension and declination by (194).

α Andromedæ.

$T = 20^h\ 26^m\ 3^s.6$			
$\Delta T = -\ 22\ 50.$			
$\theta = 20\ 3\ 13.6$			
$\alpha = 0\ 1\ 51.8$			
$t = -\ 3^h\ 58^m\ 38^s.2$			
$t = -59°\ 39'\ 33''$	$\cos t = 9.70341$	$\tan t = 0.23262_n$	$\cos t = 9.70341$
$\varphi = 49\ 1\ 2.4$	$\cot \varphi = 9.93890$		$\cos \varphi = 9.81679$
$N = 23\ 41\ 39$	$\tan N = 9.64231$	$\sin N = 9.60407$	9.52020
$\delta = 28\ 23\ 31$			
$\delta + N = 52\ 5\ 10$	$\sec(\delta + N) = .21150$	$\cot = 9.89147$	
$q = -48\ 10\ 21$	$\tan q = .04819_n$	$\cos q = 9.82405$	$\cos q = 9.82405$
$s = 49\ 25\ 46$		$\tan s = .06742$	$\sin s = 9.88059$
			9.70464
	9.81557	—Proof—	9.81556

From table, mean refraction = 68″.1

Factor = .960 Therefore $r = 65''.4$

α Pegasi.

$T = 20^h\ 26^m\ 3^s.6$			
$\Delta T = -\ 22\ 50.$			
$\theta = 20\ 3\ 13.6$			
$\alpha = 22\ 58\ 28.5$			
$t = -\ 2^h\ 55^m\ 14^s.9$			
$t = -43°\ 48'\ 44''$	$\cos t = 9.85830$	$\tan t = 9.98199_n$	$\cos t = 9.85830$
$\varphi = 49\ 1\ 2.4$	$\cot \varphi = 9.93890$		$\cos \varphi = 9.81679$
$N = 32\ 5\ 2$	$\tan N = 9.79720$	$\sin N = 9.72523$	9.67509
$\delta = 14\ 31\ 33$			
$\delta + N = 46\ 36\ 35$	$\sec(\delta + N) = .16307$	$\cot = 9.97558$	
$q = -36\ 34\ 5$	$\tan q = 9.87029_n$	$\cos = 9.90479$	$\cos q = 9.90479$
$s = 49\ 39\ 0$		$\tan s = .07079$	$\sin s = 9.88201$
			9.78680
	9.88830	—Proof—	9.88829

Mean refraction $= 68''.6$
Factor $= .960$ Therefore $r = 65''.9$

By (194)—

α. Andromedæ.		α. Pegasi.	
$\cos q = 9.82405$		$\cos q = 9.90479$	
$\log r = 1.81558$		$\log r = 1.81889$	
$\sin q = 9.87225_n$		$\sin q = 9.77508_n$	
$\log d\delta = 1.63963_n$	$d\delta = -\ 43''.6$	$\log d\delta = 1.72368_n$	$d\delta = -\ 52''.9$
$\cos \delta d\alpha = 1.68783$	$\delta_0 = 28\ 23\ 30\ .8$	$\cos \delta d\alpha = 1.59397$	$\delta_0 = 14\ 31\ 33\ .2$
$15 \cos \delta = 1.12043$	$\delta = 28\ 24\ 14\ .4$	$15 \cos \delta = 1.16198$	$\delta = 14\ 32\ 26\ .1$
$\log d\alpha = .56740$	$d\alpha = +\ 3\ .69$	$\log d\alpha = .43199$	$d\alpha = +\ 2\ .70$
	$\alpha_0 = 0\ 1\ 51\ .78$		$\alpha_0 = 22\ 58\ 28\ .50$
	$\alpha = 0^h\ 1^m\ 48^s.09$		$\alpha = 22^h 58^m 25^s.80$

These values of the right ascensions and declinations of the stars are the ones to be employed in computing the apparent distance between the two stars by equations $(IV)_1$.

$\alpha' = 24^h\ 1^m\ 48^s.09$			
$\alpha = 22\ 58\ 25\ .80$			
$\alpha' - \alpha = 1^h\ 3^m\ 22^s.29$			
$\alpha' - \alpha = 15°\ 50'\ 34\ .35$	$\cos(\alpha' - \alpha) = 9.983181$	$\tan(\alpha' - \alpha) =$	9.452982
$\delta = 14\ 32\ 26\ .1$	$\cot \delta = .586075$	$\sin N =$	9.984762
$N = 74\ 54\ 39\ .6$	$\tan N = .569256$	$\sec(N + \delta') =$	$.637698_n$
$\delta' = 28\ 24\ 14\ .4$		$\tan B =$	$.075442_n$
$N + \delta' = 103\ 18\ 54\ .0$	$\cot(N + \delta') = 9.374136_n$		
$B = -\ 49\ 57\ 5\ .9$	$\cos B = 9.808504$	Proof	$.622460$
$d = 20\ 11\ 39\ .8$	$\tan d = 9.565632$		
$I = 3\ 43\ .$		$\cos(\alpha' - \alpha) =$	9.983181
$n = 20\ 15\ 22\ .8$		$\cos \delta =$	9.985862
$n' = 20\ 15\ 20\ .5$			9.969043
$n - n' = +\ 2''.3 = N.$			
		$\cos B =$	9.808504
		$\sin d =$	9.538079
			9.346583
		Proof	$.622460$

The value of N obtained by the original computation, and which is employed in our equations, is $2''.2$. The difference is of no importance here.

N is now the absolute term of equation (212). For the coefficients $A = \sin \frac{1}{2}n$, $\cos \frac{1}{2}n$, and $B = \sin^2 \frac{1}{2}n$ we must employ for n not the above angles, but the angle corresponding to the point on the limb which coincides with the vernier-scale. For example, the first measured angle of the first series is $63°\ 25'\ 50''$. The limb

was graduated directly to 10′; these intervals were subdivided by the vernier to 10″. The zero-point of the vernier falls between 63° 20′ and 63° 30′; then reading along the vernier to the point where coincidence takes place, we find this to be at the reading 69° 10′ of the limb. It is therefore the eccentricity of this point by which our angle is affected, and not that of the point 63° 25′ +.

In this way we find the point of contact for each reading of our series as follows:

63° 25′ 50″	Point of contact = 69° 10′		
15′ 50″	= 69° 00′		
6′ 45″	= 69° 45′		
62° 57′ 10″	= 70° 00′		
48′ 10″	= 70° 50′		
39′ 45″	= 72° 15′		
29′ 50″	= 72° 10′	$\frac{1}{4}n = 17^\circ\ 11'\frac{3}{4}$	$l \sin = 9.47075$
21′ 10″	= 63° 30′		$l \cos = 9.98014$
13′ 5″	= 65° 15′	$A = 0.2824$	$\log A = 9.45089$
62° 3′ 55″	= 65° 55′	$B = 0.0874$	$\log B = 8.94150$
	Mean = n = 68° 47′		

Therefore from this series we derive the equation

$$0.2824x + 0.0874y + z = -\ 9''.1.$$

By proceeding in a similar manner with each of the eleven angles measured, the following equations of condition are obtained:

$$\begin{aligned}
.0703x + .0050y + z &= -\ \ 5.5;\\
.1104x + .0123y + z &= +\ \ 2.2;\\
.2019x + .0425y + z &= -\ \ 7.3;\\
.2341x + .0582y + z &= -\ 17.5;\\
.2824x + .0874y + z &= -\ \ 9.1;\\
.3295x + .1239y + z &= -\ 18.5;\\
.3586x + .1515y + z &= -\ 10.5;\\
.3933x + .1913y + z &= -\ 14.0;\\
.3997x + .1996y + z &= -\ 24.0;\\
.4244x + .2357y + z &= -\ 46.2;\\
.4423x + .2668y + z &= -\ 28.6.
\end{aligned}$$

It will be seen that the coefficients of x and y are much smaller throughout than those of z, while the absolute terms are relatively large. It would therefore be a little more systematic to render the equations homogeneous, as ex-

plained in Art. 24, before forming the normal equations. This has not been done, however.

The details of the formation of the normal equations (Articles 21 and 25) are as follows: As the number of unknown quantities is three, we rule our sheet into $\frac{(3+2)(3+3)}{2} - 1 = 14$ vertical columns (Art. 25), to which we have added two columns for the residuals (v) and their squares (vv). These will be filled in after the unknown quantities have been determined.

No.	*ac*	*bc*	*cc*	*cn*	*cs*	*aa*	*ab*	*an*
1	.0703	.0050	1.	− 5.5	6.5753	.00494	.00035	− .3867
2	.1104	.0123	1.	+ 2.2	− 1.0773	.01218	.00136	+ .2429
3	.2019	.0425	1.	− 7.3	8.5444	.04074	.00859	− 1.4739
4	.2341	.0582	1.	− 17.5	18.7923	.05480	.01362	− 4.0967
5	.2824	.0874	1.	− 9.1	10.4698	.07976	.02468	− 2.5698
6	.3295	.1239	1.	− 18.5	19.9534	.10859	.04084	− 6.0957
7	.3586	.1515	1.	− 10.5	12.0101	.12854	.05432	− 3.7653
8	.3933	.1913	1.	− 14.0	15.5846	.15467	.07522	− 5.5062
9	.3997	.1996	1.	− 24.0	25.5993	.15974	.07976	− 9.5928
10	.4244	.2357	1.	− 46.2	47.8601	.18014	.10002	− 19.6073
11	.4423	.2668	1.	− 28.6	30.3091	.19561	.11801	− 12.6498
	3.2469 [*ac*]	1.3742 [*bc*]	11.0 [*cc*]	− 179.0 [*cn*]	194.6211 [*cs*]	1.11971 [*aa*]	.51677 [*ab*]	− 65.5013 [*an*]

No.	*as*	*bb*	*bn*	*bs*	*nn*	*ns*	*v*	*vv*
1	+ .4622	.00002	− .0275	+ .0329	30.25	− 36.16	+ 1.7	2.89
2	− .1189	.00015	+ .0271	− .0133	4.84	− 2.37	− 6.1	37.21
3	+ 1.7251	.00181	− .3103	+ .3631	53.29	− 62.37	+ 1.0	1.00
4	4.3993	.00339	− 1.0185	1.0937	306.25	− 328.87	+ 9.7	94.09
5	2.9567	.00764	− .7953	.9151	82.81	− 95.28	− 1.9	3.61
6	6.5746	.01536	− 2.2922	2.4722	342.25	− 369.14	+ 3.2	10.24
7	4.3068	.02296	− 1.5907	1.8195	110.25	− 126.11	− 8.2	67.24
8	6.1294	.03657	− 2.6782	2.9813	196.00	− 218.18	− 9.8	96.04
9	10.2319	.03982	− 4.7904	5.1096	576.00	− 614.38	− 0.9	0.81
10	20.3118	.05554	− 10.8893	11.2806	2134.44	− 2211.14	+ 16.6	275.56
11	13.4057	.07118	− 7.6305	8.0865	817.96	− 866.84	− 5.2	27.04
	70.3846 [*as*]	.25444 [*bb*]	− 31.9958 [*bn*]	34.1412 [*bs*]	4654.34 [*nn*]	− 4930.84 [*ns*]		615.73 [*vv*]

The correctness of the work up to this point is now verified by substitution in proof-formulæ (44).

Therefore the normal equations are as follows:

$$\begin{aligned}
1.1197x + .5168y + 3.2469z &= - 65.5013; \\
.5168x + .2544y + 1.3742z &= - 31.9958; \\
3.2469x + 1.3742y + 11.0000z &= - 179.0000.
\end{aligned}$$

For the solution of these equations we make use of the form given in Art. 32.

$[aa]$ 1.1197 $l =$ 0.049102	$[ab]$ .5168 $l =$ 9.713323	$[ac] =$ 3.2469 $l =$ 0.511469	$[an]$ −65.5013 $l =$ 1.816250$_n$	$[as]$ 70.3846 $l =$ 1.847478	E
$l\frac{[ab]}{[aa]} =$ 9.664221	$[bb]$ .2544 .2385	$[bc]$ 1.3742 1.4986	$[bn]$ −31.9958 −30.2323	$[bs]$ 34.1412 32.4862	
	$[bb\ 1]$ .0159 $l =$ 8.20140	$[bc\ 1]$ −.1244 $l =$ 9.09482$_n$	$[bn\ 1]$ − 1.7635 $l =$ 0.24638$_n$	$[bs\ 1]$ 1.6550 $l =$ 0.21880	I' E
$l\frac{[ac]}{[aa]} =$ 0.462367		$[cc] =$ 11.0000 9.4153	$[cn]$ −179.0000 −189.9403	$[cs]$ 194.6211 204.1009	
$l\frac{[bc\ 1]}{[bb\ 1]} =$ 0.89342$_n$		$[cc\ 1] =$ 1.5847 9733	$[cn\ 1] =$ 10.9403 13.7974	$[cs\ 1] =$ − 9.4798 −12.9485	II
		$[cc\ 2] =$ .6114 $l =$ 9.78633	$[cn\ 2]$ − 2.8571 $l =$ 0.45592$_n$	$[cs\ 2]$ + 3.4687	III'
		$lz =$ 0.66959$_n$	$z = -$ 4''.673		
$l\frac{[an]}{[aa]} =$ 1.767148	$[nn] =$ 4654.34 3831.76	$[ns] =$ −4930.84 −4117.43			
$l\frac{[bn\ 1]}{[bb\ 1]} =$ 2.04498$_n$	$[nn\ 1] =$ 822.58 195.59	$[ns\ 1]$ − 813.41 − 183.56	VII		
$l\frac{[cn\ 2]}{[cc\ 2]} =$ 0.66959$_n$	$[nn\ 2] =$ 626.99 13.35	$[ns\ 2]$ − 629.85 − 16.21	VIII		
	$[nn\ 3] =$ 613.64	$[ns\ 3]$ − 613.64	IX		

Proof-Formulæ.

I'. $[bs\ 1] = [bb\ 1] + [bc\ 1] - [bn\ 1]$;
II. $[cs\ 1] = [bc\ 1] + [cc\ 1] - [cn\ 1]$;
III'. $[cs\ 2] = [cc\ 2] - [cn\ 2]$;
VII. $[ns\ 1] = [bn\ 1] + [cn\ 1] - [nn\ 1]$;
VIII. $[ns\ 2] = [cn\ 2] - [nn\ 2]$;
IX. $[ns\ 3] = -[nn\ 3]$.

The work is checked at the various stages by substitution in any or all of the above proof-formulæ.

The *elimination equations* (56) are here rewritten for convenience:

$$[aa]x + [ab]y + [ac]z = [an];$$
$$[bb\ 1]y + [bc\ 1]z = [bn\ 1].$$

By substituting in these the coefficients, the logarithms of which are in the horizontal lines marked E in the foregoing scheme, we find

$$y = -\ 147''.47; \qquad x = +\ 23''.12.$$

These values substituted in the equations of condition give the residuals v. For the final proof of the accuracy of the entire computation we have, Eq. (62),

$$[nn\ 3] = [vv].$$

The agreement, though not exact, is sufficiently close for our purpose, and as close as could be expected when the magnitude of some of the numerical quantities involved in the equations is considered.

For determining the weights of x, y, and z we employ equations (76), by means of which we find

$$p_z = .6114; \qquad p_y = .006135; \qquad p_x = .01196.$$

The mean error of an observation we obtain by formula (88), viz.,

$$\epsilon = \pm \sqrt{\frac{[vv]}{m-3}} = 8''.7725.$$

The mean errors of x, y, and z are then given by equations (89):

$$\epsilon_x = \frac{\epsilon}{\sqrt{p_x}} = 80''.21; \qquad \epsilon_y = \frac{\epsilon}{\sqrt{p_y}} = 112''.00; \qquad \epsilon_z = \frac{\epsilon}{\sqrt{p_z}} = 11''.22.$$

These quantities multiplied by .6745 give the probable errors.

Collecting our results, we have the following values of x, y, z, with their probable errors:

$$x = + \ 23''.1 \pm 52''.9;$$
$$y = - \ 147''.5 \pm 75''.5;$$
$$z = - \ 4''.7 \pm 7''.6.$$

We next compute a table of corrections, to be employed with this instrument, by formulæ (211) and (210), viz.:

$$4e'' \cos \alpha = x;$$
$$4e'' \sin \alpha = y;$$
$$n - n' = 4e'' \sin \tfrac{1}{2}n \cos (\tfrac{1}{2}n - \alpha).$$

We find $\qquad 4e'' = 149''.3; \qquad \alpha = - 81° \ 6'.$

Substituting for n successively 10°, 20°, etc., we have the following table of corrections:

Angle.	Correction.	Angle.	Correction.
0°	0''.0	80°	− 9''.8
10°	+ 0''.7	90°	− 13''.4
20°	+ 0''.9	100°	− 17''.5
30°	+ 0''.5	110°	− 22''.0
40°	− 0''.5	120°	− 26''.9
50°	− 2''.0	130°	− 32''.1
60°	− 4''.1	140°	− 37''.7
70°	− 6''.7		

Other Theoretical Errors.

117. In a complete theoretical discussion of the sextant there are several other sources of error which require consideration. The more important of these are the following: *prismatic form of the index-glass, of the colored glass shades, and of the horizon-roof; want of perpendicularity of the planes of the index and horizon glass to the plane of the instrument; inclination of line of collimation of telescope to plane of instrument; errors of graduation of the limb.*

With a good instrument well adjusted the effect of any one of these will be small, although they may combine together in such a way as to produce a very appreciable effect on the value of a measured angle. Not much can be gained, however, practically by investigating in detail the forms of the corrections required. The experienced observer will avoid these errors as far as can be done by careful adjustment, and then will arrange his observations with a view to eliminating from the results such of them as remain undetermined. See Art. 127.

The Chronometer.

118. The chronometer is simply a watch made with special care, and in which the balance-wheel is so constructed that changes of temperature will produce the least possible effect on its time of oscillation. The test of a good chronometer is the uniformity of its rate from day to day. It is impossible to make an instrument so perfect that 24^h as shown by it shall exactly correspond to one day, but its excellence is indicated by the uniformity with which it gains or loses.

The daily rate of a chronometer is the amount which it gains or loses in 24 hours.

The error of the chronometer is the difference between the time as shown by the face of the instrument and the true time.

The chronometer correction is the amount which must be added to the reading of the chronometer-face at any instant to give the true time; it is equal to the error with its sign changed.

It is a convenience to have the error and rate small, but it

is not essential. Chronometers are made in two different forms, viz., box-chronometers and pocket-chronometers. The first form of instrument is generally suspended by means of gimbals in a wooden box, in such a manner that, whatever the position of the box, the face of the instrument will maintain a horizontal position. This arrangement is useful at sea, but for transportation on land the instrument must be securely fastened, as otherwise the violent agitation produced by sudden shocks would be injurious. The balance-wheel of this form of instrument oscillates at half-second intervals.

The pocket-chronometer is generally somewhat larger than an ordinary watch. The oscillation or beat is a little more rapid than with the box-chronometer; thus the pocket-instruments of T. S. and J. D. Negus beat five times in two seconds.

A chronometer regulated to sidereal time is more convenient for observation on stars. With the sun a mean time chronometer is preferable.

The error and rate will be considered more fully in connection with the subject of determining time. Most chronometers require winding every 24 hours. This should be done at about the same time each day, as if they are allowed to run much longer than the usual time a different part of the spring comes into action, which may affect the rate. Such instruments will run for 48^h or more before stopping, so that in case the winding should be neglected for one day they will be found running the next; but for the reason just stated this should not occur.

Comparison of Chronometers.

119. When the errors of several chronometers are to be determined at the same time, the error of one of them is ob-

tained by observation, and of the others by comparison with this. When two sidereal or two mean solar chronometers are compared together the beats will be sensibly of the same length, but generally the two will not beat exactly together; the fraction of a second by which the beat of one falls behind that of the other must therefore be estimated. With some practice this can be done so that the error in the estimation will not much exceed $0^s.1$.

When a sidereal is to be compared with a mean time chronometer the error of comparison will be much smaller. Since 1^s of sidereal time is equal to $0^s.99727$ mean solar time, it follows that the sidereal gains $0^s.00273$ on the mean time chronometer in one second; this gain will amount to one entire beat, or $0^s.5$, in 183^s, or approximately 3^m. Therefore practically once every three minutes the beat of the two will coincide. It is found that with a little practice the ear can detect a discordance in the beats as long as they differ by $0^s.02$ or $0^s.03$, and therefore the comparison can be made within this limit of error.

When a number of chronometers are to be compared with a standard clock, it may be done very conveniently by means of the chronograph.* The clock being connected with the chronograph, the observer taps the signal-key in coincidence with one or more even beats of the chronometer, and thus the time by both clock and chronometer are recorded on the same sheet.

The Astronomical Clock.

120. In a fixed observatory the clock is an instrument of great importance. It is generally regulated for sidereal time. The only part of the mechanism which requires notice

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* See Art. 121.

here is the pendulum, which is made of the necessary length to beat seconds.

The rate of the clock depends upon the length of the pendulum; and since a rod of metal changes its length with every change of temperature, some method of compensation is necessary in order to keep the centre of oscillation at a constant distance from the point of suspension. For accomplishing this two different forms are used, viz., the *gridiron* and the *mercurial* pendulum.

In the gridiron pendulum the rod is composed of a number of parallel bars, alternately of brass and steel. These are so arranged that the expansion of the steel bars tends to *increase* the length, while that of the brass bars tends to diminish it. As these metals expand and contract by different amounts when subjected to changes of temperature, the relative lengths of the two may be so adjusted as to maintain a constant length for the system.

With the mercurial pendulum the rod consists of a single bar of steel. The "bob" is a cylindrical vessel of glass or metal filled with mercury. The expansion of the rod depresses the centre of oscillation, while that of the mercury raises it. Thus by making the cylinder of proper proportions, as compared with the rod, the necessary compensation is effected.

With a clock which is exposed to sudden changes of temperature the gridiron pendulum will give a more uniform rate than the mercurial, as the comparatively thin bars of metal will accommodate themselves to the temperature of the air much sooner than the comparatively large mass of mercury.

The density of the air as indicated by the barometer also affects the rate of the clock by its variable resistance to the motion of the pendulum. Struve found for the standard clock of the Poulkova observatory a change of $0^s.32$ in rate

for a variation of one inch in the barometer. It is therefore very important to protect the standard clock from sudden and extreme atmospheric changes. In some observatories this is done by placing it in an air-tight compartment below the surface of the ground.

The Chronograph.

121. The chronograph is used in connection with the clock for registering graphically on a strip or sheet of paper the beats of the latter. Fig. 22 shows a common form of this instrument. The sheet of paper on which the record is to be made is wrapped around the cylinder, which in this instrument is 14 inches long and 6 or 7 inches in diameter. The cylinder is given one revolution per minute by means of the clockwork. The pen which is shown above the cylinder is supplied with aniline ink, and being moved slowly along in the direction of the axis of the cylinder it traces a continuous spiral on the surface.

The apparatus is placed in an electric circuit passing through the clock, and so arranged that the pendulum breaks the circuit for an instant at the beginning of each second.* By means of a spring which acts in the direction contrary to that of the electro-magnet shown in the figure, the pen is thus given a slight lateral motion at each beat of the clock, producing instead of a continuous line a line graduated as shown in the folding plate, Fig. *22a*.

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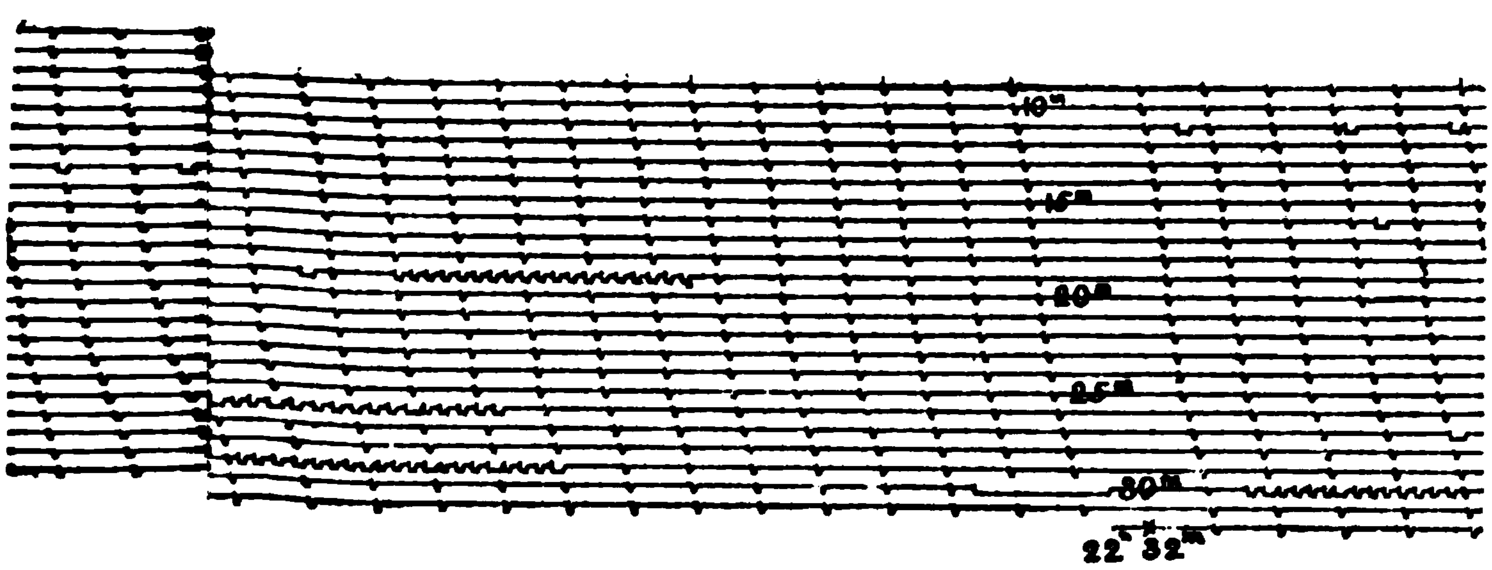
* The arrangement may be such that the circuit will be closed for an instant at the beginning of each second, remaining open during the remainder. The break-circuit plan is the one more commonly employed. Various mechanical devices are employed by different makers for causing the clock to open or close the circuit.

FIG. 22.—The Chronograph.


10^m
15^m
20^m
25^m
30^m
$22^h\ 32^m$

Each of these spaces is the graphic record of one second of time as shown by the clock. The beginning of the minute is marked by the omission of one of the points. The instrument here shown will run 2½ hours. When the paper is removed from the cylinder and spread out it is marked with parallel lines, each line being the record of one minute of clock time.

In order to make use of this apparatus for recording the time of the occurrence of any phenomenon, the wire which forms the circuit, passing from the battery through the clock and chronograph, is made to pass through a signal-key held in the hand of the observer, and by means of which the circuit can be instantly broken.

In Fig. 23, *aa'* is the wire through which the circuit passes. When the point *b* touches the metallic plate *c* the circuit is closed. A key is so arranged that by tapping it with the finger this point is raised and the circuit broken; this produces a mark on the chronograph-sheet similar to that made by the clock, and the position of which is the record of the instant when the key was pressed.

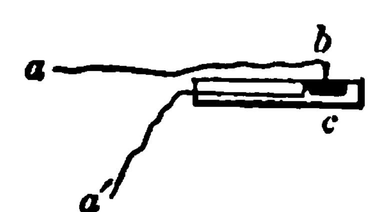


FIG. 23.

Fig. 22*a* is a reduced copy of the chronograph record of transits of the stars θ *Aquarii*, γ *Aquarii*, π *Aquarii*, σ *Aquarii*, α *Lacertæ*, and η *Aquarii* observed with the transit-circle of the Washington observatory, 1884, December 7.

Each star is observed over eleven threads.* The record begins by striking the signal-key several times in quick succession before the star reaches the first thread, in order to mark the beginning of the series; then it is tapped in exact coincidence with the star's passage over each thread in succession.

* See Art. 170.

Taking the first of the above stars, θ *Aquarii*, our chronograph-sheet gives the following record:

$22^h\ 10^m\ 33^s.4$	$22^h\ 10^m\ 47^s.9$
$36^s.0$	$50^s.0$
$37^s.6$	$54^s.1$
$41^s.7$	$55^s.7$
$43^s.8$	$22^h\ 10^m\ 58^s.3$
$22^h\ 10^m\ 45^s.8$	

For reading the record a scale long enough to reach the entire length of the sheet is used, the spaces of which are the same as those of the sheet. These spaces are numbered continuously from 0 up to 60; each space being divided to tenths, the fractional parts of these subdivisions may be estimated.

While the paper is on the cylinder it is necessary to mark somewhere on the sheet the hour and minute shown by the clock; this serves as a starting-point for reading the record.

For the purpose of determining longitude, chronometers are sometimes provided with a break-circuit attachment, when they can be used with a chronograph in the same manner as a clock.

The main advantages which the chronograph possesses over the methods employed before its introduction are, *first*, a comparatively inexperienced observer can record astronomical phenomena by its use with a degree of accuracy which it would take months or perhaps years of practice to acquire without it; and *second*, the record is made by simply pressing a key with the finger: thus many more observations can be made in a given time than is possible when everything must be written down with a pencil.

CHAPTER V.

DETERMINATION OF TIME AND LATITUDE.—METHODS ADAPTED TO THE USE OF THE SEXTANT.*

122. In a spherical triangle, when three parts are known any other part may be determined. Let us consider the triangle *PZS*, where *P* is the pole of the heavens, *Z* the observer's zenith, and *S* a known star (the word star here including the sun, moon, or a planet).

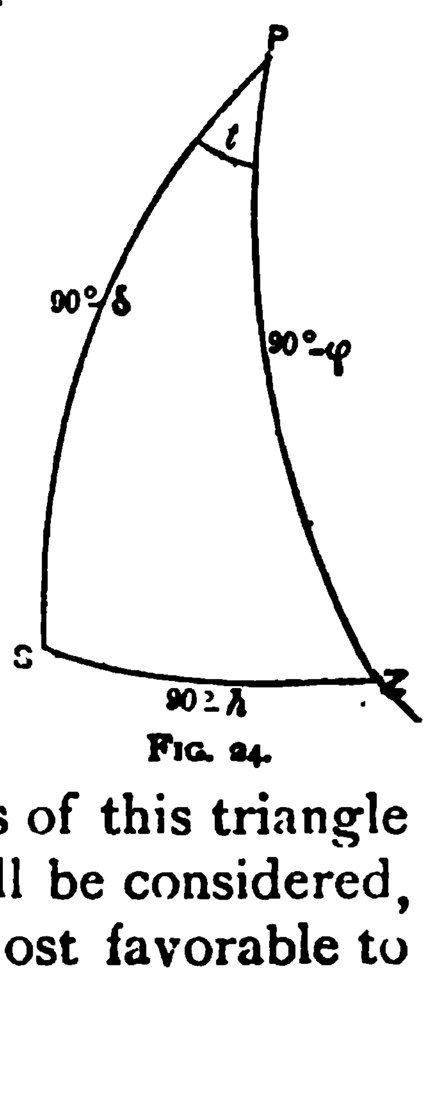


FIG. 24.

If we measure the altitude of *S*, the side *SZ* of our triangle is known. The declination δ is taken from the Nautical Almanac. If then we know the hour-angle t, we have the data for determining the latitude φ. If φ is known, we have the hour-angle t by computation, and therefore the true local time, from (197).

We have then simply to give the solutions of this triangle best adapted to the different cases which will be considered, and to determine what conditions will be most favorable to accuracy.

Determination of Time.

123. *By a single altitude of the sun.*

Let h' = the observed altitude of the sun's limb, corrected for index error;

* The methods of this chapter are of course equally adapted to the use of any instrument for measuring altitudes.

h = the true altitude of the sun's centre;
z = the true zenith distance of the sun's centre $= 90° - h$;
r = the correction for refraction;
p = the correction for parallax;
s = the correction for semidiameter.

Then $$h = h' - r + p \pm s. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (213)$$

s is $\pm$ when the $\left\{ \begin{matrix} \text{lower} \\ \text{upper} \end{matrix} \right\}$ limb is observed.

The required solution of the triangle may now be deduced from the last of equations (121), viz.,

$$\cos z = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t;$$

from which

$$\cos t = \frac{\cos z - \sin \varphi \sin \delta}{\cos \varphi \cos \delta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (214)$$

In some cases this equation may be conveniently employed for computing t, as when the same star is observed on several successive days at the same place. $\sin \varphi \sin \delta$ and $\cos \varphi \cos \delta$ may then be considered constant for a week or more in ordinary sextant work. The numerator will be computed with addition and subtraction logarithms.

As t is given in terms of the cosine, this equation should not be used when the angle is less than 45°.

124. To place (214) in a form more generally applicable, first subtract both members from unity, then add both members to unity, viz.:

$$1 - \cos t = \frac{\cos \varphi \cos \delta + \sin \varphi \sin \delta - \cos z}{\cos \varphi \cos \delta};$$

$$1 + \cos t = \frac{\cos \varphi \cos \delta - \sin \varphi \sin \delta + \cos z}{\cos \varphi \cos \delta};$$

from which we easily obtain

$$\sin \tfrac{1}{2}t = \sqrt{\frac{\sin \frac{1}{2}[z + (\varphi - \delta)] \sin \frac{1}{2}[z - (\varphi - \delta)]}{\cos \varphi \cos \delta}}; \quad (215)$$

$$\cos \tfrac{1}{2}t = \sqrt{\frac{\cos \frac{1}{2}[z + (\varphi + \delta)] \cos \frac{1}{2}[z - (\varphi + \delta)]}{\cos \varphi \cos \delta}}; \quad (216)$$

$$\tan \tfrac{1}{2}t = \sqrt{\frac{\sin \frac{1}{2}[z + (\varphi - \delta)] \sin \frac{1}{2}[z - (\varphi - \delta)]}{\cos \frac{1}{2}[z + (\varphi + \delta)] \cos \frac{1}{2}[z - (\varphi + \delta)]}}. \quad (217)$$

For most purposes equation (215) will give the necessary degree of precision.

When the extremest accuracy is required (217) should be used.

These equations give t in degrees, minutes, and seconds of arc. For our purposes it must be reduced to time by dividing by 15.

Then let T_0 = the chronometer time of observation;
ΔT = the chronometer correction;
E = the equation of time.

Then the apparent time of observation is t (Art. 90).

$$\left.\begin{array}{ll}\text{Mean time of observation} & = t + E = T_0 + \Delta T; \\ \text{from which} & \Delta T = t + E - T_0.\end{array}\right\} \quad (218)$$

ΔT is the quantity required.

In the above, where the object observed was the sun, we have supposed the chronometer to be regulated to mean time. If a sidereal chronometer has been used, the mean time $(t + E)$ must be converted into sidereal time by (200) or (201) and the resulting value compared with the chronometer time.

Example 1. **West Las Animas,**

Observation of sun for time.

	Sextant.	Chronometer.
	$\underline{\odot}$ 88° 50′ 00″	$3^h\ 35^m\ 12^s.$
	89 00 0	35 39 .5
	10 0	36 3 .5
	20 0	36 30 .5
	89 30 0	36 56 .5
	$\overline{\odot}$ 88° 50′ 0″	$3^h\ 37^m\ 55^s.5$
	89 0 0	38 22 .
	10 0	38 48 .
	20 0	39 14 .5
	89 30 0	39 41 .0
Means	89° 10′ 0″	$3^h\ 37^m\ 26^s.3$
I	− 11	
Eccentricity	− 45	
$2A =$	89° 9′ 4″	
$A =$	44 34 32	
Refraction $r =$	− 49	
Parallax $p =$	6	
$h =$	44° 33′ 49″	
$z =$	45 26 11	= zenith distance of sun's centre.

We have now the data for applying formulæ (215) and (218).

			Δ*
$\varphi =$	38° 4′ 0″	sec = 0.10386	9.9
$\delta =$	18 42 17	sec = .02357	4.3
$\varphi - \delta =$	19 21 43		
$z =$	45 26 11		
$z + (\varphi - \delta) =$	64 47 54		
$z - (\varphi - \delta) =$	26 4 28		
$\frac{1}{2}[z + (\varphi - \delta)] =$	32 23 57	$= S$. sin = 9.72901	19.9
$\frac{1}{2}[z - (\varphi - \delta)] =$	13 2 14	$= D$. sin = 9.35331	54.6
		$\sin^2 \frac{1}{2}t =$ 9.20975	
$\frac{1}{2}t =$	23° 44′ 28″	$\sin \frac{1}{2}t =$ 9.60487	28.7
$t =$	47 28 56		
$t =$	$- 3^h\ 9^m\ 55^s.7$		
$t =$	20 50 4 .3		
$E =$	+ 6 13 .0		
$t + E =$	20 56 17 .3	= mean solar time.	
$T =$	3 37 26 .3	= observed time.	
$\Delta T =$	− 6 41 9 .0	= chron. correction [Eq. (218)].	

This value differs but little from the value assumed above. If the difference had been large it would have been necessary to take from the ephemeris the value of δ for this more correct time, and to repeat the computation for a more correct value of ΔT. Or, if the difference were not too great, the necessary correction could be determined by a differential formula.

* These values are written down for the purpose of computing the differential formulæ in case it is thought desirable. See Articles 128–131.

Colorado, 1878, July 28.9.

Mean solar chronometer. Observer B.
Negus 1326.

Thermometer 78°.
Barometer 26.05

Latitude $\varphi = 38° \; 4' \; 0''$
Longitude $L = 1^h \; 44^m \; 41^s$ w. of Washington.
Assumed $\varDelta T = -6 \; 41 \; 7$

INDEX CORRECTION.

On Arc.	Off Arc.
31′ 50″	359° 28′ 45″
31 30	28 40
31 40	28 40
31′ 40″	359° 28′ 42″

Index correction $= I = -11''$

From the refraction table we find

Mean refraction $= 59''.1$
Barometer factor $= .880$
Thermometer $= .946$
Therefore $r = 49''.2$

From the American Ephemeris we find—

p. 218, eq. hor. parallax $\pi = 8''.72$
p. 327, $\delta = +18° \; 42' \; 16''.7$
p. 327, equation of time $E = + \; 6^m \; 12^s.99$
p. 327, semidiameter $s = 15' \; 47''.7$

δ is interpolated from the ephemeris by the method explained in Art. 52.

The ephemeris is given for the meridian of Washington; therefore we require the Washington time of our observation.

Time of observation	$T =$	$3^h \; 37^m \; 26^s.3$
Approximate correction	$\varDelta T =$	$-6 \; 41 \; 7$
Approximate local time	$=$	$20 \; 56 \; 19$
Longitude	$=$	$1 \; 44 \; 41$
Washington time, July 28	$=$	$22 \; 41 \; 0$

$= 1^h \; 19^m \; 0^s$ before noon of July 29
$= 1^h.317$
$= {}^d.055$

At noon, July 29, $\delta = 18° \; 41' \; 29''.6$

Hourly change July 28 $= -35''.00$
Hourly change July 29 $= -35''.77$
Therefore the correction to δ $= -1^h.317[-35.77 + \tfrac{1}{2}.77 \times {}^d.055]$,
$= +47''.1$

At time of observation $\delta = 18° \; 42' \; 16''.7$

At noon July 29, eq. of time $= + 6^m \; 12^s.89$
Correction for ${}^d.055$ $= .10$
$E = 6^m \; 12^s.99$

In taking E from the ephemeris, second differences need not be considered for this purpose, though it has been done in this case.

If a sidereal chronometer had been used we should have had only to convert the mean time $t + E$ into sidereal time, when we should have had ΔT by comparing with the observed time as now. It may be remarked also that in using a sidereal chronometer the observed sidereal time must be converted into mean solar time for the purpose of taking δ and E from the ephemeris, since these are given for mean solar time.

In reducing such a series as this it is perhaps a little better to reduce the readings on the two limbs separately; the two reductions will then mutually check each other. Of course the altitudes must be corrected for semidiameter. If a considerable number of series have been reduced in this way the observer can see, by comparing results, whether his personal equation is the same for both limbs.

125. *By a single altitude of a star.*

It will be convenient to use a sidereal chronometer when practicable.

Let Θ = the true sidereal time of observation;
Θ_c = the chronometer time of observation;
$\Delta\Theta$ = the chronometer correction.

Then t is computed the same as above; recollecting that for a star the semidiameter and parallax will be inappreciable, we have

$$z = 90° - (h' - r); \quad . \quad . \quad . \quad . \quad . \quad . \quad (219)$$

$$\Theta = (t + \alpha) = \Theta_c + \Delta\Theta;$$

$$\Delta\Theta = (t + \alpha) - \Theta_c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220)$$

Example 2. West Las Animas, Colorado. 1878, July 29.3.

Observation of *Arcturus* for time. Observer B.

Sidereal chronometer. Negus 1590.

	Sextant.	Chronometer.	
	$87°\ 40'$	$18^h\ 11^m\ 29^s.0$	Latitude $\varphi = 38°\ 4'\ 00''$
	30	11 55 .0	Longitude $L = 1^h\ 44^m\ 41^s$ w. of Wash.
	20	12 21 .0	Thermometer $74°.0$
	10	12 46 .5	Barometer 25 .91
	87 00	13 13 .0	
Means	$87°\ 20'\ 00''$	$18^h\ 12^m\ 20^s.9$	From ephemeris, $\alpha = 14^h\ 10^m\ 8^s.2$, $\delta = 19°\ 48'\ 58''$

$$I \qquad -18$$
$$E \qquad -42$$

$$2A = 87°\ 19'\ 00''$$
$$A = 43\ \ 39\ \ 30$$
$$r = \qquad -46$$
$$h = 43\ \ 38\ \ 44$$
$$s = 46\ \ 21\ \ 16$$

		Δ
$\varphi = 38°\ 4'\ 0''$	$\sec\varphi = 0.10386$	$+\ 9.9$
$\delta = 19\ \ 48\ \ 58$	$\sec\delta = .02651$	
$\varphi - \delta = 18°\ 15'\ 2''$		
$s + (\varphi - \delta) = 64\ \ 36\ \ 18$		
$s - (\varphi - \delta) = 28\ \ 6\ \ 14$		
$S = 32\ \ 18\ \ 9$	$\sin S = 9.72786$	$+\ 20.0$
$D = 14\ \ 3\ \ 7$	$\sin D = 9.38525$	$+\ 50.5$
	$\sin^2 \frac{1}{2}t = 9.24348$	
$\frac{1}{2}t = 24°\ 44'\ 33''.3$	$\sin \frac{1}{2}t = 9.62174$	$+\ 27.5$

$$t = 49\ \ 29\ \ 7$$
$$t = 3^h\ 17^m 56^s.5$$
$$\alpha = 14\ \ 10\ \ 8\ .2$$
$$\theta = 17\ \ 28\ \ 4\ .7 = \text{sidereal time}$$
$$\text{Observed } \theta_0 = 18\ \ 12\ \ 20\ .9 = \text{chron. reading}$$

$$\Delta\theta = -\ 44^m 16^s.2 = \text{chron. cor. [Eq. (220)]}.$$

It will be seen that the numerical work is somewhat less in case of a star than of the sun.

In case a mean solar chronometer has been used, the sidereal time $(t + \alpha)$ must be converted into mean solar time by (202), and the resulting value compared with the chronometer time.

Example 3. West Las Animas, Colorado. 1878, July 27.3.
Observation of α *Coronæ Borealis* for time. Observer B.
Mean solar chronometer.
Negus 1326.

	Sextant.	Chronometer.	
	95° 50′	$17^h\ 3^m\ 16^s.0$	Latitude $\varphi = 38°\ 4'\ 00''$
	40	3 40 .0	Longitude $L = 1^h\ 44^m\ 41^s$ w. of Wash.
	30	4 5 .0	Thermometer 62°.0
	20	4 32 .5	Barometer 26 .11
	95 10	17 4 57 .5	
Means	95° 30′ 0″	$17^h\ 4^m\ 6^s.2$	From the ephemeris, $\alpha = 15^h 29^m 34^s.1$
I	0		$\delta = 27°\ 7'\ 32''$
E	− 52		

$2A =$	95 29 8		
$A =$	47 44 34		
$r =$	− 46		
$h =$	47° 43′ 48″		
$z =$	42 16 12		Δ*
$\varphi =$	38° 4′ 0″	$\sec \varphi = 0.10386$	+ 9.9
$\delta =$	27 7 32	$\sec \delta = .05061$	+ 4.5
$\varphi - \delta =$	10 56 28		
$z + (\varphi - \delta) =$	53 12 40		
$z - (\varphi - \delta) =$	31 19 44		
$S =$	26 36 20	$\sin S = 9.65113$	+ 25.2
$D =$	15 39 52	$\sin D = 9.43137$	+ 45.0
		$\sin^2 \frac{1}{2}t = 9.23697$	
$\frac{1}{2}t =$	24° 32′ 43″	$\sin \frac{1}{2}t = 9.61848$	+ 27.7
$t =$	49 5 26		
$t =$	$3^h\ 16^m 21^s.7$		
$\alpha =$	15 29 34 .1		
$\theta =$	18 45 55 .8	= sidereal time. This is now converted into mean solar time by equation (202).	
$V =$	8 21 15 .7	= sidereal time of mean noon from ephemeris	
$\theta - V =$	10 24 40 .1		
	1 42 .3	Table II, Appendix to Ephemeris.	
M. S. time $=$	10 22 57 .8		
Chronom. $=$	17 4 6 .2		
$\Delta T =$	− 6 41 8 .4		

126. ***Conditions most favorable to accuracy in determining time by a single altitude.***

As our data will always be liable to more or less uncertainty, it becomes a matter of great practical importance to so arrange our observations that small errors in the quantities regarded as known shall have the least effect on the computed value of t.

* These quantities are written down so that we may employ them in computing the differential formulæ when desirable. (See Articles 128–131.)

As we require equations (121), we rewrite them here for convenience of reference.

$$\left.\begin{aligned} \cos h \cos a &= \cos \delta \cos t \sin \varphi - \sin \delta \cos \varphi; & (e) \\ \cos h \sin a &= \cos \delta \sin t; & (f) \\ \sin h &= \cos \delta \cos t \cos \varphi + \sin \delta \sin \varphi. & (g) \end{aligned}\right\} \quad . \quad . \quad (121)$$

To determine the effect upon t of a small error in the measured altitude. Differentiating (g) with respect to h and t and reducing by means of (f), we readily find

$$dt = -\frac{1}{\cos \varphi \sin a} dh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (221)$$

From this we see that for a given latitude φ a small error dh in the altitude will produce the least effect when $\sin a$ has its greatest value, viz., when the star is on the prime vertical. Also, that for a constant positive error dh the error produced in t will be $\mp$ when the star is $\left\{\begin{matrix}\text{west}\\ \text{east}\end{matrix}\right\}$ of the meridian, and may therefore be eliminated by observing both east and west stars.

(221) also shows that dt will be least when $\cos \varphi$ is greatest, that is, when φ is small; the most favorable part of the earth's surface for this kind of determination being the equator.

Effect of a small error in the assumed latitude φ. Differentiating (g) with respect to φ and t and reducing by means of (e) and (f), we find

$$dt = -\frac{1}{\tan a \cos \varphi} d\varphi; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (222)$$

from which it appears that when the star is near the prime vertical dt is relatively small. If the star is on the prime vertical, dt is zero, as $\tan a$ is then infinite.

If the star is not observed on the prime vertical, dt will disappear from the mean of two observations at the same distance east and west of the meridian. Also, we see that an error $d\varphi$ will have the least effect on t when the latitude is near zero.

In the same way we may discuss the effect of a small error in δ; but as no stars will ever be likely to be used for this purpose whose declination is uncertain to any appreciable amount, this is not practically a source of error.

127. From this discussion we see that a determination of time should always depend on observations of stars both east and west of the meridian; the observations should be made at as nearly the same azimuth as possible east and west, and if two stars are employed it will be better if the declinations are nearly equal.

dh may be regarded as including all of the undetermined errors of the instrument—see Articles 115, 116, and 117—as well as constant errors of observation and refraction.

Differential Formulæ.

128. The numerical values of the differential coefficients of t with respect to φ, δ, and $2h$ are often convenient where the time has been determined in the

manner just explained. Sometimes values of φ, δ, or $2h$ are employed in the computation which are afterwards found to require small corrections. If these are so small that the second and higher powers may be neglected, the necessary correction of the hour-angle may be found by the differential formula. Otherwise the computation must be repeated.

Let $\Delta\varphi$, $\Delta\delta$, $\Delta 2h$ = small corrections to the values of the latitude, declination, and double altitude employed;

Δt = the resulting correction to the hour-angle.

Then, neglecting terms of the second and higher orders,

$$\Delta t = \frac{dt}{d\varphi}\Delta\varphi + \frac{dt}{d\delta}\Delta\delta + \frac{dt}{d2h}\Delta 2h. \quad . \quad . \quad . \quad . \quad . \qquad (223)$$

The differential coefficients may be computed by the formulæ of the previous article, but they are not convenient since they require a knowledge of the azimuth.

129. For practical purposes a more convenient process is the following, where the numerical values of these coefficients are expressed in terms of the differences of the logarithms employed. Taking logarithms of both members of (215), we have

$$2 \log \sin \tfrac{1}{2}t = \log \sin S + \log \sin D + \log \sec \varphi + \log \sec \delta; \quad . \qquad (224)$$

where
$$\left.\begin{aligned} S &= \tfrac{1}{2}[z + (\varphi - \delta)] = \tfrac{1}{2}90^\circ - \tfrac{1}{2}2h + \tfrac{1}{2}(\varphi - \delta); \\ D &= \tfrac{1}{2}[z - (\varphi - \delta)] = \tfrac{1}{2}90^\circ - \tfrac{1}{2}2h - \tfrac{1}{2}(\varphi - \delta). \end{aligned}\right\} \quad . \quad . \qquad (225)$$

First differentiate (224) with respect to $2h$ and $\frac{1}{2}t$. We find

$$\frac{2dl \sin \frac{1}{2}t}{d\frac{1}{2}t} = \frac{dl \sin S}{dS} \cdot \frac{dS}{d2h} \cdot \frac{d2h}{d\frac{1}{2}t} + \frac{dl \sin D}{dD} \cdot \frac{dD}{d2h} \cdot \frac{d2h}{d\frac{1}{2}t}.$$

From (225),
$$\frac{dS}{d2h} = \frac{dD}{d2h} = -\frac{1}{4}.$$

Therefore we have, writing $\dfrac{dl \sin \frac{1}{2}t}{d\frac{1}{2}t} = \Delta l \sin \tfrac{1}{2}t \; \dfrac{dl \sin S}{dS} = \Delta l \sin S \ldots,$

$$\frac{dt}{d2h} = -\frac{\Delta l \sin S + \Delta l \sin D}{4\Delta l \sin \frac{1}{2}t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (226)$$

The quantities $\Delta l \sin S$, $\Delta l \sin D \ldots$ are the rates of change of the logarithms for the values of S, D, etc., employed. It requires, therefore, very little time to take these from the tables while computing t, as we have done in the examples in the foregoing pages.

Thus, in example 1 we have found $\Delta l \sin S = 19.9$, which is the change expressed in units of the last decimal place of log sin S produced by a change of 1′ in S. In practice the l sin of the angle 5′ less than S is subtracted from that of the angle 5′ greater, and the difference divided by 10. This is a little more accurate than to take the difference between consecutive logarithms.

In our example $S = 32° 24'$

$$\begin{aligned} l \sin 32° 19' &= 9.72803 \\ l \sin 32° 29' &= \underline{9.73002} \\ \text{Difference for } 10' &= \quad 199 \\ \text{Difference for } \ 1' &= \Delta = 19.9 \end{aligned}$$

In like manner we have found $\quad \Delta l \sin D = \quad 54.6$
$\Delta l \sin \tfrac{1}{2}t = -\ 28.7$

Therefore, by (226), $\dfrac{dt}{d2h} = -\dfrac{19.9 + 54.6}{-4 \times 28.7} = +.649.$

A correction to the assumed value of $2h$ may result from a variety of causes, such as the employment of values of the refraction, parallax, index error, or eccentricity, which are only approximately correct, or from errors in the preliminary computation.

Suppose the value of $2h$ employed in example 1 was found to require the correction $\Delta 2h = 1'$. Then the resulting correction to the hour-angle would be

$$\Delta t = .649 \times \frac{60''}{15} = 2^s.596.$$

130. For the value of $\dfrac{dt}{d\delta}$ we differentiate (224) with respect to t and δ, viz.,

$$\frac{2dl \sin \frac{1}{2}t}{d\frac{1}{2}t} = \frac{dl \sec \delta}{d\delta} \cdot \frac{d\delta}{d\frac{1}{2}t} + \frac{dl \sin S}{dS} \cdot \frac{dS}{d\delta} \cdot \frac{d\delta}{d\frac{1}{2}t} + \frac{dl \sin D}{dD} \cdot \frac{dD}{d\delta} \cdot \frac{d\delta}{d\frac{1}{2}t};$$

and from (225), $\qquad \dfrac{dS}{d\delta} = -\dfrac{1}{2}; \qquad \dfrac{dD}{d\delta} = +\dfrac{1}{2}.$

Therefore

$$\frac{dt}{d\delta} = \frac{2\Delta l \sec \delta - \Delta l \sin S + \Delta l \sin D}{2\Delta l \sin \frac{1}{2}t}. \quad . \quad . \quad . \quad . \quad (227)$$

Substituting the numerical values of $\Delta l \sec \delta$, $\Delta l \sin S$, etc., given in example 1, we find

$$\frac{dt}{d\delta} = \frac{8.6 - 19.9 + 54.6}{-57.4} = -.754.$$

If now, for example, the δ with which the reduction is made were found to require the correction $\Delta\delta = 1'$, we should have

$$\Delta t = \frac{dt}{d\delta}\Delta\delta = -.754 \times \frac{60''}{15} = -3^s.02.$$

131. For $\dfrac{dt}{d\varphi}$ we differentiate with respect to φ and t, viz.,

$$\frac{2dl \sin \frac{1}{2}t}{d\frac{1}{2}t} = \frac{dl \sin S}{dS} \cdot \frac{dS}{d\varphi} \cdot \frac{d\varphi}{d\frac{1}{2}t} + \frac{dl \sin D}{dD} \frac{dD}{d\varphi} \cdot \frac{d\varphi}{d\frac{1}{2}t} + \frac{dl \sec \varphi}{d\varphi} \cdot \frac{d\varphi}{d\frac{1}{2}t};$$

Also, $$\frac{dS}{d\varphi} = \frac{1}{2}; \qquad \frac{dD}{d\varphi} = -\frac{1}{2}.$$

Therefore $$\frac{dt}{d\varphi} = \frac{2\varDelta l \sec \varphi + \varDelta l \sin S - \varDelta l \sin D}{2\varDelta \sin \frac{1}{2}t}. \quad . \quad . \quad . \quad . \quad (228)$$

For our example 1 we have by this formula

$$\frac{dt}{d\varphi} = \frac{19.8 + 19.9 - 54.6}{57.4} = -.260,$$

and a correction of 1′ to the assumed latitude produces a corresponding correction to the time of

$$\varDelta t = -.260 \frac{60''}{4} = -1^s.04.$$

Probable Error.

132. By means of formula (226) we may reduce the time of each altitude to the time of the mean altitude for the purpose of comparing the individual measurements and computing the probable error. The application to example 1 will sufficiently explain the process.

The mean value of $2h$ is 89° 10′, so that each time will be reduced to the time corresponding to this altitude. Further, as one half the readings were made on the lower limb and one half on the upper limb, we must add to the latter and subtract from the former the time required for the sun to move in altitude over an arc equal to the sun's semidiameter, or in double altitude a space equal to the diameter.

Thus we have—see example 1—

Semidiameter of sun $= S = 15' \, 47''.7$;
Diameter of sun $= 31'.590$.

From previous article, $$\frac{dt}{d2h} = .649.$$

Therefore reduction for semidiameter $= .649 \times \dfrac{31'.590 \times 60}{15} = 82^s.01.$

The reduction is now as follows:

Limb.	Observed $2h$.	$\Delta 2h$.	Δt.	Correction for Semi-diameter.	Observed Time.	Reduced Time.	v $o - c$.	vv.
Upper	88° 50′	+ 20′	+ 51s.9	+ 1m 22s.0	3h 35m 12s.0	3h 37m 25s.9	− .4	16
	89 0	+ 10	+ 26 .0		35 39 .5	27 .5	+ 1.2	144
	89 10	0	.0		36 3 5	25 .5	− .8	64
	89 20	− 10	− 26 .0		36 30 .5	26 .5	+ .2	4
	89 30	− 20	− 51 .9		36 56 .5	26 .6	+ .3	9
Lower	88 50	+ 20	+ 51 .9	− 1 22 .0	3 37 55 .5	25 .4	− .9	81
	89 0	+ 10	+ 26 .0		38 22 .	26 .0	− .3	9
	89 10	0	.0		38 48 .	26 .0	− .3	9
	89 20	− 10	− 26 .0		39 14 .5	26 .5	+ .2	4
	89 30	− 20	− 51 .9		39 41 .0	3 37 27 .1	+ .8	64

Mean $3^h \, 37^m \, 26^s.3$ $[vv] = 4.04$

Then by formulæ (27), probable error of single observation $= r = {}^s.43$;
probable error of mean $= r_0 = {}^s.14$.

The reader must not fall into the error of supposing that this quantity represents the actual probable error of a determination of time by this method, since no account is here taken of the relatively large *constant* errors to which observations of this kind are liable. The subject will be considered more at length hereafter. (See Art. 156.)

Corrections for Refraction and Motion in Declination.

133. The refraction of the atmosphere and the sun's motion in declination affect the computed value of Δt by small quantities, which it may be considered desirable to take into account in a more refined discussion.

Correction for Refraction. Since refraction decreases with the altitude, it follows that when the sun's altitude increases by a given quantity—10′ for example—as measured with the instrument, the actual space passed over is greater than 10′ by the difference of refraction for the first and last position. Thus, instead of simply $\Delta 2h$ as used in our formula, we should employ $\Delta 2h + 2\Delta r$, Δr being the difference between the refraction for altitude h and that for $h + \Delta h$.

For our example we find for the mean altitude of the sun, viz., 44° 34′,

Change in refraction corresponding to 10′ altitude $= 0''.30 = 2\Delta r$.

Therefore the correction to Δt corresponding to $\Delta 2h = 10'$ is

$$.649 \times \frac{''.30}{15} = {}^s.013$$

This must be added to the computed interval, viz., $\Delta t = 25^s.96$

$$\Delta' t = 25^s.973$$

134. *Correction for Sun's Motion in Declination.* Since the sun's declination is not constant, but is ever increasing or diminishing, the time required for the altitude to change by a given amount will be slightly modified by this cause.

For our example with $\Delta 2h = 10'$ we find $\Delta t = 25^s.97$. Referring to the example, we have found the hourly motion in declination to be $-35''.7$; therefore in the interval $25^s.97$ the change is $-''.26$.

By formula (227) we have found for this example $\frac{dt}{d\delta} = -.754$.

Therefore correction to $\Delta t = -.754 \times \frac{-''.26}{15} = +{}^s.013$.

Therefore the final value of Δt corresponding to $\Delta 2h = 10'$ is $25^s.986$.

If both limbs are reduced together, as in our example, the reduction for semi-diameter should be corrected for motion in declination, but not for refraction, since both limbs are observed at the same altitude.

Determination of Time by Equal Altitudes.

135. *By a star observed at equal altitudes east and west of the meridian.*

Method of observing. When the star is at some distance east of the meridian (the nearer the prime vertical the better), measure with the sextant a series of five or more altitudes in the manner already explained (Arts. 111, 112, and 113); then, a short time before the star reaches the same altitude in the west, set the vernier at the reading of the last altitude and observe the same number of altitudes as before at the same readings. Some observers prefer to take only one reading east and then lay the instrument where nothing will disturb it until it is time for the west observation. In this way both observations are secured at absolutely the same altitude so far as it depends on the reading of the instrument; but there is the objection that only one reading can be made, which more than neutralizes the advantage. No correction for index error, refraction, or parallax is required.

Now, as the declination is constant and the altitudes the same, the numerical values of the hour-angle measured east and west of the meridian will be equal. Suppose a sidereal chronometer used. Let

Θ' = the chronometer time of the first observation;
Θ'' = the chronometer time of the second observation;
$\Delta\Theta$ = the chronometer correction.

Then the sidereal time of the star's meridian passage equals its right ascension α.

For the first observation $\alpha = \Theta' + \Delta\Theta + t$;
For the second observation $\alpha = \Theta'' + \Delta\Theta - t$.

From which $\Delta\Theta = \alpha - \frac{1}{2}(\Theta' + \Theta'')$. . . . . (229)

Example 1. 1856, March 19th, equal altitudes of Arcturus east and west of the meridian were observed as follows:

East of meridian,	$\Theta' =$	11^h	4^m	$51^s.5$
West of meridian,	$\Theta'' =$	17	21	30 .0
	$\frac{1}{2}(\Theta' + \Theta'') =$	14	13	10 .75
From ephemeris,	$\alpha =$	14	9	7 .11
	Therefore $\Delta\Theta =$		-4^m	$3^s.64$

136. If a mean time chronometer is employed, the sidereal time of the star's culmination (which is equal to the right ascension) must be converted into mean time, and this compared with the mean of the observed times as before.

Example 2. 1856, March 15th, equal altitudes of Spica were observed as below, the time being noted by a mean time chronometer:

Latitude $\varphi = -33° 56'$
Longitude $L = -1^h 13^m 56^s$ from Greenwich.

Chronometer. East.	Sextant. Double Alt.	Chronometer. West.
$10^h 20^m 0^s.5$	104° 0′	$2^h 40^m 38^s$.
20 28	10	40 10 .5
20 55	20	39 42
$T' = 10^h 20^m 27^s.83$		$T'' = 2^h 40^m 10^s.17$

$\frac{1}{2}(T' + T'') = 12\ 30\ 19 .0$

From ephemeris, $\alpha = \Theta =$	13	17	37 .92
Then—Art. 95—from ephemeris $V =$	23	32	53 .22
$\Theta - V =$	13	44	44 .70
Table II, ephemeris,		− 2	15 .12
Mean time =	13	42	29 .58
$\frac{1}{2}(T' + T'') =$	12	30	19 .00
Therefore $\Delta T = +$	1	12	10 .58

137. *By equal altitudes of the sun.*

This method is less simple when applied to the sun, for the reason that the sun's declination cannot be considered constant for the interval of time between the morning and afternoon observations. The mean of the observed times will not therefore be the time of meridian passage as in case of a star. The correction due to this cause is called the *equation of equal altitudes.* To determine its value we proceed as follows:

Let $\Delta\delta$ = the hourly change in declination taken from the Nautical Almanac.

Then $t\Delta\delta$ = the total change in δ in the time t;

δt = change produced in t by the increment $t\Delta\delta$ of δ.

Then since $t = f(\delta)$,

$$t + \delta t = f(\delta + t\Delta\delta);$$

and neglecting terms of higher order than the first,

$$\delta t = \frac{dt}{d\delta} \cdot t\Delta\delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (230)$$

To determine $\frac{dt}{d\delta}$ we differentiate the last of equations (121) with respect to t and δ, viz.,

$$\frac{dt}{d\delta} = \frac{\sin\varphi\cos\delta - \cos\varphi\sin\delta\cos t}{\cos\varphi\cos\delta\sin t} = \frac{\tan\varphi}{\sin t} - \frac{\tan\delta}{\tan t}.$$

Therefore substituting this value in (230), and dividing by 15, as δt is required in seconds of time, we find

$$\delta t = \left[\frac{\tan\varphi}{\sin t} - \frac{\tan\delta}{\tan t}\right] t\,\frac{\Delta\delta}{15}. \quad . \quad . \quad . \quad . \quad (231)$$

Now suppose a mean time chronometer used, and let

T' and T'' = chronometer times of east and west observation.

Then will

$t - \delta t =$ the hour-angle of the A.M. observation;
$t + \delta t =$ the hour-angle of the P.M. observation;
$E =$ equation of time.

Then $E = T' + \varDelta T + (t - \delta t)$ from A.M. observation;
$E = T'' + \varDelta T - (t + \delta t)$ from P.M. observation.

From which

$$\varDelta T = E - [\tfrac{1}{2}(T' + T'') - \delta t]. \quad . \quad . \quad . \quad . \quad (232)$$

Example 3. 1856, March 5th, at the U. S. Naval Academy the sun was observed east and west of the meridian as follows:

East, $T' = 1^h\ 8^m\ 26^s.6$
West, $T'' = 8\ \ 45\ \ 41.7$

$t = \tfrac{1}{2}(T'' - T') = 3^h\ 48^m\ 37^s.5$
$= 57°\ 9'$
$= 3^h.810$

$\tfrac{1}{2}(T' + T'') = 4^h\ 57^m\ \ 4^s.15$
$\delta t = +\ \ \ \ 15.18$
$E = +11\ \ 35.11$

$\varDelta T = -\ 4^h\ 45^m\ 13^s.86$

Latitude $\varphi =\ \ 38°\ 59'$
Longitude $L = -\ \ 2^m\ 16^s$
from Washington
From ephemeris, $\delta = -\ \ 5°\ 46'$
Equation of time $E = +11^m\ 35^s.11$
$\varDelta\delta = +\ \ \ 58''.10$

$\tan \varphi = 9.9081$
$\sin t = 9.9243$
9.9838
$^*A = 1.1696$

$\tan \delta = 9.0042_n$
$\tan t = .1900$
8.8142_n
$^*B = 1.1980$
$\log t = .5809$
$\log \varDelta\delta = 1.7642$
$\log \tfrac{1}{15} = 8.8239$
$\log \delta t = 1.1812$

138. *Equal altitudes of the sun observed in the afternoon of one day and the morning of the day following.*

In this case the mean of the observed times plus the neces-

* See tables of addition and subtraction logarithms.

sary corrections will be the time of the sun's passing the lower branch of the meridian, or midnight.

Let $t' =$ the sun's hour-angle, reckoned from the lower branch of the meridian.

Then $t' = t + 180°$; $\sin t = - \sin t'$; $\tan t = \tan t'$.

Therefore for this case (231) becomes

$$\delta t = - \left[\frac{\tan \varphi}{\sin t'} + \frac{\tan \delta}{\tan t'}\right] t' \frac{\Delta\delta}{15}; \quad . \quad . \quad . \quad (233)$$

and the clock correction will be given by (232), as before, except that for E we write $12^h + E$.

Example 4. 1856, May 3d. The altitude of the sun being observed on the afternoon of the 3d and the morning of the 4th as follows, required the correction of the chronometer at midnight.

$$\begin{aligned}
T' &= 6^h\ 54^m\ 10^s.3 \\
T'' &= 21\ \ 9\ \ 17\ .5 \\
\hline
\tfrac{1}{2}(T'' - T') = t' &= 7^h\ 7^m\ 34^s. \\
t' &= 106°\ 53' \\
t' &= 7^h.126 \\
\hline
\tfrac{1}{2}(T'' + T') &= 14^h\ 1^m\ 43^s.9 \\
\delta t &= \qquad\quad 22\ .2 \\
12^h + E &= 11\ \ 56\ \ 41\ .33 \\
\hline
\Delta T &= -\ 2^h\ 4^m\ 40^s.4
\end{aligned}$$

$$\begin{aligned}
\text{Latitude south} = \varphi &= -\ 43°\ 21' \\
\text{Longitude W. of Wash.} = L &= +\ 9^h\ 1^m\ 40^s \\
\text{From ephemeris, } \delta &= 15°\ 15' \\
\Delta\delta &= +\ 43''.76 \\
\text{Equation of time } E &= -\ 3^m\ 18^s.67
\end{aligned}$$

$$\begin{aligned}
\tan \varphi &= 9.9750_n \\
\sin t' &= 9.9809 \\
\hline
& \ \ 9.9941_n \\
A &= 1.0764
\end{aligned}
\qquad\qquad
\begin{aligned}
\tan \delta &= 9.4356 \\
\tan t' &= .5179_n \\
\hline
& \ \ 8.9177_n \\
B &= 1.1114 \\
\log t &= .8528 \\
\log \Delta\delta &= 1.6411 \\
\log \tfrac{1}{15} &= 8.8239 \\
\hline
\log (-\ \delta t) &= 1.3469_n
\end{aligned}$$

139. The chief advantages possessed by the method of determining time by equal altitudes are the following: the computation is very simple, and no corrections are required for parallax, refraction, semidiameter, or instrumental errors, nor is a knowledge of the latitude required, except very roughly, when the sun is employed. The disadvantages are the difficulty and often impossibility of obtaining the observations at exactly the same altitude, owing to clouds or other hinderances; also, the changes which often take place in the refraction between the morning and afternoon. A correction for this last mentioned source of error may be computed by means of a differential formula, but it has not been thought necessary to develop it here.

Latitude.

140. We have seen (Art. 63) that the astronomical latitude of any place is equal to the declination of the zenith of that place, or to the elevation of the pole above the horizon. The distinctions between the different kinds of latitude, as defined in Art. 73, must be borne in mind. We are at present only dealing with the *astronomical latitude* as there defined. It is perhaps unnecessary to state that all formulæ derived will be applicable to either north or south latitude, care being taken to use the proper algebraic signs: $\left.\begin{matrix}\text{north}\\ \text{south}\end{matrix}\right\}$ latitudes and declinations being $\left\{\begin{matrix}\text{plus.}\\ \text{minus.}\end{matrix}\right.$

First Method.

141. *By the zenith distance of a star observed on the meridian.* Resuming the last of equations (121),

$$\cos z = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t,$$

we know that when the star is on the meridian,

$$t = 0; \qquad \cos t = 1.$$

Therefore we have

$$\cos z = \cos (\varphi - \delta);$$
$$\pm z = \varphi - \delta \quad \text{and} \quad \varphi = \delta \pm z. \quad . \quad . \quad (234)$$

By referring to the figure, $ES = \delta$, $zS = z$, and we readily see that in the above formula the sign will be $\pm$ for a star $\left\{\begin{matrix}\text{south}\\ \text{north}\end{matrix}\right\}$ of the zenith.

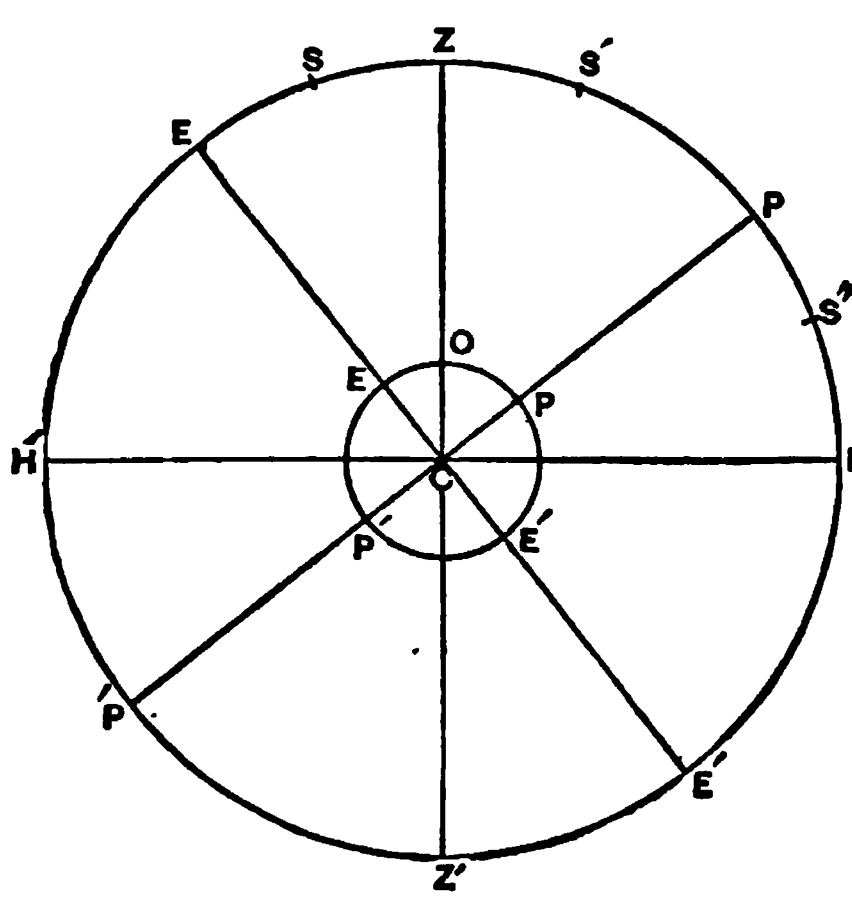


FIG. 25.

The same formula applies to a star S'' observed below the pole. If we reckon the declination on that branch of the meridian which contains the observer's zenith, or, what is the

same thing, if we replace δ in formula (234) by $(180° - \delta)$, it then becomes

$$\varphi = (180° - \delta) - z. \quad . \quad . \quad . \quad . \quad . \quad (235)$$

Second Method.

142. *By a circumpolar star observed at both upper and lower culmination.*

From (234) we have—

$$\text{For upper culmination } \varphi = \delta - z;$$
$$\text{For lower culmination } \varphi = 180° - \delta - z'.$$

The mean of which gives $\varphi = 90° - \frac{1}{2}(z + z')$. (236)

The method has this advantage, viz., that the latitude determined in this way does not require a knowledge of the place of the star; it is therefore especially adapted to the determination of the latitude of a fixed observatory, where it is desirable to make the results independent of what has been done at other places. As will appear hereafter, when extreme accuracy is required there will be a small correction necessary for the change in δ between the first and second observation. The result is also affected by whatever error there may be in the tabular value of the refraction used.

The following example will illustrate both the above methods:

1875, November 11th, at the Washington observatory the zenith distance of Polaris was observed as follows:

$$\text{Upper culmination } z = 49° \ 45' \ 22''.2;$$
$$\text{Lower culmination } z' = 52° \ 27' \ 20''.0.$$

From the Nautical Almanac we find for the declination of Polaris at the time of upper culmination at Washington:

	Nov. 11.4,	$\delta = 88° \; 39' \; 2''.8$
		$z = 49 \;\; 45 \;\; 22 \;.2$
Therefore, formula (234),		$\varphi = \delta - z = 38° \; 53' \; 40''.6$
Also for lower culmination,	Nov. 11.9,	$\delta = 88 \;\; 39 \;\;\; 3 \;.0$
		$z' = 52 \;\; 27 \;\; 20 \;.0$
Then formula (236) gives		$\varphi = 180° - \delta - z' = 38° \; 53' \; 37''.0$
The mean of these values gives us		$\varphi = 38° \; 53' \; 38''.8$

By the second method we have

$$\varphi = 90° - \tfrac{1}{2}(z + z') = 38° \; 53' \; 38''.9$$

Third Method.

143. *By an altitude of a star observed in any position, the time being known.*

Θ, the sidereal time, is known; α, the right ascension, and δ, the declination, are taken from the Nautical Almanac.

We then have $$t = \Theta - \alpha.$$

This will be given in time, and must be multiplied by 15 to reduce it to arc. We then have

$$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t;$$

in which φ is the only unknown quantity.

For solving the equation introduce two auxiliaries, d and D, determined by the equations

$$d \sin D = \sin \delta; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$
$$d \cos D = \cos \delta \cos t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a')$$

The above equation then becomes, by substituting the value of d from (a),

$$\cos(\varphi - D) = \sin h \sin D \operatorname{cosec} \delta.$$

Dividing (a) by (a') to determine D, we have the following formulæ for determining φ:

$$\left.\begin{aligned} \tan D &= \tan \delta \sec t; \\ \cos(\varphi - D) &= \sin h \sin D \operatorname{cosec} \delta. \end{aligned}\right\} \quad . \quad . \quad (237)$$

D is taken less than 90°, + or − according to the algebraic sign of the tangent. ($\varphi - D$), being determined in terms of the cosine, may be either + or −. There will therefore be two values of the latitude which will satisfy the above conditions. Practically an approximate value of the latitude will always be known with accuracy sufficient for deciding this ambiguity.

Example. On March 4th, 1882, I observed the following double altitudes of Polaris with a Pistor & Martins prismatic sextant and artificial horizon:

	Sextant.	Clock.	
	79° 12′ 0″	$10^h\ 43^m\ 4^s$	
	10 50	43 56	
	10 30	45 2	
	10 5	45 50	
	9 50	47 45	From Nautical Almanac:
Means	79° 10′ 39″	$10^h\ 45^m\ 7^s.4$	$\alpha = 1^h\ 15^m\ 6^s.0$
Index correction I	− 1 2.0	$\Delta\Theta$ + 1 .5	$\delta = 88°\ 41'\ 6''.2$
	$2h' = 79°\ 9'\ 37''$	$\Theta = 10^h\ 45^m\ 8^s.9$	
	$h' = 39\ \ 34\ \ 48.5$	$\alpha = 1\ \ 15\ \ 6.0$	
Refraction	− 1 9.7	$t = 9^h\ 30^m\ 2^s.9$	
	$h = 39\ \ 33\ \ 38.8$	$t = 142°\ 30'\ 43''.5$	

$\delta = 88°\ 41'\ 6''.2$	tan = 1.6391390	cosec δ = .0001144
$t = 142\ \ 30\ \ 43\ .5$	cos = 9.8995369_n	
$D = -\ 88\ \ 57\ \ 23\ .6$	tan D = 1.7396021_n	sin D = 9.9999279_n
$h = 39\ \ 33\ \ 38\ .8$		sin h = 9.8040688
$\varphi - D = 129\ \ 33\ \ 55\ .4$		cos ($\varphi - D$) = 9.8041111_n
$\varphi = 40\ \ 36\ \ 31\ .8$		

In this example there is no ambiguity: $\cos(\varphi - D)$ being negative, the angle must be in the second or third quadrant. If we had taken it in the third quadrant we should have found $\varphi = 141° +$. As φ is never greater than 90°, this value is in any case excluded.

144. *Effect of Errors in the Data upon the Latitude determined by an Altitude of a Star.*

Differentiating equation (g), Art. 126, regarding h and φ as variable, and reducing by equation (e), we readily find

$$d\varphi = -\frac{1}{\cos a}dh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (238)$$

From this we see that a small error in the measured altitude will have the least effect on the latitude when the star is on the meridian.

Again, differentiating the same equation with respect to φ and t, and reducing, we readily find

$$d\varphi = -\tan a \cos \varphi dt; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (239)$$

from which it appears that the effect upon φ of a small error, dt, in the hour-angle will be least when a is zero or 180°.

It appears, therefore, that the latitude will be determined with greater accuracy the nearer the star is to the meridian. When the star is very near the meridian the method which follows will be preferable.

Fourth Method.

145. *By circummeridian altitudes.* When the latitude is determined by the altitude of a star observed on the meridian, the accuracy is greater than in any other position, and at the same time the computation is extremely simple. We can, however, only measure one altitude when the star is on the meridian; and frequently at the time when the observation is made we shall not know the chronometer correction with sufficient accuracy for determining the exact instant when this observation should be taken. If, however, altitudes are measured near the meridian (how near we shall discuss later), the observed altitudes may be reduced to the meridian altitude by a simple computation. It will thus be possible to

make a considerable number of measurements instead of relying on one alone. When this method is applied observation is begun if possible a few minutes before culmination, and a series of altitudes measured in quick succession so as to have about the same number on each side of the meridian.

Altitudes measured in this manner are called *circummeridian altitudes.*

It is not essential, however, that the series should be symmetrical with respect to the meridian; the method is equally applicable to the reduction of one or more altitudes taken on either side of the meridian if sufficiently near.

Let h = any altitude of a star corresponding to the hour-angle t;
h_0 = the altitude when the star is on the meridian;
z_0 = the zenith distance $= 90° - h_0 = \varphi - \delta$

Then

$$\sin h = \sin\varphi \sin\delta + \cos\varphi \cos\delta \cos t.$$

Let us write for $\cos t$ its value, $1 - 2\sin^2\frac{1}{2}t$.

Then the above equation becomes

$$\sin h = \cos z = \cos(\varphi - \delta) - \cos\varphi \cos\delta \, 2\sin^2\tfrac{1}{2}t. \qquad (a)$$

Let us write $\quad \cos\varphi \cos\delta \, 2\sin^2\frac{1}{2}t = y. \quad . \quad . \quad . \quad . \quad . \quad (b)$

Then (a) becomes $\quad \cos z = \cos z_0 - y, \quad . \quad . \quad . \quad . \quad . \quad (c)$

or $\quad z = f(y).$

This expression may now be expanded into a series in terms of ascending powers of y, and when t is small the series will converge rapidly if z_0 is not too small.

Maclaurin's formula applied to this case is as follows:

$$z = z_0 + \left(\frac{dz}{dy}\right)y + \left(\frac{d^2z}{dy^2}\right)\frac{y^2}{2} + \left(\frac{d^3z}{dy^3}\right)\frac{y^3}{1.2.3}, \text{ etc.} \quad . \quad (d)$$

Differentiating (c) and observing that when $y = 0$, $z = z_0$, we find the following values of the differential coefficients:

$$\left(\frac{dz}{dy}\right) = \frac{1}{\sin z_0}; \quad \left(\frac{d^2z}{dy^2}\right) = -\frac{\cot z_0}{\sin^2 z_0}; \quad \frac{d^3z}{dy^3} = \frac{1 + 3\cot^2 z_0}{\sin^3 z_0}.$$

Substituting these values in (d) and restoring the value of y, we find

$$z = z_0 + \frac{\cos\varphi\cos\delta}{\sin z_0} 2\sin^2\tfrac{1}{2}t - \left(\frac{\cos\varphi\cos\delta}{\sin z_0}\right)^2 \cot z_0\, 2\sin^4\tfrac{1}{2}t$$
$$+ \left(\frac{\cos\varphi\cos\delta}{\sin z_0}\right)^3 \tfrac{2}{3}(1 + 3\cot^2 z_0) 2\sin^6\tfrac{1}{2}t. \quad . \quad . \quad . \quad (240)$$

In this equation $2\sin^2\frac{1}{2}t$, $2\sin^4\frac{1}{2}t$, etc., are expressed in terms of the radius. The equation must be made homogeneous by introducing the divisor $\sin 1''$ where necessary.

$$\left.\begin{array}{lll} \text{Let} \quad \dfrac{\cos\varphi\cos\delta}{\sin z_0} = A; & \dfrac{2\sin^2\frac{1}{2}t}{\sin 1''} = m; \\ A^2\cot z_0 = B; & \dfrac{2\sin^4\frac{1}{2}t}{\sin 1''} = n; \\ A^3\tfrac{2}{3}(1 + 3\cot^2 z_0) = C; & \dfrac{2\sin^6\frac{1}{2}t}{\sin 1''} = o. \end{array}\right\} \quad . \quad . \quad (241)$$

Then we have

$$\varphi = \delta \pm z \mp Am \pm Bn \mp Co. \quad . \quad . \quad . \quad (242)$$

146. This computation is made very simple by the use of table VIII, where m and n are given with the argument t expressed in time (the last term, Co, is seldom used).

As A and B will be constant for the entire series, we shall have,

If z_1, z_2, z_3, etc., z_μ, are the observed zenith distances,
m_1, m_2, m_3, etc., m_μ, the corresponding values of m taken from the table,

n_1, n_2, n_3, etc., n_μ, the corresponding values of n,

$$\varphi = \delta \pm z_1 \mp Am_1 \pm Bn_1;$$
$$\varphi = \delta \pm z_2 \mp Am_2 \pm Bn_2;$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\varphi = \delta \pm z_\mu \mp Am_\mu \pm Bn_\mu.$$

The mean of these equations will then be

$$\varphi = \delta \pm \frac{z_1 + z_2 + \ldots + z_\mu}{\mu} \mp A\frac{m_1 + m_2 + \ldots + m_\mu}{\mu} \pm B\frac{n_1 + n_2 + \ldots + n_\mu}{\mu}. \quad (243)$$

147. It will be observed that an approximate value of the latitude is required for computing A. When the observations extend on both sides of the meridian a sufficiently close approximation may always be obtained by taking the largest measured altitude and calling this the meridian altitude; or, better, take the mean of this in connection with that immediately preceding and following it. If the altitudes are all measured on one side of the meridian, or if for any reason a value of φ has been used which proves to be considerably in error, it may be necessary to repeat the computation of A, using for φ the value found from the first computation. In that case only the correction Am need be computed in the first approximation, and only three or four altitudes reduced.

148. Let us now examine separately the terms of equation (240) in order to see how far from the meridian the observations may be extended without introducing into the resulting latitude inadmissible errors.

Taking the last term, viz.,

$$\left(\frac{\cos\varphi\cos\delta}{\sin(\varphi - \delta)}\right)^2 \tfrac{1}{3}(1 + 3\cot^2 z_0)\frac{2\sin^4\frac{1}{2}t}{\sin 1''} = Co,$$

for any given values of φ and δ, we can compute the value of t, for which this

quantity will have any value, as, for instance, 1″. We readily see that when the zenith distance of the star is large the observations may be extended much further from the meridian than when it is small. The following table gives the hour-angle, for which this term has the value 1″ for different values of φ and δ. Thus, referring to the table, we see that if $\varphi = 40°$ and $\delta = 0$, then $t = 40^m$; or, in this case, the error committed in neglecting this term amounts to 1″ only when the star is 40^m from the meridian. If $\varphi = 40°$ and $\delta = 23°$ about the maximum declination of the sun, then $t = 20^m$.

LIMITING HOUR-ANGLE AT WHICH THE THIRD REDUCTION AMOUNTS TO ONE SECOND.

Latitude.	Declination same sign as Latitude.								Declination different sign from Latitude.								
	80°	70°	60°	50°	40°	30°	20°	10°	0°	10°	20°	30°	40°	50°	60°	70°	80°
0°	135^m	90^m	67^m	51^m	40^m	29^m	20^m	11^m	0^m	11^m	20^m	29^m	40^m	51^m	67^m	90^m	135^m
10	128	82	59	43	32	21	11	0	11	20	28	37	47	59	75	96	
20	118	73	51	35	23	12	0	11	20	28	37	46	56	67	82		
30	107	64	42	26	14	0	12	21	29	37	46	55	64	75			
40	95	54	32	16	0	14	23	32	40	47	56	64	73				
50	82	42	19	0	16	26	35	43	51	59	67	75					
60	67	27	0	19	32	42	51	59	67	75	82						
70	45	0	27	42	54	64	73	82	90	96							

Let us now consider the term

$$\left(\frac{\cos\varphi\cos\delta}{\sin(\varphi-\delta)}\right)^2 \cot z_0 \frac{2\sin^4\frac{1}{2}t}{\sin 1''} = Bb.$$

In a precisely similar manner we can compute the limiting values of t, within which this term is less than 1″. The table is computed in this way; from it we find that in the first of the above cases $t = 16^m$; in the second, $t = 9^m$.

LIMITING HOUR-ANGLE AT WHICH THE SECOND REDUCTION AMOUNTS TO ONE SECOND.

Latitude.	Declination same sign as Latitude.								Declination different sign from Latitude.								
	80°	70°	60°	50°	40°	30°	20°	10°	0°	10°	20°	30°	40°	50°	60°	70°	80°
0°	67^m	39^m	27^m	21^m	16^m	12^m	8^m	5^m	0^m	5^m	8^m	12^m	16^m	21^m	27^m	39^m	67^m
10	54	33	24	17	13	9	5	0	5	8	12	15	19	24	32	48	
20	48	29	20	14	10	5	0	5	8	12	15	18	23	29	40		
30	43	26	17	11	6	0	5	9	12	15	18	22	28	37			
40	38	22	13	7	0	6	10	13	16	19	23	28	36				
50	33	18	9	0	7	11	14	17	21	24	29	37					
60	28	12	0	9	13	17	20	24	27	32	40						
70	20	0	12	18	22	26	29	33	39	48							

If we are able to choose our own times for observing, we can always make our measurements so near the meridian that these terms may be neglected.

As $1''$ is much within the error of an ordinary sextant measurement, the limits may be extended somewhat beyond those of the table without serious error. We may, in a similar manner, determine for what values of t Co or Bn will have the values $0''.1$, $0''.01$, or any other value.

Lower Culmination.

149. When the star is observed near the meridian at lower culmination, the hour-angles should be reckoned from the lower branch of the meridian. This is equivalent to substituting $180° + t$ in the formula in place of t. We then have

$$\cos z = \sin \varphi \sin \delta - \cos \varphi \cos \delta \cos t.$$

Writing, as before, $\cos t = 1 - 2 \sin^2 \frac{1}{2}t$,

this becomes

$$\cos z = -\cos(\varphi + \delta) + \cos \varphi \cos \delta \; 2 \sin^2 \tfrac{1}{2}t.$$

Expanding this as before, and remembering that for lower culmination we have, from (235),

$$z_0 = 180° - (\varphi + \delta),$$

and therefore $\cos z_0 = \cos(\varphi + \delta)$,

we readily obtain

$$z_0 = z + \frac{\cos \varphi \cos \delta}{\sin z_0} \frac{2 \sin^2 \frac{1}{2}t}{\sin 1''} + \left(\frac{\cos \varphi \cos \delta}{\sin z_0}\right)^2 \cot \tfrac{1}{2}z_0 \frac{2 \sin^4 \frac{1}{2}t}{\sin 1''}, \quad (244)$$

or

$$z_0 = z + Am + Bn, \quad . \quad . \quad . \quad . \quad . \quad (245)$$

and

$$\varphi = 180° - \delta - (z + Am + Bn). \quad . \quad . \quad (246)$$

This formula might have been obtained from (240) exactly as (235) is from (234), viz., by simply changing δ into $180° - \delta$.

The hour-angle is obtained by simply taking the difference

between the chronometer time of observation and of culmination.*

Let α = star's right ascension = sidereal time of culmination;
$\Delta\Theta$ = chronometer correction, + when chronometer is slow.
Then $(\alpha - \Delta\Theta)$ = chronometer time of culmination.

If then Θ' is the chronometer time of any observation,

$$t = \Theta' - (\alpha - \Delta\Theta). \quad . \quad . \quad . \quad . \quad . \qquad (247)$$

Formulæ for Latitude by Circummeridian Altitudes of a Star.

$$\left.\begin{aligned}
&z = 90^\circ - (h - r); \\
&t = \Theta' - (\alpha - \Theta\Delta); \\
&A = \frac{\cos\varphi\cos\delta}{\sin z_0}; \qquad B = A^2 \cot z_0; \\
&m = \frac{2\sin^2\frac{1}{2}t}{\sin 1''}; \qquad n = \frac{2\sin^4\frac{1}{2}t}{\sin 1''}; \\
&\varphi = \quad \delta \pm (z - Am + Bn), \text{ upper culmination}; \\
&\varphi = 180^\circ - \delta - (z + Am + Bn), \text{ lower culmination.}
\end{aligned}\right\} \text{(XIII)}$$

Example of Latitude by Circummeridian Altitudes.

1873, August 20. *α Aquilæ* observed for Latitude. Observer Boss.
Instruments: Sextant and Sidereal Chronometer.

Assumed latitude φ =	49° 01′
Assumed longitude L =	+ $1^h\ 41^m\ 18^s$
Chronometer correction $\Delta\Theta$ =	− 22 50
From ephemeris, right ascension of star α =	$19^h\ 44^m\ 37^s.5$
Therefore chronometer time of culmination = $\alpha - \Delta\Theta$ =	20 7 27 .5
Star's declination δ =	8° 32′ 11″.5

* If the rate of the chronometer is appreciable it must be taken into account. For the simplest manner of doing this see Art. 152.

$\varphi = 49^\circ\ 01'.$ $\cos\varphi = 9.8168$ $\log A^2 = 9.9992$

$\delta = 8\ \ 32.2$ $\cos\delta = 9.9952$ $\cot z_0 = .0688$

$z_0 = 40\ \ 28.8$ $\operatorname{cosec} z_0 = .1876$ $^*\log B = 0.0680$

$A = .9991$ $\log A = 9.9996$

$B = 1.169$

The observations and method of reduction are shown in the following tabular statement, which will be sufficiently explained by reference to formulæ (XIII).

	Sextant. $2h$.	h.	Chronometer. Θ'.	t.
1	99° 5′ 35″	49° 32′ 47″.5	$20^h\ 1^m\ 35^s$	$-\ 5^m\ 52^s.5$
2	6 10	33 5	2 37	4 50 .5
3	7 5	33 32 .5	3 57	3 30 .5
4	7 55	33 57 .5	5 5	2 22 .5
5	8 10	34 5	6 41	− 46 .5
6	8 0	34 0	7 52	+ 24 .5
7	7 50	33 55	8 51	1 23 .5
8	7 40	33 50	9 47	2 19 .5
9	7 5	33 32 .5	10 41	3 13 .5
10	99 6 55	49 33 27 .5	20 12 0	+ 4 32 .5

	m.	Am.	n.*	Bn.*	$h + Am - Bn$.	v.	vv.
1	67″.8	67″.7	″.01	″.01	49° 33′ 55″.2	4.6	21.16
2	46 .0	46 .0	.01	.01	51 .0	8.8	77.44
3	24 .2	24 .2			56 .7	3.1	9.61
4	11 .1	11 .1			68 .6	8.8	77.44
5	1 .2	1 .2			66 .2	6.4	40.96
6	.3	.3			60 .3	.5	.25
7	3 .8	3 .8			58 .8	1.0	1.00
8	10 .6	10 .6			60 .6	.8	.64
9	20 .4	20 .4			52 .9	6.9	47.61
10	40 .5	40 .5			49 33 68 .0	8.2	67.24

Mean $h = 49^\circ\ 33'\ 59''.8$ $[vv] = 343.35$

Index error $= \frac{1}{2}I = -\ \ 1\ 51\ .5$ $r = 3''.9$

Eccentricity $= \frac{1}{2}E = -\ \ 10\ .1$ $r_0 = 1\ .3$

Refraction $r = -\ \ 47\ .3$

* It is easy to see in advance than the term Bn is inappreciable in this case. It is introduced here to illustrate the method.

Corrected altitude	$= 49°$	$31'$	$10''.9$		
Zenith distance	$z = 40$	28	49 .1		
Declination	$\delta = 8$	32	11 .4		
Resulting latitude	$\varphi = 49$	1	0 .5	$\pm 1''.3$	

If it is not considered necessary to reduce each observation separately, the work is abridged somewhat by the following process [see Art. (146)]:

Mean of	$2h = 99°$	$7'$	$14''.5$		
Index	$I = -$	3	43 .0		
Eccentricity	$E = -$		20 .2		
Corrected	$2h = 99$	3	11 .3		
	$h = 49$	31	35 .6		Mean of $m = 22''.6 = m'$
	$Am' = +$		22 .6		
Refraction	$= -$		47 .3		$Am' = 22''.6$
Corrected	$h = 49$	31	10 .9		
Zenith distance	$z = 40$	28	49 .1		
Declination	$\delta = 8$	32	11 .4		
Latitude	$\varphi = 49$	1	0 .5		

150. In the formulæ which we have derived for circum-meridian altitudes we have supposed the declination practically constant during the interval of observation.

With the sun this is not the case; but the same method may be used if we take for δ the mean of the declinations corresponding to each time of observation, or, what is practically the same, the declination corresponding to the mean of the times. It is, however, better to reduce each altitude separately for the purpose of estimating the accuracy of the final result and as a partial check against error of computation. If formulæ (XIII) are used, the declination must be interpolated for the time of each altitude; this considerably augments the labor of reduction. This additional labor may be avoided by the method which follows.

Gauss' Method of Reducing Circummeridian Observations of the Sun.

151. In this method the hour-angle is reckoned from the point where the sun reaches his maximum altitude instead of from the meridian. The meridian declination may then be used in reducing all of the observations.

Let δ_0 = the sun's meridian declination;
δ = the declination corresponding to hour-angle t;
$\Delta\delta$ = hourly change in δ given in the Nautical Almanac, + when the sun is moving N.;
t = the hour-angle given in seconds of time.

Then $\frac{\Delta\delta}{3600}$ = the change in δ in one second,

and
$$\delta = \delta_0 + t\frac{\Delta\delta}{3600}. \quad . \quad . \quad . \quad . \quad . \quad (248)$$

Also, since $\delta = f(t)$,
$$\delta = \delta_0 + t\frac{d\delta}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (249)$$

by neglecting terms of higher order than the first. Then

$$\varphi = z + \delta_0 + t\frac{d\delta}{dt} - \frac{\cos\varphi\cos\delta}{\sin z_0} . 2\sin^2\tfrac{1}{2}t, \text{ etc.} \quad (250)$$

The peculiarity of the process is in the method by which the small term $t . \frac{d\delta}{dt}$ is taken into account. For this purpose we determine the value of t corresponding to the maximum value of h by placing $\frac{dh}{dt}$ equal to zero and solving for t.

Take the equation

$$\sin h = \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos t.$$

Differentiating with respect to h, δ, and t, and placing $\frac{dh}{dt} = 0$, we have

$$\cos h \frac{dh}{dt} = (\sin \varphi \cos \delta - \cos \varphi \sin \delta \cos t) \frac{d\delta}{dt} - \cos \varphi \cos \delta \sin t = 0. \quad (251)$$

As t will be very small, no appreciable error will be introduced by making $\cos t = 1$, when the above equation readily gives

$$\frac{d\delta}{dt} = \frac{\cos \varphi \cos \delta}{\sin (\varphi - \delta)} \sin t. \quad . \quad . \quad . \quad (252)$$

In this t is the hour-angle of the sun corresponding to the maximum altitude. To distinguish it from the general value of t call it y, and as it is small we may write

$$\frac{d\delta}{dt} = \frac{\cos \varphi \cos \delta}{\sin z_0} . y. \quad . \quad . \quad . \quad . \quad (253)$$

Substituting this value of $\frac{d\delta}{dt}$ in equation (250), it becomes

$$\varphi = z + \delta_0 - \frac{\cos \varphi \cos \delta}{\sin z_0} (2 \sin^2 \tfrac{1}{2}t - ty). \quad . \quad (254)$$

Since t will always be small when this method is used, let us write

$$\sin \tfrac{1}{2}t = \tfrac{1}{2}t, \qquad \text{whence} \qquad 2 \sin^2 \tfrac{1}{2}t = \tfrac{1}{2}t^2.$$

Then
$$\begin{aligned} 2 \sin^2 \tfrac{1}{2}t - ty &= \tfrac{1}{2}(t^2 - 2ty + y^2) - \tfrac{1}{2}y^2 \\ &= \tfrac{1}{2}(t - y)^2 - \tfrac{1}{2}y^2. \end{aligned}$$

Passing back from the angles to the sines and making the

terms homogeneous by introducing the divisor sin 1″, equation (254) becomes

$$\varphi = z + \delta_0 + \frac{\cos\varphi\cos\delta}{\sin z_0}\cdot\frac{2\sin^2\frac{1}{2}y}{\sin 1''}$$
$$- \frac{\cos\varphi\cos\delta}{\sin z_0}\cdot\frac{2\sin^2\frac{1}{2}(t-y)}{\sin 1''}. \quad . \quad (255)$$

The term $\frac{\cos\varphi\cos\delta}{\sin z_0}\cdot\frac{2\sin^2\frac{1}{2}y}{\sin 1''}$ is always very small, and in the solution of the problem as given by Gauss it was neglected. Its computation only requires one additional logarithm, and is therefore very simple; but in reducing sextant work it is perhaps an excess of refinement to retain it.

We now require a convenient formula for computing y.

Equation (253) may be written

$$y\ 15 \sin 1'' = \frac{\sin z_0}{\cos\varphi\cos\delta}\cdot\frac{d\delta}{dt}, \quad . \quad . \quad . \quad (256)$$

since y will be required in seconds of time.

If we replace $d\delta$ by the number of seconds of arc which δ increases in one hour, and dt by one hour expressed in seconds of arc, we have

$$\frac{d\delta}{dt} = \frac{\Delta\delta}{54000}.$$

Then from (256)

$$y = \frac{\sin z_0}{\cos\varphi\cos\delta}\,\Delta\delta\,.\,\frac{206265}{15\times 54000} = \frac{\sin z_0}{\cos\varphi\cos\delta}\,\Delta\delta\,.25465. \quad (257)$$

$$y = [9.40594]\,\frac{\Delta\delta}{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (258)$$

It will frequently be accurate enough to take $y = \frac{\frac{1}{4}\Delta\delta}{A}$.

y is added algebraically to the chronometer time of culmination; the result is the chronometer time of maximum altitude. The difference between this and the chronometer time of observation is $(t - y)$.

Formulæ for Latitude by Circummeridian Altitudes of the Sun.

$$\left.\begin{aligned}
&y = \frac{.25465}{A}\Delta\delta = [9.40594]\frac{\Delta\delta}{A};\\
&{}^*x = A\frac{2\sin^2\frac{1}{2}y}{\sin 1''};\ (t-y) = T' - (E - \Delta T + y);\\
&m = \frac{2\sin^2\frac{1}{2}(t-y)}{\sin 1''};\qquad n = \frac{2\sin^4\frac{1}{2}(t-y)}{\sin 1''};\\
&\varphi = z + \delta_0 + x^* - Am + Bn.
\end{aligned}\right\}\ \text{(XIV)}$$

Correction for Rate of Chronometer.

152. If the times are recorded by a chronometer which has a large rate, the hour-angle used in formulæ (XIII) and (XIV) may require a correction. This correction can be applied in a very simple manner, as follows:

Suppose first a star to be observed by a sidereal chronometer which has a daily rate $\delta\Theta$, + when the chronometer is losing. Then 24 actual sidereal hours correspond to $24^h - \delta\Theta$, as shown by the chronometer, and all hour-angles given in units of chronometer time will be in error in a like ratio.

Let t = any hour-angle as shown by the chronometer;
t' = true value of the hour-angle.

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* x may always be neglected without serious error when z_0 is not too small.

Then
$$\frac{t'}{t} = \frac{24^h}{24^h - \delta\Theta} = \frac{86400^s}{86400^s - \delta\Theta};$$
$$\left(\frac{t'}{t}\right)^2 = \frac{1}{\left[1 - \frac{\delta\Theta}{86400}\right]^2} = k. \quad . \quad . \quad . \quad . \quad . \quad (259)$$

Then in formula (XIII) we shall have with practical accuracy

$$\sin \tfrac{1}{2}t' : \sin \tfrac{1}{2}t = t' : t;$$
$$\sin^2 \tfrac{1}{2}t' = k . \sin^2 \tfrac{1}{2}t.$$

The factor k or log k may be conveniently tabulated with the argument *rate;* and as it will be constant in any series of observations, it may be combined with the factor A, which will then be computed by the formula

$$A = k\frac{\cos\varphi\cos\delta}{\sin z_0}. \quad . \quad . \quad . \quad . \quad . \quad (260)$$

k is given in table VIII, C.

If a star is observed with a mean time chronometer whose rate is δT, the factor $\sqrt{k}$ will convert the chronometer intervals into mean time intervals; we then require the factor μ* = 1.00273791 to convert these mean time intervals into sidereal intervals. The formula for computing A will then be

$$A = k\mu^2\frac{\cos\varphi\cos\delta}{\sin z_0}, \quad . \quad . \quad . \quad . \quad . \quad (261)$$

where log $\mu = .0011874$.

If the sun is observed with a mean time chronometer the intervals of the chronometer corrected for rate will not correspond exactly to the solar intervals, as these will be apparent time intervals.

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* See Art. 93.

If we let δE = the increase of the equation of time in one day, then (one apparent solar day) = (one mean solar day) − δE, and $\delta T - \delta E$ = the chronometer rate on apparent time. k will then be given by the formula

$$k' = \frac{1}{\left[1 - \frac{\delta T - \delta E}{86400}\right]^2}. \quad . \quad . \quad . \quad . \quad . \quad (262)$$

Finally, if the sun is observed with a sidereal chronometer, we must introduce the factor $\frac{1}{\mu}$ to convert the sidereal intervals into mean time intervals.

The $\log \frac{1}{\mu} = 9.9988126$.

The formulæ for the four cases are then as follows:

$$\left.\begin{array}{ll}
k = \dfrac{1}{\left[1 - \dfrac{\delta\Theta}{86400}\right]^2}; & k' = \dfrac{1}{\left[1 - \dfrac{\delta T - \delta E}{86400}\right]^2}; \\
\text{Star with sidereal chronometer,} & A = k\,\dfrac{\cos\varphi\cos\delta}{\sin z_0}; \\
\text{Star with mean time chronometer,} & A = [0.002375]k\,\dfrac{\cos\varphi\cos\delta}{\sin z_0}; \\
\text{Sun with mean time chronometer,} & A = k'\,\dfrac{\cos\varphi\cos\delta}{\sin z_0}; \\
\text{Sun with sidereal chronometer,} & A = [9.997625]k'\,\dfrac{\cos\varphi\cos\delta}{\sin z_0}.
\end{array}\right\}\text{(XV)}$$

k and k' are taken from table VIII, C.

Example. Determination of latitude by circummeridian altitudes of the sun.

1869, July 24th. Des Moines, Iowa. Observer Harkness.

Instruments: Sextant and Mean Time Chronometer.

The declination, equation of time, etc., are taken from the ephemeris for the

instant of the sun's meridian passage at Des Moines = $1^h\ 6^m\ 16^s$ apparent time at Washington.

Assumed Latitude	$\varphi =$	$41°\ 35'.5$
Longitude	$L = +$	$1^h\ 6^m\ 16^s$
Chronometer correction	$\varDelta T = -$	$6\ 18\ 8.9$
From ephemeris,	$\delta =$	$19°\ 46'\ 16''.1$
	$\varDelta\delta =$	-31.94
Equation of time	$E = +$	$6^m\ 12^s.0$
Semidiameter	$S =$	$15'\ 47''.2$
Equatorial hor. parallax	$\pi =$	8.44

Computation of A and B.

$\varphi = 41°\ 35.5$	$\cos = 9.8738$	
$\delta = 19\ 46.3$	$\cos = 9.9736$	$\log A^2 = 0.5544$
$z_0 = 21\ 49.2$	$\text{cosec} = .4298$	$\cot z_0 = .3975$
$A = 1.893$	$\log A = .2772$	$\log B = .9519$
$B = 8.95$		

Computation of y.	Computation of x*.	INDEX ERROR. On arc.	Off arc.
Constant log = 9 4059	$\dfrac{2\sin^2 \frac{1}{2}y}{\sin 1''} = ''.01$	$29'\ 5''$	$33'\ 60''$
$\log \varDelta\delta = 1.5043_n$		10	40
$\log \dfrac{1}{A} = 9.7228$	$x = ''.02$	10	50
$\log y = .6330_n$		$29'\ 8''.3$	$33'\ 50''$
$y = -4^s.3$		$I = +2'\ 20''.8$	

For the chronometer time of culmination we have

Equation of time	$E =$	$0^h\ 6^m\ 12^s.0$
	$\varDelta T = -$	$6\ 18\ 8.9$
	$y = -$	4.3
Chronometer time of max. alt. =		$6^h\ 24^m\ 16^s.6$

The difference between this quantity and the observed time T is the quantity $(t - y)$.

* In reducing sextant observations x may always be disregarded.

The observations and reductions are now as follows:

		Sextant $2h$.	h.	Chronometer T.	$t - y$.	m.	Am.	n.	Bn.	$h + Am - Bn$
Upper limb	1	136°17′45″	68° 8′52″.5	6h 7m51s.5	16m25s.1	529″.1	1001″.6	.68	6″.1	68°25′28″.0
	2	20 10	10 5	8 32	15 44 .6	486 .5	920 .9	.58	5 .2	20 .7
	3	22 20	11 10	9 9 .5	15 7 .1	448 .6	849 .2	.48	4 .3	14 .9
Lower limb	4	135 23 10	67 41 35	10 11 .3	14 5 .3	389 .6	737 .5	.37	3 .3	67 53 49 .2
	5	25 10	42 35	10 55 .3	13 21 .3	350 .1	662 .7	.30	2 .7	35 .0
	6	29 30	44 45	6 12 7 .0	12 9 .6	290 .3	549 .5	.20	1 .8	52 .7

Mean h	$\bar{\odot} =$	68° 25′ 21″.2	$\underline{\odot} =$	67° 53′ 45″.6
Semidiameter	$S = -$	15 47 .2	$+$	15 47 .2
Refraction	$r = -$	21 .6	$-$	21 .8
Parallax	$p = +$	3 .1	$+$	3 .2
Index cor.	$\frac{1}{2}I = +$	1 10 .4	$+$	1 10 .4
Eccentricity	$\frac{1}{2}E = +$	14 .8	$+$	14 .8
Corrected	$h =$	68° 10′ 40″.7		68° 10′ 39″.4
Mean	$h =$	68° 10′ 40″.0		
	$z_0 =$	21 49 20		
	$\delta =$	19 46 16		
Resulting latitude	$\varphi =$	41° 35′ 36″		

The observations of the above series, it will be noticed, were all taken before the sun reached the meridian, and so far from the meridian that the term Bn has a very appreciable value. It is a little better to take the observations near the meridian when practicable, as then small errors in ΔT will produce less effect on the resulting latitude. (See Art. 144.)

The above observations may be reduced by the method of Art. 146 if it is not considered necessary to compare the individual results. The labor is considerably less, as will be seen by the following:

Mean of chronometer times	$=$	6h 9m 47s.8
	$\Delta T = -$	6 18 8 .9
True mean time	$=$	23 51 38 .9
Longitude from Washington	$L =$	1 6 16
Washington mean time	$=$	0 57 54 .9

The declination of the sun is now to be taken from the ephemeris for this mean time of observation, instead of the instant of meridian passage as in the previous method.

Thus
$$\delta = 19^\circ\ 46'\ 23''.8;$$
$$E = \quad 6^m 12^s.0.$$

This value of E is now the mean time of the sun's meridian passage. For the chronometer time we have

$$E = \quad 0^h\ 6^m\ 12^s.0;$$
$$\Delta T = -6\ \ 18\ \ \ 8.9;$$
Chronometer time of the sun's meridian passage $= \quad 6\ \ 24\ \ 20.9.$

Chronometer T.	t.	m.	n.
$6^h\ 7^m\ 51^s.5$	$16^m\ 29^s.4$	$533''.7$	.69
8 32.0	15 48.9	491 .0	.59
9 9.5	15 11.4	452 .9	.49
10 11.3	14 9.6	393 .6	.38
10 55.3	13 25.6	353 .9	.31
6 12 7.0	12 13.9	293 .7	.21

Means: $m' = 419''.8 \qquad n' = .44$
$Am' = 794 .7 \qquad Bn' = 3''.9$

The number of observations on the two limbs being the same, the semi-diameter will be eliminated by taking the mean of the individual values.

Mean of sextant readings	$= 2h =$	$135^\circ\ 53'\ 00''.8$
Index correction	$= I = +$	2 20 .8
Eccentricity	$E = +$	29 .7
Corrected reading	$=$	$135^\circ\ 55'\ 51''.3$
	$h =$	67 57 55 .6
Refraction	$r = -$	21 .7
Parallax	$p = +$	3 .2
	$+ Am = +$	13 14 .7
	$- Bn = -$	3 .9
Corrected altitude	$=$	$68^\circ\ 10'\ 47''.9$
	$z_0 =$	21 49 12
	$\delta =$	19 46 24
Resulting latitude	$\varphi =$	$41^\circ\ 35'\ 36''$

The rate of the chronometer was $\delta T = -\ ^s.47$
The daily increase of the equation of time $\delta E = +\ .63$
$\delta T - \delta E = -\ 1.10$

Therefore the log $k = 9.999989$. (See Art. 152.)
The correction for rate is therefore absolutely inappreciable.

Fifth Method.

153. *By Polaris observed at any hour-angle.* We have already seen (method third) how the latitude may be obtained by an altitude of a star, observed in any position. We have also applied the formulæ deduced to a series of altitudes of Polaris.

A more convenient formula than the one there used is obtained by expanding the expression for the latitude into a series in terms of ascending powers of the polar distance. The latter, in case of Polaris, being at present only about $1° 20'$, the series will converge rapidly, and a very few terms give an approximation sufficiently accurate for every practical purpose.

Let
$$p = 90° - \delta = \text{the polar distance};$$
$$\varphi = h - x.$$

Then x is the correction which is to be applied to the measured altitude—corrected for refraction—to produce the latitude. x can never be greater than p.

Substituting these values in

$$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t,$$

it becomes

$$\sin h = \sin (h - x) \cos p + \cos (h - x) \sin p \cos t. \quad (a)$$

Expanding $\sin (h - x)$ and $\cos (h - x)$ by Taylor's, and

$\sin p$ and $\cos p$ by Maclaurin's formula, we have, as far as terms of the order p^4 and x^4,

$$\sin(h-x) = \sin h - x\cos h - \tfrac{1}{2}x^2\sin h + \tfrac{1}{6}x^3\cos h + \tfrac{1}{24}x^4\sin h;$$
$$\cos(h-x) = \cos h + x\sin h - \tfrac{1}{2}x^2\cos h - \tfrac{1}{6}x^3\sin h + \tfrac{1}{24}x^4\cos h;$$
$$\sin p = p - \tfrac{1}{6}p^3;$$
$$\cos p = 1 - \tfrac{1}{2}p^2 + \tfrac{1}{24}p^4.$$

Substituting these values in (a), we readily obtain

$$\begin{aligned} x = p\cos t &- \tfrac{1}{2}(x^2 - 2xp\cos t + p^2)\tan h \\ &+ \tfrac{1}{6}(x^3 - 3x^2p\cos t + 3xp^2 - p^3\cos t) \qquad (b) \\ &+ \tfrac{1}{24}(x^4 - 4x^3p\cos t + 6x^2p^2 - 4xp^3\cos t + p^4)\tan h. \end{aligned}$$

Which contains all terms in p and x, from the first to the fourth orders inclusive. x must now be determined from (b) by successive approximations. For the first approximation let

$$x = p\cos t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Substituting this value in the second term of (b) and retaining terms of the order p^2, we find for the second approximation

$$x = p\cos t - \tfrac{1}{2}p^2\sin^2 t\tan h. \quad . \quad . \quad . \quad . \quad (d)$$

Substituting this value in the second and third terms of (b) and retaining terms of the order p^3, we find the third approximation, viz.,

$$x = p\cos t - \tfrac{1}{2}p^2\sin^2 t\tan h + \tfrac{1}{3}p^3\cos t\sin^2 t. \quad . \quad (e)$$

Similarly for the fourth and final approximation,

$$\begin{aligned} x = p\cos t &- \tfrac{1}{2}p^2\sin^2 t\tan h + \tfrac{1}{3}p^3\cos t\sin^2 t \\ &- \tfrac{1}{2}p^4\sin^4 t\tan^2 h + \tfrac{1}{24}p^4(4 - 9\sin^2 t)\sin^2 t\tan h. \quad (f) \end{aligned}$$

As x and p will be expressed in seconds of arc, the series must be made homogeneous by multiplying p^2 by $\sin 1''$, p^3 by $\sin^2 1''$, and p^4 by $\sin^3 1''$.

Then the expression for the latitude is

$$\begin{aligned}\varphi = h - p \cos t &+ \tfrac{1}{2} p^2 \sin 1'' \sin^2 t \tan h \\ &- \tfrac{1}{6} p^3 \sin^2 1'' \cos t \sin^2 t + \tfrac{1}{8} p^4 \sin^3 1'' \sin^4 t \tan^3 h \\ &- \tfrac{1}{24} p^4 \sin^3 1'' (4 - 9 \sin^2 t) \sin^2 t \tan h. \quad (263)\end{aligned}$$

Let us now examine separately the last three terms of (263) in order to see when they may be neglected.

Let us write the last term equal to u, viz.,

$$u = \tfrac{1}{24} p^4 \sin^3 1'' (4 - 9 \sin^2 t) \sin^2 t \tan h.$$

Forming the differential coefficient of u with respect to t, placing it equal to zero in order to determine what value of t will make u a maximum, we find

$$\sin t \cos t (2 - 9 \sin^2 t) = 0;$$

from which

$$\sin t = 0; \qquad \cos t = 0; \qquad \sin^2 t = \tfrac{2}{9}.$$

The last of these corresponds to a maximum, as will be found by substituting this value in the second differential coefficient.

The maximum value of this term is then found to be (p being $1° 20'$)

$$u' = 0''.0011 \tan h.$$

It will therefore always be inappreciable.

The next term, viz., $\tfrac{1}{8} p^4 \sin^3 1'' \sin^4 t \tan^3 h$, is a maximum when $\sin t = 1$.

Its greatest value is therefore $0''.0076 \tan^3 h$.

This term will then be only $0''.01$ in latitude $48°$, and $0''.1$ in latitude $67°$. It may therefore always be neglected when the instrument used is the sextant.

Writing $$v = \tfrac{1}{8} p^3 \sin^3 1'' \cos t \sin^2 t,$$

forming $\frac{dv}{dt}$, placing it equal to zero, we readily find that v is a maximum when $\sin^2 t = \frac{2}{3}$. The maximum value of this term will then be $0''.333$. If then we drop this term with those which follow, the error introduced in this way will seldom amount to half a second, and will generally be much smaller as the maxima values of the different terms occur for different values of t.

Therefore for determining the latitude by Polaris by sextant observation,

$$\left.\begin{aligned} t &= \Theta' - (\alpha - \varDelta\Theta); \\ \varphi &= h - p \cos t + [4.38454] p^2 \sin^2 t \tan h. \end{aligned}\right\} \quad \text{(XVI)}$$

Let us apply this method to the example solved in Art. 143. We have given—

From Nautical Almanac.		By Observation.	
$\alpha =$	$1^h\ 15^m\ 6^s.0$	$h =$	$39°\ 33'\ 38''.8$
$\delta =$	$88°\ 41'\ 6''.2$	$\Theta' =$	$10^h\ 45^m\ 7^s.4$
Therefore $p =$	$4733''.8$	$\varDelta\Theta =$	$+1.5$
		Therefore $t =$	$142°\ 30'\ 43''.5$

			constant log	4.38454
		$\log p = 3.675210$	$\log p^2$	7.35042
		$\cos t = 9.899537_n$	$\sin^2 t$	9.56866
			$\tan h$	9.91704
First correction	$-1°\ 2'\ 36''.2$	$\log = 3.574747_n$		
Second correction +	$16\ .6$		log 2d cor. =	1.22066
	Therefore $\varphi = 40°\ 36'\ 31''.6$			

We find the third correction to be 0″.24, which makes the value of φ agree exactly with the value before found (Art. 143).

Tables have been prepared with the design of abridging this computation, but the direct application of the formula is so simple that tables are of no great advantage, especially if the third and fourth corrections are not required.

Correction for Second Differences.

154. When a series of, say, ten altitudes is observed, if the measurements are made in quick succession, so that the arc of the circle in which the apparent motion of the star takes place does not differ appreciably from a straight line, then the mean of the observed altitudes will be the altitude corresponding to the mean of the times. If, however, the deviation from a straight line is appreciable, this mean altitude will require a correction which may be obtained as follows:

Let $t_1, t_2, t_3, \ldots t_n$ be the times of observation;

$h_1, h_2, h_3, \ldots h_n$ be the observed altitudes;

$$t_0 = \frac{t_1 + t_2 + \ldots + t_n}{n}; \qquad \text{(a)}$$

h_0 = the altitude corresponding to the time t_0;

$$\left.\begin{array}{lll} \Delta t_1 = t_0 - t_1, & \text{from which} & t_0 = t_1 + \Delta t_1; \\ \Delta t_2 = t_0 - t_2, & & t_0 = t_2 + \Delta t_2; \\ \cdot \quad \cdot \quad \cdot & & \cdot \quad \cdot \quad \cdot \\ \Delta t_n = t_0 - t_n, & & t_0 = t_n + \Delta t_n. \end{array}\right\} \qquad \text{(b)}$$

Then $\qquad h_0 = f(t_0); \quad h_1 = f(t_1); \quad h_n = f(t_n);$

from (*b*), $\qquad h_0 = f(t_1 + \Delta t_1) = f(t_2 + \Delta t_2) = f(t_n + \Delta t_n). \qquad$ (*c*)

Expanding these expressions by Taylor's formula, we find

$$\left.\begin{array}{l} h_0 = h_1 + \dfrac{dh}{dt}\Delta t_1 + \dfrac{1}{2}\dfrac{d^2h}{dt^2}\overline{\Delta t_1}^2; \\ h_0 = h_2 + \dfrac{dh}{dt}\Delta t_2 + \dfrac{1}{2}\dfrac{d^2h}{dt^2}\overline{\Delta t_2}^2; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ h_0 = h_n + \dfrac{dh}{dt}\Delta t_n + \dfrac{1}{2}\dfrac{d^2h}{dt^2}\overline{\Delta t_n}^2. \end{array}\right\} \qquad \text{(d)}$$

The mean of these values will be

$$h_0 = \frac{h_1 + h_2 + \ldots + h_n}{n} + \frac{dh}{dt}\frac{\Delta t_1 + \Delta t_2 + \ldots + \Delta t_n}{n}$$
$$+ \frac{1}{2}\frac{d^2h}{dt^2}\frac{\overline{\Delta t_1}^2 + \overline{\Delta t_2}^2 + \ldots + \overline{\Delta t_n}^2}{n}. \quad (264)$$

From the values Δt_1, Δt_2, etc., by (*b*), the term multiplied by $\frac{dh}{dt}$ will be zero; but as the quantities $\overline{\Delta t_1}^2$, $\overline{\Delta t_2}^2$, etc., will all be plus, the term multiplied by $\frac{d^2h}{dt^2}$ will not be zero. It should always be taken into account when large enough to be appreciable.

To determine $\frac{d^2h}{dt^2}$ we differentiate the equation

$$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t;$$

when we readily find

$$\frac{d^2h}{dt^2} = -\left(\frac{\cos \varphi \cos \delta}{\cos h}\right) \cos t + \left(\frac{\cos \varphi \cos \delta}{\cos h}\right)^2 \sin^2 t \tan h. \quad . \quad . \quad (265)$$

And since $\cos h = \sin z$, this equation becomes

$$\frac{d^2h}{dt^2} = -A \cos t + A^2 \sin^2 t \tan h. \quad . \quad . \quad . \quad . \quad . \quad . \quad (266)$$

The quantities Δt_1, Δt_2, etc., will be expressed in seconds of time; they must be reduced to arc by multiplying by 15. Also, $\overline{15\Delta t_1}^2$, etc., must be multiplied by $\sin 1''$ in order to make formula (264) homogeneous. The last term will therefore be multiplied by $\frac{1}{2}(15)^2 \sin 1''$, the logarithm of which is $6.73673 - 10$. Therefore formula (264) becomes

$$h_0 = \frac{h_1 + h_2 + \ldots + h_n}{n} + [6.73673]\frac{d^2h}{dt^2}\frac{\overline{\Delta t_1}^2 + \overline{\Delta t_2}^2 + \ldots + \overline{\Delta t_n}^2}{n}. \quad (267)$$

As an example, we may apply formula (267) to the observations of Polaris given in Art. 143, where we have

$$\begin{array}{rrrr}
\Delta t_1 = & 124^s.9 & \overline{\Delta t_1}^2 = & 1560.0 \\
\Delta t_2 = & 72.9 & \overline{\Delta t_2}^2 = & 531.4 \\
\Delta t_3 = & 6.9 & \overline{\Delta t_3}^2 = & 47.6 \\
\Delta t_4 = & -\ 41.1 & \overline{\Delta t_4}^2 = & 168.9 \\
\Delta t_5 = & -\ 156.1 & \overline{\Delta t_5}^2 = & 2436.7 \\
\hline
& & \text{Mean} = & 948.9 \qquad \log = 2.9772
\end{array}$$

By formula (265), with the data given in Art. 143, $\log \dfrac{d^2h}{dt^2} = 8.2898$

constant logarithm $= 6.7367$

Correction $= 0''.01$ $\qquad \log = 8.0037$

In this case the correction is therefore inappreciable.

We may in a manner precisely similar derive the correction to be applied to the mean of the times, to obtain the time corresponding to the mean of the zenith distances: this may be more convenient in certain cases.

The necessity for applying a correction for second differences may generally be avoided by dividing a long series of observations into two or more parts, neither of which shall embrace an interval of time long enough to require such correction. This proceeding has the advantage that in reducing the two halves of the series separately they will mutually check each other.

155. The methods of determining time and latitude which have been given in this chapter are especially adapted to the requirements of the explorer. The observations can generally be obtained more conveniently at night, and both time and latitude will be required. From the observed time the longitude will be obtained, as will be explained more fully hereafter. As we have already shown, the time will be best determined by observing two stars, one east and one west of the meridian, both as near the prime vertical as practicable.

The latitude will generally be most conveniently determined in the northern hemisphere by observing Polaris

north, and another star south, by circummeridian altitudes. Then, with the best attainable approximation to the latitude, the time can be computed by the method of Art. 125. With this value of the time the correct value of the latitude may then be determined by (XIII) and (XVI), and if this differs much from the assumed latitude the time must be recomputed. In extreme cases it may be necessary to recompute the latitude, but with proper care this need not often occur.

As a survey of the line of travel is generally made by means of a compass and odometer (which is a little instrument for recording the number of revolutions of a cart-wheel), the observer always knows his position approximately. The same process, essentially, is followed at sea, where the approximate place of the vessel is always known from the "dead reckoning," which is the course as indicated by the compass and log.

The methods of this chapter are those which are most convenient and useful in practice. On land, where the observer has a certain degree of choice as to time of observation and methods, and where the results must have a considerable degree of accuracy to be of any value, it will seldom be desirable to employ others. At sea, however, the case is somewhat different. It sometimes happens that the determination of the place of the vessel is of the greatest importance when, from cloudy weather or other causes, observations cannot be obtained which are suitable for the employment of the methods of this chapter. Further, a high degree of accuracy is not required for purposes of navigation. Various methods of determining the place of a vessel are therefore given in works on navigation, in order that the mariner may be in a position to utilize any data which he may obtain.

It can readily be seen that by varying the conditions a great variety of solutions of the problem may be obtained. Some of these are exceedingly elegant from a mathematical

point of view. Such, for instance, is the method given by Gauss for determining both the time and latitude from observation of three stars at the same altitude. Thus if h is the common altitude, δ, δ', δ'' the declinations, t, $t + \lambda$, $t + \lambda'$ the hour-angles of the three stars respectively, we have

$$\left.\begin{array}{l} \sin h = \sin \varphi \sin \delta \;\; + \cos \varphi \cos \delta \;\; \cos t; \\ \sin h = \sin \varphi \sin \delta' \; + \cos \varphi \cos \delta' \; \cos (t + \lambda); \\ \sin h = \sin \varphi \sin \delta'' + \cos \varphi \cos \delta'' \cos (t + \lambda'). \end{array}\right\} \quad (268)$$

Three equations from which t and φ may be found. Further than this, as there are three equations, we can also determine h from them, so that the altitude need not be measured at all, but only the instant of time observed when each star reaches the altitude h. If, however, the altitude is measured by the instrument, this process shows the error of the instrument, thus giving us one equation for determining the eccentricity by Art. 116.

If three altitudes of the same star are measured, a similar process gives us three equations for determining the latitude, hour-angle, and declination of the star.

Also, it is evident that two measured altitudes either of the same star or of different stars will give two equations of the form of (268), from which the latitude and hour-angle may be determined.*

A variety of cases may also be considered in which the measured quantity is the azimuth of a star, or three different altitudes of the same star and the differences of the azimuths, or the data may be varied in many ways; but these solutions are of little practical value.

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* For a solution of this problem graphically, see Captain Sumner's New Method of Determining the Place of a Ship at Sea.

Probable Error of Sextant Observations.

156. In all instrumental measurements the error of the result obtained consists of two parts: *first*, that due to the observer; and *second*, that due to instrumental and other sources with which the observer has nothing to do. When the instrument employed is the sextant, the latter consists for the most part of the various undetermined errors noticed in Articles 114–117. In any given series of observations these affect all alike, and therefore nothing is gained in this direction by increasing the number of individual measurements.

With the first class, however, the case is different. These form the accidental errors of observation, and, as they occur in accordance with the law of least squares, their effect diminishes with an increase in the number of measurements.

Let R_0 = the probable error of the mean of a series of observed altitudes;
R_1 = the error due to the observer, not including personal equation;
R_2 = the error due to instrument and causes other than the observer.

Then, by Art. 16, $$R_0 = \sqrt{R_1^2 + R_2^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (269)$$

Thus if the observer could do his part perfectly, he could never diminish the probable error of a single series below R_2.

The values of R_0, R_1, and R_2 for a given instrument and observer may be determined by methods which we have already employed.

Thus (Art. 132) we have found for the probable error of the time determined by a series of ten double altitudes of the sun, $R_1{}^s = \pm\ {}^s.14$. The corresponding error in the double altitude $2h$ is found by the differential formula, viz.,

$$\varDelta 2h = \frac{d2h}{dt}\varDelta t,$$

and for this case we have found $\dfrac{dt}{d2h} = .649$.

Therefore $$\varDelta 2h = \frac{{}^s.14 \times 15}{.649} = 3''.2 = R_1''.$$

From the latitude observations (Art. 149) we have found $2''.6 = R_1''$.

By a discussion of the ninety individual measurements of altitude employed in the investigation of the eccentricity of the sextant (example, Art. 116), Prof. Boss finds the probable error of a single measurement of double altitude to be $\pm\ 14''$, and of the mean of ten measurements $\pm\ 4''.4 = R_1$. From the solution of the equations of condition of the same example we found for the probable

error of a single equation $R_0 = 5''.9$. Therefore by equation (269) $R_2 = 3''.93$. Thus the instrumental probable error is nearly equal to the observer's probable error of a mean of ten measurements.

If now we assume the probable error of a single measurement to be $\pm 14''$ as above, we have for the observer's probable error of the mean of m measurements, by equation (25),

$$R_1 = \frac{14''}{\sqrt{m}},$$

and the total probable error $R_0 = \sqrt{\frac{196}{m} + 15.45}$.

If $m = 1$, $R_0 = 14''.5$; $m = 10$, $R_0 = 5.9$; $m = 50$, $R_0 = 4.4$;
$m = 5$, $R_0 = 7.4$; $m = 20$, $R_0 = 5.0$; $m = 100$, $R_0 = 4.2$.

Thus it appears that with a skilled observer almost nothing is gained by extending the number of observations of a given series beyond ten. Instead, therefore, of multiplying observations in the same circumstances, when accuracy is desired, the circumstances must be varied with a view to eliminating the instrumental errors.

Thus for good results a determination of time or latitude should never depend on a single series, no matter how carefully made or how elaborately the instrumental errors have been investigated. Latitude should be determined by both north and south observations, giving both equal weight, no matter whether determined from an equal number of measurements or not. In like manner time should be determined from observations both east and west combined with equal weights. (See also Harkness, Washington Observations, 1869, Appendix I, page 51.)

CHAPTER VI.

THE TRANSIT INSTRUMENT.

157. When the time is required with extreme accuracy, as in a careful determination of longitude, the methods of the preceding chapter are not adapted to the purpose. The instrument used will then be the transit.

The common form of transit instrument consists essentially of a telescope attached to an axis perpendicularly. As it revolves with the axis the line of collimation produced to the celestial sphere describes a great circle. The instrument is generally mounted so that this great circle is the meridian, and it is used in connection with the sidereal clock or chronometer for determining the instant of a star's transit over the meridian. If our clock is accurately regulated to show sidereal time, such an observed transit gives us at once the star's right ascension, the latter being, as we have seen, the same as the sidereal time of culmination. If, however, we observe a star whose right ascension is already known, this process gives us the error of the clock. The field-transit mounted in the meridian, with which we are at present more particularly concerned, is always used for this latter purpose.

Theoretically the instrument may be used in any vertical plane. It is sometimes used in the plane of the prime vertical for finding the latitude, or in a fixed observatory for finding the declinations of stars. When speaking of the transit instrument simply we understand it to be mounted in the meridian.

FIG. 26.

Description of the Instrument.

158. The transit instrument designed for a fixed observatory, where it is permanently mounted, is much larger and more complete than one designed for use in the field, where it must be transported from place to place. The transit-circle of the Washington observatory, for instance, has a telescope of twelve feet focal length, the aperture being eight and one half inches; it is mounted on massive piers of marble, which rest on a foundation of masonry extending ten feet below the surface of the ground.

Figs. 26, 27, 28, and 29 show different forms of the field-transit used by the coast and other government surveys. Fig. 26 is a very common form. The telescope is 26 inches focal length and 2 inches aperture. It is provided with a diagonal eye-piece for observing transits of stars near the zenith, the magnifying power being about 40 diameters. As may be seen from the figure, the frame folds up so that the entire instrument may be packed in a single box of comparatively small dimensions. The frame rests on three foot-screws by means of which it is levelled, the final adjustment in this direction being made by a fine screw at the right end of the axis, as shown in the figure. At the opposite end is a screw, or pair of screws acting against each other, by means of which the final adjustment in azimuth is made. The two lamps at opposite ends of the axis are for illuminating the field. The axis being perforated, the light enters it, falling on a small mirror at the intersection with the telescope, by which it is reflected down the tube to the eye-piece. The threads of the reticule then appear as dark lines in a bright field. With some instruments there is only one lamp: with two the unequal heating and consequent expansion of the

Fig. 27.

two pivots is to a great extent avoided, also the inconvenience of changing the lamp from one side to the other when the instrument is reversed.

The two small circles attached to the telescope below the axis are called *finding-circles;* they are used for setting the telescope at the proper elevation. They are about 6 inches in diameter. The alidade carries a level, as shown in the figure. The index is generally adjusted so as to read zero when the telescope is horizontal. If then the vernier is set at the meridian altitude of a star and the telescope revolved until the bubble stands in the middle of the tube, the star will be seen in the middle of the field when it passes the meridian. One circle could be made to answer every purpose, but it would read differently in the two positions of the axis, and this would be likely to prove a fruitful source of annoyance. The instrument is reversed by lifting the axis up out of the supports by hand, turning it around and carefully replacing it.

159. Fig. 27 shows a larger and more complete instrument designed for longitude work. The focal length of the telescope is 46 inches, aperture 2¾ inches. Magnifying powers varying from 80 to 120 diameters are used. A special apparatus is provided for reversing the instrument, which will be understood by reference to the figure. The cam worked by the crank below the frame raises the axis out of its supports, when it is turned around and again lowered into its place. One of the finders has two levels attached, one the ordinary finding-level, the other a much finer one for use in determining latitude, as will be explained hereafter.

160. Fig. 28 is a somewhat common form of transit, one end of the axis being made to take the place of the lower half of the telescope. A reflecting prism is placed at the intersection of the telescope with the axis, which bends the

FIG. [illegible],

rays of light at an angle of 90°, the eye-piece being at the end of the axis.

The instrument shown in the figure may be used as a transit, zenith telescope, or azimuth instrument, and is very convenient for use in positions where it is not practicable to have two or three separate instruments. It has, besides, the advantage that, for stars of all zenith distances, the observer occupies the same position: with the common form of instrument the position of the observer is sometimes uncomfortable, which is prejudicial to accuracy.

161. Fig. 29 shows another form of instrument, made for the Coast Survey by Fauth & Co. of Washington. This form was first proposed by Steinheil (*Astronomische Nachrichten*, vol. xxix. page 177). Here a separate tube for the telescope is dispensed with entirely, the axis being made to serve this purpose by placing the object-glass at one end and the eye-piece at the other. The reflecting prism is placed in front of the objective, as shown in the figure, and almost in contact with it. The tube is placed horizontally and in the prime vertical. When the reflecting surface of the prism is adjusted at the proper angle, the image of any star may be made to transit across the threads of the reticule, precisely as in the other forms of instruments.

The instrument shown in the figure has a focal length of 25 inches, and 2 inches aperture. It is fitted with the appliances necessary to adapt it to use as a zenith telescope. It is very compact and portable, and is therefore particularly adapted for use in a rough country where transportation is difficult.

The portable transit instrument is mounted when practicable on a pier of brick or stone, set into the ground deep enough to insure stability. Where such a foundation is not available a log sawed off square and firmly planted in the ground answers a very good purpose. The observatory may be a shed made of boards or a canvas tent.

FIG. [illegible]

The Reticule.

162. This consists of a number of spider-lines arranged as shown in the figure. The middle line is placed as nearly as may be so that a line joining it with the optical centre of the object-glass shall be perpendicular to the axis.

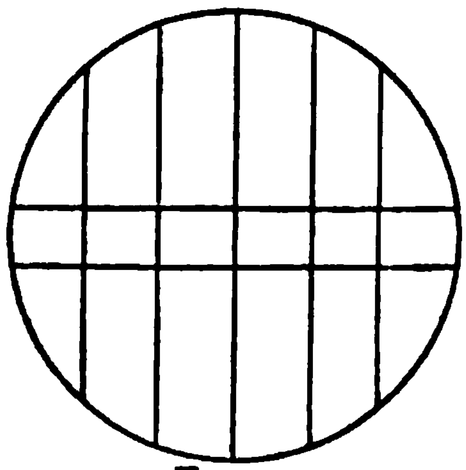

FIG. 30.

In field-instruments a very thin piece of glass ruled with fine lines is often used, and is found more satisfactory in some respects than the spider-threads. In the larger instruments intended to be used with the chronograph there are sometimes as many as twenty-five lines; in the smaller instruments there are usually five or seven—always an odd number. The two horizontal lines are for marking the centre of the field. The instrument should always be set so that the star will pass across the field midway between them.

The Level.

163. Every transit instrument is provided with a delicate striding-level. It is supported by two legs, the bottoms of which are V-shaped. The length is such that these V's rest on the pivots of the axis when the level is placed in the position shown in Figs. 27, 28, and 29. The tube—which is nearly filled with alcohol or sulphuric ether—is apparently cylindrical, but in reality has a curvature of large radius. The bubble of air which is allowed to remain in the tube will always occupy the highest point, and so any change in the relative elevation of the two ends will cause a change in the position of the bubble. It may therefore be used not only for determining when the axis is horizontal, but, by ascertaining the angle corresponding to a motion over one division of

the graduated scale, we may by reading the two ends of the bubble determine the small outstanding deviation from perfect adjustment. The level when so used is a very delicate instrument for angular measurement.

164. *To find the value of one division of the level.* This is most easily accomplished by the use of a little instrument called a *level-trier*, which is simply a bar of wood one end of which rests on two pivots, while the other is supported by a micrometer-screw.

Let $d =$ the distance between two consecutive threads of the screw;
$L =$ the length of the bar between the points of support;
$r =$ the angle corresponding to one revolution of the screw.

Then
$$r = \frac{d}{L \sin 1''}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (270)$$

Suppose the scale of the level to read from the middle in both directions. Call the two ends of the level E. and W. The readings in the direction W. may be considered $+$; those in the direction E., $-$. Let the level be placed on the bar of the trier, and both ends of the bubble read; then let the micrometer-screw be turned so as to cause the bubble to move from its first position, and the two ends read again.

Let e and w be the readings of the bubble in the first position;
e' and w' be the readings of the bubble in the second position;
d, the value of one division of the level;
v, the true angle through which the bar has been moved, as given by the micrometer-screw.

Then $\frac{1}{2}(w - e)$ will be the reading for the middle of the bubble in the first position;

$\frac{1}{2}(w' - e')$ will be the reading for the middle of the bubble in the second position.

$$v = \frac{d}{2}[(w' - e') - (w - e)];$$

from which
$$d = \frac{2v}{(w' - e') - (w - e)}. \quad . \quad . \quad . \quad . \quad . \qquad (271)$$

The operation should be repeated many times in different parts of the tube to insure greater accuracy in the final result, and to test the tube for irregularities.

The following example of determining the value of one division of a level is given by Schott, of the Coast Survey; for brevity only one half of the series is given here:

Coast Survey Office, December 8, 1868. Determination of value of one division of level B, belonging to Transit No. 6. Value of one division of level-trier $= 0''.99$.

	Level-trier.	Level B.		Change for 10 divisions of Trier.	Tempera-ture.
		W.	E.		
$12^h\ 39^m$	210	91.5	9.0		62°.5
				10.75	
	220	80.5	19.5		
				12.00	
	230	68.5	31.5		
				11.25	
	240	57.0	42.5		
				9.25	
	250	47.5	51.5		
				9.00	
	260	38.5	60.5		
				9.25	
	270	29.0	69.5		
				8.75	
	280	20.0	78.0		
				8.75	
$52^h 12^m$	290	11.0	86.5		62°.5

12' 02

The numbers in the last column but one show that the level is not uniform, but there appears to be a gradual change of curvature from one end towards the other. With such a level the extreme divisions ought never to be used. If we take the mean of the quantities in this column we find

10 divisions of level-trier $= 9''.9 = 9.875$ divisions of level.
Therefore $\quad$ 1 division $= 1''.003$.

The determination should be repeated at different temperatures to ascertain whether change of temperature affects the curvature of the tube.

All fine levels are furnished with an air-chamber for regulating the length of the bubble. When using the level this should be kept at about the length which it had when the value of the scale was being determined.

The value of the level may also be determined by placing it on a finely-graduated circle and reading the circle with the bubble in different parts of the tube. Thus by means of the mural circle of the Washington observatory I found the value of one division of the level of a zenith telescope to be $1''.059$, with a probable error of $0''.018$.

165. *Adjustment of the Level of the Transit Instrument.*

The level is used for testing the horizontality of the axis; therefore when it is placed on the axis the tube should be parallel to the latter. If such is the case—

First. The bubble must be in the middle of the tube when the axis is horizontal. Place the level on the axis, and bring the latter approximately horizontal, read the scale, reverse the level and again read the scale. If this adjustment is perfect the reading will ~~be the same~~ * in both positions, otherwise one half the difference of the two readings must be corrected by raising or lowering one end of the tube. The screws for this purpose are shown on the right in Fig. 27. Repeat the process until the adjustment is satisfactory.

Second. The vertical plane passed through the axis must be parallel to that passed through the tube. Let the level be revolved or rocked in both directions around the pivots of the axis. If the reading changes in consequence of this motion the adjustment is not perfect. The direction in which the adjusting-screws must be moved will readily appear from the motion of the bubble. The first adjustment should afterwards be examined, as it may have been disturbed by this operation.

Adjustment of the Instrument.

166. *First. The threads of the reticule must be in the common focus of the object-glass and eye-piece.* First adjust the eye-piece by sliding it in and out of the tube until the position is found where the threads are most distinctly seen. (A mark should then be made on the tube of the eye-piece so that it may be at once set to the proper focus, or a collar may be fitted to it so that when it is pushed "home" it will be in focus.) The instrument should then be turned to a distant terrestrial object, or a star, and the tube carrying the threads set so that the image will remain constantly on one of the threads when the eye is moved to one side or the other of the eye-piece. In some small instruments the threads are fixed at the principal focus of the objective by the maker, with no provision for further adjustment.

167. *Second. The threads must be parallel to a plane perpendicular to the axis of the instrument.* Direct the telescope to a distant well-defined point, and bisect it with the middle thread; move the telescope up and down through a small angle (the axis having been previously levelled). If the thread is vertical it will bisect the object throughout its entire extent.

With some instruments there is an arrangement for revolv-

ing the reticule and consequently for perfecting this adjustment; with others there is none. In any case care should be taken to observe all transits over the same part of the field when a small deviation from true verticality will not be a source of error.

168. *Third. To adjust the line of collimation.* Direct the telescope to a distant terrestrial point, and bisect it with the middle thread; then carefully reverse the telescope, and if the thread does not then bisect the object, bring it half way by means of the adjusting-screws found on each side of the tube which contains the reticule. The operation must be repeated until the adjustment is satisfactory.

Instead of a distant terrestrial point various instrumental devices have been used, particularly in fixed observatories. One of these is the collimating telescope, or collimator as it is called. This is a small telescope placed north or south of the transit instrument, so that when the telescope of the latter is horizontal the observer may look through the eye-piece into the object-glass of the collimator. A thread in the principal focus of the latter will then appear precisely as if seen from an infinite distance, since the rays of light coming from the thread through the object-glass will all emerge in parallel lines. A sharply-defined image of this thread will therefore be found at the principal focus of the transit telescope, and as the thread itself is only a few feet distant, this image will not be disturbed by atmospheric undulations as in the case of a distant mark. By using two collimators, one north and one south, the adjustment may be made without reversing the instrument; this process, however, cannot be conveniently applied to a field-instrument.

The mercury collimator is also much used with the fixed instruments of observatories. This is simply a basin of mercury placed directly under the telescope, so that when the latter is placed vertical with the objective down the

observer can look through the eye-piece into the mercury. The threads will then be seen in the field, together with their images reflected from the mercury. The axis having been carefully levelled, the thread and its reflected image will coincide if there is no error of collimation. If the collimation has been previously adjusted by the collimating telescope, this process may be employed for measuring the inclination of the axis; it is not, however, a suitable method to employ with the portable instrument.

169. *Fourth. To adjust the instrument in the plane of the meridian.* The transit is used in connection with the sidereal chronometer. The observations will be made for determining the error of the chronometer; this is, therefore, presumably not known with any degree of accuracy.

If nothing whatever is known of the chronometer error, it may in certain cases be advisable to determine it approximately by the sextant, or by the altitude of a star measured with the vertical circle of an engineer's theodolite. Such a preliminary determination will very seldom be necessary.

As the approximate time may therefore be known by some process, we first take the best value available. Suppose, for simplicity, the chronometer to be set for this approximate time—or, in other words, that to the best of our knowledge the time shown by the chronometer is correct. We then take from the Nautical Almanac the right ascension of a close circumpolar star, and as this is equal to the sidereal time of culmination, we direct the telescope to the star, level the axis, and at the instant when the time shown by the chronometer equals this right ascension bring the middle thread of the reticule on the star, using the fine-motion screw at the end of the axis for the final adjustment. The instrument will now be approximately in the meridian. We next level the instrument carefully by the fine-motion screw at the end of the axis, and select from the almanac a star which culminates

near the zenith for determining a more correct value of the time, or of the chronometer correction. As all vertical circles pass through the zenith, by selecting a star which passes as near as possible to this point we determine a very close approximation to the true chronometer correction, even when the instrument has a large azimuth error. It is better to use two stars for this purpose, one culminating north of the zenith, and one south (as it will very seldom be possible to find a star culminating exactly in the zenith). If the operations already described have been carefully attended to we shall now know our chronometer correction within a second, which will be accurate enough for perfecting the adjustment in the meridian by another circumpolar star.

Let $\Delta\Theta$ = the value of the chronometer correction just determined;
α = the right ascension of any star.

Then $\alpha - \Delta\Theta$ = the chronometer time of culmination.

When the chronometer indicates this time, the star must be carefully bisected by the middle thread, the axis having been previously levelled. If the observer does not yet feel sufficient confidence in the adjustment, the operation must be repeated for a closer approximation.

The circumpolar stars most suitable for this adjustment are the four standard stars of the Nautical Almanac, viz., α, δ, and λ Ursæ Minoris and 51 Cephei. Besides these the ephemeris for 1885 and following years gives a number of other stars near the pole reduced to apparent place for intervals of ten days.

Methods of Observing.

170. The immediate aim of the observer is to obtain as accurately as possible the instant of time, as shown by the clock or chronometer, when the star crosses each thread of the reticule. These times may then be reduced by a method to be explained hereafter to the time over the middle thread. If then r is the probable error of a transit observed over a single thread, and n the number of threads observed, the probable error of the mean will be $\frac{r}{\sqrt{n}}$.

There are two methods of observing transits, viz., the *eye and ear method* and the *chronographic method.* The latter method is more accurate except with an observer of long experience, and is now used almost universally in fixed observatories. It is also employed in the field when the time is required with great accuracy for longitude work.

In other cases, when the portable instrument is used, the observations will be made by the *eye and ear method,* which is as follows: A few seconds before the star to be observed reaches the thread the observer takes the time from the chronometer and watches the star as it approaches the thread, at the same time counting the beats of the chronometer. When the star crosses the thread the exact instant is noted; if the thread is crossed between two beats, the fractional part of a second is estimated to the nearest tenth. This estimation is made more by the eye than the ear; thus, suppose when the observer counts 10^s the star is at a, and when 11^s at b; the distance from a to the thread will be compared with the distance from a to b, and the ratio will be expressed in tenths. In this case the time will be $10^s.4$. A skilful observer will seldom

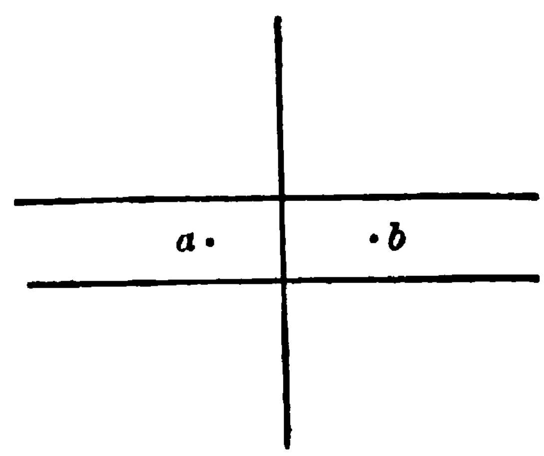

FIG 32.

be in error by so much as $\frac{2}{10}$ of a second in estimating the time over a single thread for a star near the equator.

By the chronographic method the observer registers the instant when the star is on the thread by simply pressing the key which closes or breaks, as the case may be, the galvanic circuit.* This instant is recorded by a mark on the cylinder of the chronograph, and may be read off at leisure. As the observer is not obliged to count the seconds as in the other method, the threads may be placed much closer together and a larger number of readings taken. A practical limit will, however, soon be reached beyond which nothing will be gained in accuracy by increasing the number of threads.

Formerly the large transits of the Coast Survey were provided with twenty-five threads arranged in five groups, or tallies of five threads each. Of late this number has been reduced to thirteen, the central tally containing five threads, the two on each side three each, and the two extreme tallies only one each. The middle threads of the tallies are at equal distances and may be used for eye and ear observation, while the middle tally is convenient for observing close circumpolar stars, which may be best observed by the eye and ear method.

Mathematical Theory of the Transit Instrument.

171. We have shown how to adjust the instrument and place it in the plane of the meridian. With whatever care these adjustments are made, there will always remain small outstanding errors, the existence of which will affect the observed time of a star's transit. The amount of these errors must then be determined, and the necessary corrections applied to the observed time to reduce it to the true time of meridian passage.

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* See Art. 121.

We shall call a line passing through the centres of the pivots and produced indefinitely the *rotation axis*. Also, the line drawn through the optical centre of the object-glass and perpendicular to the rotation axis is the *collimation axis*. When the instrument is revolved this line describes a great circle of the celestial sphere, the poles of which are the points where the rotation axis pierces the sphere. When these poles are known the position of the circle itself is known.

Let $90° - a =$ the azimuth of the point where the west end of the axis pierces the sphere;
$b =$ the altitude of the same point.

Then a will be the deviation of the axis from the true east and west position, plus when the west end deviates to the south; and b is the deviation from the true horizontal position, plus when the west end is high.

Let $90° - m =$ the hour-angle of this point;
$n =$ the declination.
Let x, y, z be the rectangular co-ordinates of this point referred to the horizon.

Then Δ, $90° - a$, and b will be the polar co-ordinates, and we have *

$$\left.\begin{aligned} x &= \Delta \cos b \cos (90° - a) = \Delta \cos b \sin a; \\ y &= \Delta \cos b \sin (90° - a) = \Delta \cos b \cos a; \\ z &= \Delta \sin b. \end{aligned}\right\} \quad (272)$$

Let x', y', z' be the rectangular co-ordinates referred to the equator.

* See equations (110).

Then Δ, $(90° - m)$, and n are the polar co-ordinates, and

$$\left.\begin{aligned} x' &= \Delta \cos n \cos (90° - m) = \Delta \cos n \sin m; \\ y' &= \Delta \cos n \sin (90° - m) = \Delta \cos n \cos m; \\ z' &= \Delta \sin n. \end{aligned}\right\} \quad (273)$$

The formulæ for transformation of co-ordinates will be*

$$\left.\begin{aligned} x' &= x \sin \varphi + z \cos \varphi; \\ y' &= y; \\ z' &= - x \cos \varphi + z \sin \varphi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (274)$$

Substituting for x, y, z and x', y', z' their values, and dropping the common factor Δ, we have

$$\left.\begin{aligned} \cos n \sin m &= \cos b \sin a \sin \varphi + \sin b \cos \varphi; \\ \cos n \cos m &= \cos b \cos a; \\ \sin n &= - \cos b \sin a \cos \varphi + \sin b \sin \varphi. \end{aligned}\right\} \quad (275)$$

Equations (275) give m and n when a and b are known. No limit has been placed to the values of a, b, m, and n, which may therefore be of any magnitude, and consequently the instrument in any position. By careful adjustment, however, these quantities may always be made very small, and there will therefore be no appreciable error in writing the quantities themselves for their sines, and writing for the cosines unity. Therefore

For the transit instrument in the meridian,

$$\left.\begin{aligned} m &= a \sin \varphi + b \cos \varphi; \\ n &= - a \cos \varphi + b \sin \varphi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (276)$$

From these we readily derive

$$\left.\begin{aligned} a &= m \sin \varphi - n \cos \varphi; \\ b &= m \cos \varphi + n \sin \varphi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (277)$$

* See equations (112.)

172. Now let τ = the east hour-angle of a star when seen on the middle thread;
c = the error of collimation; plus when the star reaches the thread too soon.*

Now let the star when on the middle thread be referred to a system of rectangular co-ordinates, the plane of x, y being the plane of the equator, the axis of x being perpendicular to the rotation axis.

Then δ = the star's declination is the angle formed with the plane of x, y, by the radius vector;
$\tau - m$ = the angle formed with the axis of x by the projection of the radius vector on the plane of x, y.

Then
$$\left.\begin{aligned} x &= \Delta \cos\delta \cos(\tau - m); \\ y &= \Delta \cos\delta \sin(\tau - m); \\ z &= \Delta \sin\delta; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (278)$$

y being reckoned towards the east.

Let the star be now referred to a new system of co-ordinates in which the axis of x coincides with that of the last system, the axis of y being the rotation axis of the instrument.

Then c = the angle formed with the plane of x, z, by the radius vector;
δ_1 = the angle formed with the axis of x by the projection of the radius vector on the plane of x, z.

Then
$$\left.\begin{aligned} x' &= \Delta \cos c \cos\delta_1; \\ y' &= \Delta \sin c; \\ z' &= \Delta \cos c \sin\delta_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (279)$$

* The star is supposed to be observed at upper culmination.

In these two systems the axes of x coincide, the axes of y' and z' make the angle n with those of y and z. Therefore

$$\left.\begin{aligned} x' &= x; \\ y' &= y \cos n - z \sin n; \\ z' &= y \sin n + z \cos n. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (280)$$

Combining (278), (279), and (280), we have

$$\left.\begin{aligned} \cos c \cos \delta_1 &= \cos \delta \cos (\tau - m); \\ \sin c &= \cos \delta \sin (\tau - m) \cos n - \sin \delta \sin n; \\ \cos c \sin \delta_1 &= \cos \delta \sin (\tau - m) \sin n + \sin \delta \cos n. \end{aligned}\right\} \quad (281)$$

With these equations, as with (275), no restrictions have been placed on the quantities involved, and they will serve for computing τ when m, n, and c are known. When these quantities are small, as with the instrument adjusted in the meridian, the second of (281) becomes

$$c = (\tau - m) \cos \delta - n \sin \delta;$$

from which $\quad \tau = m + n \tan \delta + c \sec \delta. \quad . \quad . \quad . \quad . \quad (282)$

This is *Bessel's formula* for computing the hour-angle of the star when it passes the middle thread of the reticule. In applying it, the unit in which m, n, and c are expressed must be the second of time.

If we substitute in (282) the value of m from the second of (277), viz.,

$$m = b \sec \varphi - n \tan \varphi,$$

we have $\quad \tau = b \sec \varphi + n (\tan \delta - \tan \varphi) + c \sec \delta. \quad (283)$

This is *Hansen's formula* for computing τ. We see from it that when $\delta = \varphi$, the term in n vanishes and τ depends on b and c alone. From this it follows that those stars are best

suited for determining τ—and therefore the clock correction—which culminate near the zenith.

Substituting in Bessel's formula the values of m and n from (276), we readily find

$$\tau = a\frac{\sin(\varphi - \delta)}{\cos\delta} + b\frac{\cos(\varphi - \delta)}{\cos\delta} + \frac{c}{\cos\delta}. \quad . \quad (284)$$

Which is *Mayer's formula*, and is the one best adapted for use with the portable transit.

We adapt these formulæ to the case of lower culmination by changing δ into $180° - \delta$.

Now let α = the apparent right ascension of any star;
Θ = the observed clock time of the stars passing the middle thread;
$\Delta\Theta$ = the clock correction.

Then
$$\left.\begin{aligned} \alpha &= \Theta + \Delta\Theta + \tau; \\ \Delta\Theta &= \alpha - (\Theta + \tau). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (285)$$

In which τ may be computed by either (282), (283), or (284).

If the star is observed at lower culmination, α becomes $12^h + \alpha$.

Correction for Diurnal Aberration.

173. Aberration is the apparent change in a star's position caused by the progressive motion of light combined with the motion of the earth itself. The displacement is in the direction of the earth's motion, and the tangent of the angle of displacement is equal to the velocity of the earth divided by the velocity of light.

Aberration is considered under two heads, viz., *annual* and *diurnal* aberration, the former resulting from the earth's an-

nual motion in its orbit, and the latter from the revolution on its axis. The subject will be treated in a subsequent chapter as fully as will be necessary for our purposes. At present we shall only consider the *diurnal aberration.*

Let k = the diurnal aberration of an equatorial star at the time of transit. The velocity of light is 186,380 miles per second. A point on the earth's equator has a linear motion of 0.2882 mile per second, in consequence of the diurnal revolution of the earth. Therefore the linear velocity of a point whose latitude is φ will be 0.2882 cos φ. Then

FIG. 33.

$$k = \frac{.2882}{186380 \,.\, \sin 1''} \cos\varphi = ''.319 \cos\varphi = {}^{s}.021 \cos\varphi . \quad (286)$$

If the star's declination is δ, the effect upon the star's hour-angle being k', we have, by applying Napier's first rule for right-angle triangles to the triangle shown in the figure,

$$\sin k = \sin k' \cos\delta;$$

or
$$k' = k \sec\delta = {}^{s}.021 \cos\varphi \sec\delta . \quad . \quad . \quad (287)$$

As this will cause the star to appear too far east, the observed time of culmination will be too late and the correction must be subtracted.

The correction for diurnal aberration may be combined with the collimation constant by making

$$c' = c - {}^{s}.021 \cos\varphi . \quad . \quad . \quad . \quad . \quad . \quad (288)$$

As observations are made in both positions of the axis, it is necessary to distinguish between them. This may be done by noting the position of the clamp, whether it is *east* or *west.* If then the sign of c is determined for *clamp west,* the alge-

braic sign must be changed when the position is *clamp east*. It must be remembered that the algebraic sign of the aberration does not change when the instrument is reversed; so if this correction has been combined with c, c' will in one case be the sum of the two, and in the other case the difference.

Equatorial Intervals of the Threads.

174. When the transit of a star over one of the side threads is observed, we may regard the distance of this thread from the collimation axis as its error of collimation, and proceed with the reduction precisely as in case of the middle thread. It is simpler in practice, however, to determine the angular distances of the side threads from the middle thread, when the times may all be reduced to the time over this thread. This angular distance when expressed in time is evidently the time required for an equatorial star to pass from the side thread to the middle thread.

Let $i =$ the equatorial interval for any thread;
$I =$ the interval for a star whose declination is δ.
Then $i + c =$ the collimation error for this thread;
$\tau + I =$ the hour-angle of a star when seen on this thread.

The second of equations (281) may be written

$$\sin(\tau - m) = \sin c \sec n \sec \delta + \tan n \tan \delta,$$

and for the side thread

$$\sin(\tau + I - m) = \sin(i + c) \sec n \sec \delta + \tan n \tan \delta.$$

By subtraction,

$$\sin(\tau + I - m) - \sin(\tau - m) = [\sin(i + c) - \sin c] \sec n \sec \delta;$$

which becomes

$$2 \cos(\tfrac{1}{2}I + \tau - m) \sin \tfrac{1}{2}I = 2 \cos(\tfrac{1}{2}i + c) \sin \tfrac{1}{2}i \sec n \sec \delta.$$

Since $\tau - m$ and n are very small quantities, the above may be written

$$\sin I = \sin i \sec \delta. \quad . \; . \; . \; . \; . \; . \quad (289)$$

For all stars not nearer the pole than 10°,

$$I = i \sec \delta. \; . \; . \; . \; . \; . \; . \; . \; . \quad (289)_1$$

When I is observed and i is required, the equations become

$$\sin i = \sin I \cos \delta; \quad . \; . \; . \; . \; . \; . \quad (290)$$

$$i = I \cos \delta. \; . \; . \; . \; . \; . \; . \; . \; . \quad (290)_1$$

When the star is nearer the pole than 10°, formulæ which are practically exact are obtained as follows: i may always be written for $\sin i$, and $(I - \frac{1}{6}I^2)$ for $\sin I$. Therefore

$$i = I(1 - \tfrac{1}{6}I^2) \cos \delta.$$

But $\cos I = 1 - \frac{1}{2}I^2$ and $(\cos I)^{\frac{1}{3}} = 1 - \frac{1}{6}I^2$;

therefore we have

$$i = I \cos \delta \sqrt[3]{\cos I}. \; . \; . \; . \; . \; . \; . \quad (291)$$

$$I = i \sec \delta \sqrt[3]{\sec I}. \; . \; . \; . \; . \; . \; . \quad (291)_1$$

The following table gives $\log \sqrt[3]{\cos I}$ and $\log \sqrt[3]{\sec I}$ with the argument I in time:

I.	$\log \sqrt[3]{\cos I}$.	$\log \sqrt[3]{\sec I}$.	I.	$\log \sqrt[3]{\cos I}$.	$\log \sqrt[3]{\sec I}$.	I.	$\log \sqrt[3]{\cos I}$.	$\log \sqrt[3]{\sec I}$.
1m	9.99999	0.00000	16m	9.99965	0.00035	31m	9.99867	0.00133
2	99	01	17	960	040	32	858	142
3	99	01	18	955	045	33	849	151
4	98	02	19	950	050	34	840	160
5	97	03	20	945	055	35	831	169
6	95	05	21	939	061	36	821	179
7	93	07	22	933	067	37	811	189
8	91	09	23	927	073	38	800	200
9	89	11	24	921	079	39	789	211
10	86	14	25	914	086	40	778	222
11	83	17	26	907	093	41	767	233
12	80	20	27	899	101	42	756	244
13	77	23	28	892	108	43	744	256
14	73	27	29	884	116	44	732	268
15	9.99969	0.00031	30	9.99876	0.00124	45	9.99719	0.00281

175. Suppose the reticule to contain five threads.

Let $T =$ the time of a star's passing the middle thread;
$t_1, t_2, t_3, t_4, t_5 =$ the times of passing the separate threads;
$i_1, i_2, i_3,$* $i_4, i_5 =$ the equatorial intervals.

The star is supposed to pass the threads in the above order when the clamp is *west*. When the position is *clamp east*, the order will be reversed, becoming i_5, i_4, i_3, i_2, i_1. At lower culmination the order will be the reverse of that of upper culmination.

$$\begin{aligned} \text{We shall have} \quad T &= t_1 + i_1 \sec \delta \\ &= t_2 + i_2 \sec \delta \\ &= t_3 + i_3 \sec \delta^* \\ &= t_4 + i_4 \sec \delta \\ &= t_5 + i_5 \sec \delta. \end{aligned}$$

* When the reduction is to the middle thread, $i_3 = 0$.

The mean is

$$T = \frac{t_1 + t_2 + t_3 + t_4 + t_5}{5} + \frac{i_1 + i_2 + i_3{}^* + i_4 + i_5}{5} \sec \delta; \quad (292)$$

or

$$T = T_0 + \Delta i \sec \delta \text{ for clamp west;}$$
$$T = T_0 - \Delta i \sec \delta \text{ for clamp east.}$$

Instead of reducing the observed times to the time over the middle thread, we may reduce them to the time over an imaginary thread, the time over which is the mean of the times over the five threads, or T_0 of the above formula. The equatorial intervals and error of collimation are then determined with reference to this *mean thread* instead of the *middle* thread. This method is more convenient than the preceding, as Δi then vanishes and the equatorial intervals are not required when all of the threads are observed.

Reduction of Imperfect Transits.

176. A transit is imperfect when the time over one or more of the threads has not been observed. Formula (292) applies equally to such a transit, by simply dropping the terms corresponding to the threads which were not observed. Thus suppose the first two threads were not observed; the formula will then be

$$T = \frac{t_3 + t_4 + t_5}{3} + \frac{i_3 + i_4 + i_5}{3} \sec \delta.$$

Correction for Rate.

177. If the rate of the chronometer is large, it may be necessary to take it into account in reducing imperfect transits.

* When the reduction is to the middle thread, $i_3 = 0$.

Let $\delta T =$ the hourly rate of the chronometer.

Then if i is given in seconds, we shall have

$$T = t + i \sec \delta \left(1 - \frac{\delta T}{3600}\right). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (293)$$

Thus if a star is observed with a mean time chronometer, $\delta T = 9^s.830$ and (293) becomes

$$\left.\begin{array}{ll} & T = t + i \sec \delta \times 0.99727; \\ \text{or} & T = t + i \sec \delta \, [9.99881]. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (294)$$

Determination of the Constants.

178. We may determine the time of the stars passing the meridian, and consequently the clock correction, from formulæ (284) and (285) when we know the values of a, b, and c, or from formulæ (282) and (285) when we know m, n, and c. The determination of these quantities will therefore now be considered.

The Level Constant, b.

Place the striding-level on the axis and read both ends of the bubble, reverse the level and read again.

Let w and e be the readings of the west and east end in first position;
- w' and e', the readings of the west and east end in second position;
- d, the value of one division of the level expressed in time;
- x, the error of the level due to any want of perfect adjustment.

Then if there were no error the inclination would be equal to the reading of the middle point of the bubble, or

$$b = \tfrac{1}{2}d(w - e) + x;$$
$$b = \tfrac{1}{2}d(w' - e') - x;$$

the mean of which is

$$b = \frac{d}{4}[(w + w') - (e + e')]. \quad . \quad . \quad . \quad (295)$$

The level is often reversed two or more times for greater accuracy. Whatever the number of reversals, the inclination is given by the formula

$$b = \frac{d}{2}[W - E]; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (296)$$

where W and E are respectively the means of the east and west readings.

Inequality of Pivots.

179. The above expression for b is obtained by applying the level to the outer surface of the pivots; it therefore gives the true inclination of the rotation axis only when the diameters of the pivots are equal. If they are unequal this value of b requires a correction determined as follows:

Fig 34b is a cross-section of one of the pivots, with the V of the level B, and of the instrument A. Suppose the clamp

Fig. 34a. Fig. 34b.

west. Formula (295) gives the inclination of the line $B'B$; that of $C'C$ is required. Suppose the V of the level to have the same angle as the of the instrument.

Let B and B' be the inclinations as shown by the level for clamp west and east respectively;
b and b', the true inclinations of $C'C$;
β, the constant inclination of $A'A$;
p, the angle $ECC' = C'CF$.

$$\left.\begin{array}{lll} \text{For clamp west,} & b = B + p; & b = \beta - p; \\ \text{For clamp east,} & b' = B' - p; & b' = \beta + p. \end{array}\right\} \qquad (a)$$

By subtraction, $b' - b = B' - B - 2p = 2p$;

$$p = \frac{B' - B}{4}. \qquad (297)$$

Which determines the value of p. In order to be reliable it must be derived from a large number of readings of the level in both positions of the axis. It will then be a correction to be added algebraically to the inclination as given by the level for the position *clamp west*, or

$$\left.\begin{array}{l} b = B + p \text{ for clamp west;} \\ b' = B' - p \text{ for clamp east.} \end{array}\right\} \qquad (b)$$

If the angle of the level V is not equal to that of instrument V, the angle ECC' will not be equal to $C'CE$ and we proceed as follows:

Let
$2i$ = the angle of the level V;
$2i_1$ = the angle of the instrument V;
r and r' = the radii of the pivots;
$d = BC$ in the figure;
$d_1 = AC$ in the figure;
L = length of level = $C'C$;
p = angle ECC';
p_1 = angle $C'CF$;

the notation in other respects remaining as before.

Then for end next the clamp $d = \frac{r}{\sin i}$; $d_1 = \frac{r}{\sin i_1}$.

Then for end remote from clamp $d' = \frac{r'}{\sin i}$; $d_1' = \frac{r'}{\sin i_1}$.

$$\sin p = \frac{d' - d}{L} = \frac{r' - r}{L \sin i}; \quad \text{therefore} \quad p = \frac{r' - r}{L \sin i \sin 15''}. \quad . \quad (c)$$

$$\sin p_1 = \frac{d_1' - d_1}{L} = \frac{r' - r}{L \sin i_1}; \quad p_1 = \frac{r' - r}{L \sin i_1 \sin 15''}. \quad . \quad . \quad (d)$$

Dividing (295) by (294) we have $\frac{p_1}{p} = \frac{\sin i}{\sin i_1}$. . . . . (e)

Then
$$b = B + p; \quad b = \beta - p_1;$$
$$b' = B' - p; \quad b' = \beta + p_1;$$
$$b' - b = B' - B - 2p = 2p_1;$$
$$\frac{B' - B}{2} = p + p_1.$$

Substituting the value of p_1 from (296) and reducing, we readily find

$$p = \frac{B' - B}{2}\left(\frac{\sin i_1}{\sin i + \sin i_1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (297)$$

Example. The following readings of the level were made for determining the inequalities of the pivots of the transit instrument of the Sayre observatory.

	Clamp East.			Clamp West.	
	E.	W.		E.	W.
Direct,	14.4	15.1		12.8	16.2
Reversed,	12.7	16.7		14.6	14.9
$(e + e') =$	27.1	31.8	$= w + w'$	27.4	31.1

By formula (295), $B' = + 1.175$; $B = + .925$;

B and B' being expressed in terms of one division of the level.

The angle of the level V was equal to that of the transit; therefore, by (297),

$$p = \frac{B' - B}{4} = + .062.$$

By a considerable number of readings made at different times the following values of p were obtained. The first and third columns show the angle of elevation of the telescope, the second and fourth the corresponding values of p.

0°	+.056	125°	.042
10	.080	130	.059
20	.068	140	.052
30	.056	150	.076
40	.077	160	.069
50	.046	170	.064
60	.062		

$$\text{Mean of 13 values } p = \mp .062 \begin{cases} \text{clamp east;} \\ \text{clamp west.} \end{cases}$$

The value of one division of the level is $d = {}^s.174$; therefore $p^s = {}^s.011$.

180. The diameters of the pivots may not only be unequal, but the forms may be irregular. This is tested by reading the level with the telescope placed at different zenith distances. If inequalities are found to exist, a table of corrections for different zenith distances from zero to 90° on each side of the zenith may be formed in case it is necessary to use the instrument in this condition. If the corrections are large enough to be appreciable, however, the instrument should be put into the hands of an instrument-maker for repairs.

181. A little instrument designed by Prof. Harkness, and called by him the "spherometer-caliper," is very convenient for measuring the inequalities and irregularities of pivots. Fig. 35*a* is a front and 35*b* a side elevation. The same

letters refer to both figures. The foundation-plate *b* carries two cylindrical guides, *dd*, which are connected at their lower end by the bar *e*. Into the foundation-plate is screwed the brass piece *m*, to which is cemented the thick circular glass plate *c*. The two V's, *aa*, are also firmly screwed to the foundation-plate. The brass plate *f* slides freely up and down

FIG. 35a. FIG. 35b.

THE SPHEROMETER-CALIPER.

between the guides *dd*, being kept in place by three loops, two of which pass around the right-hand guide and one around the left, as shown in the figure. The brass rod *g*, which passes through the piece *m* and the plate *c* without touching either of them, is firmly attached to the upper end of the plate *f*, and moves with it, while to the lower end of

f is attached a second short brass rod which passes freely through the bar e and carries the nut h.

In using the instrument, the plate f is depressed by means of the nut h until one of the pivots whose irregularity is to be measured passes freely under the V's aa. Then the V's having been properly adjusted upon the pivot, h is loosened and the flat edge of the aperture in f is pressed against the under side of the pivot by the spring i. The elevation of the rod g above the glass plate is then measured by means of the spherometer. This consists of the micrometer-screw shown in the figure, which is supported by the small tripod s, the legs of which rest on the glass plate. By means of this screw small differences in the elevation of the rod g, and consequently of the size of the pivots, may be readily measured.

Let $2v =$ the angle of the V's aa;
$n =$ the difference between the readings of the screw on the two pivots;
$R =$ the linear distance between two consecutive threads of the screw;
$L =$ the distance between the V's of the transit instrument;
$p =$ the inequality of the pivots expressed in seconds of time;
$r =$ the radius of the pivot to be measured;
$C =$ the distance from the upper surface of the glass plate to the angle of the V's.

Then the vertical distance from the upper surface of the glass plate to the flat surface of the aperture in f will be

$$C + r + \frac{r}{\sin v} = C + r\left(\frac{1 + \sin v}{\sin v}\right). \quad . \quad . \quad (298)$$

Similarly for the other pivot

$$C + r'\left(\frac{1 + \sin v}{\sin v}\right). \quad . \quad . \quad (299)$$

The difference is $(r - r')\left(\frac{1 + \sin v}{\sin v}\right)$. . . . . . (300)

This is evidently the difference in the elevation of the end of the rod g when the second pivot is substituted for the first; that is, the difference between the two micrometer readings. Therefore

$$(r - r')\left(\frac{1 + \sin v}{\sin v}\right) = nR;$$

$$r - r' = \frac{nR \sin v}{1 + \sin v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (301)$$

Then from (294)

$$p = \left(\frac{nR}{1 + \sin v}\right)\frac{1}{L \sin 15''}. \quad . \quad . \quad . \quad . \quad (302)$$

This instrument is especially to be recommended in examining the pivots for irregularities, as by measuring different diameters of the pivot the exact form may be determined. If irregularities exist they may be detected by the level, but it will not show which pivot is irregular.

The Collimation Constant, c.

182. A transit instrument of the better class is provided with a micrometer,* the movable thread of which is parallel to the threads of the reticule and so nearly in the same plane that both are in the focus of the eye-piece at the same time.

* For description of micrometer see Art. 97.

With this arrangement the error of collimation may be measured directly as follows:

By means of a distant terrestrial object. The position being *clamp west*—suppose—direct the telescope to a distant terrestrial point, and by means of the micrometer measure the distance of its image as seen in the field from the middle thread, then reverse the instrument and measure the distance again. If the object appears on the same side of the thread in both positions, the error of collimation will be half the difference of the measured distances; if on opposite sides, half their sum.

In determining c in this way care must be taken not to mistake its algebraic sign. This sign may be determined practically by remembering from which side of the field a star at upper culmination appears to enter. If then for *clamp west* the thread appears nearer that side of the field than for *clamp east*, c will be plus for *clamp west*, and minus for *clamp east*.

183. *By the collimating telescope.** The thread or cross-threads of a collimating telescope may be used in the same way as a distant terrestrial object for measuring the collimation constant, and with the advantage that there will be no appreciable atmospheric disturbance, the mark being only a few feet distant. With two collimating telescopes, one north and one south of the instrument, the error may be determined without reversing the instrument. As this method is only of practical value with the large instruments of an observatory, it will not be explained further here.

184. *By the mercury collimator.** If the telescope is directed vertically downwards, the middle thread may be seen directly, together with its image reflected from the mercury. If the axis is horizontal the constant c will be one

* Art. 168.

half the distance between the direct and reflected images, which may be measured as before.

If the axis is not horizontal,

Let b = the elevation of the west end;

M = the micrometer distance of the thread from its image, positive when the thread itself is on the side from which a star at upper culmination appears to enter.

Then
$$\tfrac{1}{2}M = c - b;$$
$$c = \tfrac{1}{2}M + b. \quad \ldots \ldots \quad (303)$$

By reversing the instrument and again measuring the distance of the thread from the reflected image we can determine both b and c, or if c has been determined by the collimating telescopes we can determine b without reversing the instrument.

185. *By a close circumpolar star.* With the portable instrument it will be found more convenient to determine the collimation constant by observation of a star in both positions of the axis, as follows:

Observe the transit of a slow-moving star over one or more threads—including the middle thread or not—then reverse the instrument and observe the transit over the same threads, now on the other side of the field. With one of the four circumpolar stars of the Nautical Almanac there will be plenty of time to reverse the instrument during the interval over two consecutive threads. It is advisable to read the level for each thread.

The times observed are then to be reduced to the times over the middle thread (or the mean thread, as the case may be) by means of the equatorial intervals, which must be well determined.

Let T = the clock time over the middle (or mean) thread for clamp west;
T' = the clock time over the middle (or mean) thread for clamp east;
b and b' = the level constants in the two positions;
$\varDelta T$ and $\varDelta T'$ = the clock corrections at times T and T';
$\varDelta T_0$ = the clock correction at time T_0;
δT = hourly rate of clock.

Then
$$\varDelta T = \varDelta T_0 + \delta T(T - T_0);$$
$$\varDelta T' = \varDelta T_0 + \delta T(T' - T_0).$$

Then applying Mayer's formula, (284) and (285),

$$\left.\begin{aligned}&\text{Cl. W. } \alpha = T + \varDelta T_0 + \delta T(T - T_0) + a\sin(\varphi - \delta)\sec\delta\\ &\quad + b\cos(\varphi - \delta)\sec\delta + c\sec\delta - {}^s.021\cos\varphi\sec\delta;\\ &\text{Cl. E. } \alpha = T' + \varDelta T_0 + \delta T(T' - T_0) + a\sin(\varphi - \delta)\sec\delta\\ &\quad + b'\cos(\varphi - \delta)\sec\delta - c\sec\delta - {}^s.021\cos\varphi\sec\delta.\end{aligned}\right\}\quad(304)$$

Subtracting the first of these from the second, we readily find

$$c = \tfrac{1}{2}(T' - T)\cos\delta + \tfrac{1}{2}(T' - T)\,\delta T\cos\delta + \tfrac{1}{2}(b' - b)\cos(\varphi - \delta). \quad (305)$$

This formula is applicable to lower culmination by changing δ into $180° - \delta$ as usual. In most cases the term in δT will be inappreciable.

The Azimuth Constant, a.

186. This can only be determined by observation of stars. Let two stars be observed which differ as widely as possible in declination.

Let T and T' be the times of observation reduced to the middle (or mean) thread;

δ and δ', the declinations of the stars;

α and α', their right ascensions.

Then equations (304) will apply to these stars, except that in the second we shall have α' and δ' in place of α and δ, and the sign of c is not changed.

Let us write

$$t = T + \delta T(T - T_0) + b \cos(\varphi - \delta) \sec\delta + c \sec\delta - {}^{s}.021 \cos\varphi \sec\delta;$$
$$t' = T' + \delta T(T' - T_0) + b' \cos(\varphi - \delta') \sec\delta' + c \sec\delta' - {}^{s}.021 \cos\varphi \sec\delta'.$$

That is, we place t and t' equal to the sum of the known quantities in the second members of the equations. Equations (304) then become

$$\alpha = t + \Delta T_0 + a \sin(\varphi - \delta) \sec\delta;$$
$$\alpha' = t' + \Delta T_0 + a \sin(\varphi - \delta') \sec\delta'.$$

From which

$$a = \frac{(\alpha' - \alpha) - (t' - t)}{\sin(\varphi - \delta')\sec\delta' - \sin(\varphi - \delta)\sec\delta}; \qquad (306)$$

which reduces to

$$a = \frac{(\alpha' - \alpha) - (t' - t)}{\cos\varphi(\tan\delta - \tan\delta')}. \quad . \quad . \quad . \quad (307)$$

The greater the denominator of this fraction the smaller will be the effect upon a of errors of observation. If two circumpolar stars are observed, one at upper and one at lower culmination, the denominator of 307 becomes

$$\cos\varphi\,[\tan\delta - \tan(180^\circ - \delta')] = \cos\varphi(\tan\delta + \tan\delta').$$

This combination is therefore most favorable for the purpose. If the rate of the clock and the stability of the instrument can be relied on for twelve hours, the same star may be observed both at upper and lower culmination. This will not be practicable, however, with a portable instrument. If two stars are observed at upper culmination, one should be near the pole and the other near the equator.

If m and n are required, they may now be computed by (276), or we may proceed as follows.

To Determine n Directly.

187. Using the same notation as in the determination of a, and applying Bessel's formula, (282),

$$\begin{aligned} \alpha &= T + \Delta T_0 + \delta T(T - T_0) + m + n \tan \delta + c \sec \delta \\ &\qquad - {}^s.021 \cos \varphi \sec \delta, \\ \alpha' &= T' + \Delta T_0 + \delta T(T' - T_0) + m + n \tan \delta' + c \sec \delta' \\ &\qquad - {}^s.021 \cos \varphi \sec \delta', \end{aligned}$$

placing the known terms of the second members equal to t and t' respectively, viz.,

$$\begin{aligned} t &= T + \delta T(T - T_0) + c \sec \delta - {}^s.021 \cos \varphi \sec \delta, \\ t' &= T' + \delta T(T' - T_0) + c \sec \delta' - {}^s.021 \cos \varphi \sec \delta', \end{aligned}$$

the above equations become

$$\begin{aligned} \alpha &= t + \Delta T_0 + m + n \tan \delta; \\ \alpha' &= t' + \Delta T_0 + m + n \tan \delta'. \end{aligned}$$

From these we derive

$$n = \frac{(\alpha' - \alpha) - (t' - t)}{\tan \delta' - \tan \delta}. \quad . \quad . \quad . \quad . \quad (308)$$

Then m is given by the second of (277), viz.,

$$m = b \sec \varphi - n \tan \varphi. \quad . \quad . \quad . \quad . \quad (309)$$

The conditions favorable for an accurate determination of n are evidently the same as in the case of a.

Recapitulation of Formulæ for Transit Instrument in the Meridian.

$$\left.\begin{array}{ll}
\text{Equatorial intervals,} & i = I \cos \delta \sqrt[3]{\cos I}; \\
 & i = I \cos \delta. \\
\text{Reduction to middle} & I = i \sec \delta \sqrt[3]{\sec I}; \\
\quad \text{(or mean) thread,} & I = i \sec \delta. \\
\text{Level constant,} & b = \frac{d}{2}[W - E]. \\
\text{Collimation constant,} & c = \frac{1}{2}(T' - T) \cos \delta + \frac{1}{2}(T' - T)\delta T \cos \delta \\
 & \qquad + \frac{1}{2}(b' - b) \cos (\varphi - \delta.) \\
\text{Azimuth constant,} & a = \dfrac{(\alpha' - \alpha) - (t' - t)}{\cos \varphi(\tan \delta - \tan \delta')}. \\
\text{Clock correction,} & \Delta T = \alpha - \left[T + a\dfrac{\sin(\varphi - \delta)}{\cos \delta} + b\dfrac{\cos(\varphi - \delta)}{\cos \delta}\right. \\
 & \qquad \left. + \dfrac{c}{\cos \delta} - \dfrac{^s.021 \cos \varphi}{\cos \delta}\right].
\end{array}\right\} \text{(XVII)}$$

For reduction by Bessel's formula we have the following:

$$\left.\begin{array}{l}
n = \dfrac{(\alpha' - \alpha) - (t' - t)}{\tan \delta' - \tan \delta}; \\
m = b \sec \varphi - n \tan \varphi; \\
\Delta T = \alpha - [T + m + n \tan \delta + c \sec \delta \\
\qquad\qquad - {}^s.021 \cos \varphi \sec \delta].
\end{array}\right\} \text{(XVIII)}$$

Transit Observations.

To illustrate the application of (XVII) let us reduce the following observations, made at the Sayre observatory, 1883, October 16. The transit is a small-sized instrument of 26 inches focal length, aperture 2 inches; magnifying

power 40 diameters. The reticule contains five threads, numbered consecutively from 1 to 5 for clamp east. As will be seen, the level was generally read two or more times in each position.

1883, October 16.

Polaris.*

$\delta = 88° 41' 23''.8$

Clamp west	V	0^h	53^m	34^s
	IV	1	5	31
	III	1	17	25
Clamp east	IV	1	29	3
	V	1	40	55
Mean clamp W.		1^h	17^m	$23^s.4$
clamp E.		1	17	7 .2
$\alpha =$		1	17	28 .83

Level. Clamp West.

E.	W.
14.9	14.5
13.0	16.6
14.6	14.7
13.0	16.5
14.7	14.7
12.9	16.6
13.0	16.8
15.2	14.3
Mean = 13.912	15.588

† $\beta = +.838$

Level. Clamp East.

E.	W.
13.0	16.7
15.3	14.5
13.0	16.8
14.7	15.0
13.1	16.8
14.8	14.9
13.983	15.783

$\beta' = +.900$

β Arietis. $\delta = 20° 14'.5$				γ Andromedæ. $\delta = 41° 46'.1$			
I			$45^s.$	I			$9^s.3$
II			2 .5	II			31 .2
III			19 .8	III			52 .9
IV			37 .1	IV			14 .8
V	1^h	48^m	$54^s.5$	V	1^h	57^m	37^s
$T =$	1	48	19 .78	$T =$	1	56	53 .04
$\alpha =$	1	48	15 .35	$\alpha =$	1	56	48 .81

ϵ Arietis. $\delta = 22° 54'.8$				ξ' Ceti. $\delta = 8° 18'.2$			
I			$8^s.9$	I			$23^s.9$
II			26 .2	II			40 .9
III			43 .9	III			56 .9
IV			1 .9	IV			13 .4
V	2^h	1^m	$19^s.2$	V	2^h	7^m	$29^s.9$
$T =$	2	0	44 .02	$T =$	2	6	57 .00
$\alpha =$	2	0	39 .54	$\alpha =$	2	6	52 .35

Level.

E.	W.
14.7	15.3
12.5	17.9
14.7	15.7
12.7	17.8
13.65	16.675

$\beta = + 1.512$

* Instrument reversed for the purpose of determining the value of c.

† $\beta = \frac{1}{2}[W - E] \therefore b = d \,.\, \beta$.

γ Trianguli.

I	52^s
II	11 .9
III	31 .1
IV	50 .8
V	$2^h\ 11^m\ 9^s.9$
$T =$	2 10 31 .14
$\alpha =$	2 10 26 .83

5 Ursæ Minoris, s.p.
$\delta = 103°\ 47'\ 8''$

V	
IV	$38^s.4$
III	$2^h\ 27^m\ 47^s.3$
II	54 .9
I	3 .5
$T =$	2 27 46 .85
$\alpha =$	14 27 40 .14

δ Ceti.
$\delta = -0°\ 10'6$

I	$5^s.3$
II	21 .8
III	37 .9
IV	54
V	$2^h\ 34^m\ 10^s.9$
$T =$	2 33 37 .98
$\alpha =$	2 33 33 .35

γ Ceti.
$\delta = 2°\ 44'.8$

I	
II	$6^s.9$
III	23
IV	39 .4
V	$2^h\ 37^m\ 55^s.9$
$T =$	2 37 23 .12
$\alpha =$	2 37 18 .54

Level.

E.	W.
12.6	17.9
15.0	15.7
12.6	17.9
15.1	15.8
13.825	16.825
$\beta' = +1.500$	

σ² Arietis.
$\delta = 14°\ 35'.9$

I	$37^s.2$
II	54 .2
III	10 .9
IV	27 .8
V	$2^h\ 45^m\ 44^s.9$
$T =$	2 45 11 .00
$\alpha =$	2 45 6 .57

47 Cephei.
$\delta = 78°\ 57'\ 18''$

I	$2^h\ 48^m\ 1^s$
II	49 27 .8
III	50 51 .5
IV	52 18
V	$2^h\ 53^m\ 42^s$
$T =$	2 50 52 .06
$\alpha =$	2 50 50 .41

Level.

E.	W.
15.0	15.4
13.6	17.3
15.0	15.3
13.7	17.0
14.325	16.25
$\beta' = +.962$	

The values of the apparent right ascensions and declinations are taken from the American Ephemeris, and are written down in connection with the observed transit of each star. α must be taken from the ephemeris with extreme accuracy, but generally δ will be sufficiently accurate if given to the nearest minute of arc.

Let us first compute the values of the equatorial intervals of the threads by the first of formulæ (XVII), taking for this purpose the observations on 47 *Cephei*. The numbers in the first column of the following table are obtained by subtract-

ing the observed time of transit over each thread from the mean of the times over all the threads. The quantities in the following columns will require no further explanation.

$\cos \delta = 9.28235.$

I.	log I.	log $\sqrt[3]{\cos I}$.*	log i.	i.
+ 171ˢ.06	2.23315	9.99999	1.51549	+ 32ˢ.77
+ 84 .26	1.92562	9.99999	1.20796	+ 16 .14
+ .56	9.74819		9.03054	+ .11
− 85 .94	1.93420	9.99999	1.21654	− 16 .46
− 169 .94	2.23029	9.99999	1.51263	− 32 .56

From a considerable number of transits the following values of the equatorial intervals were finally obtained:

Clamp east i_1	+ 32ˢ.628	log = 1.51359
i_2	+ 16 .226	1.21021
i_3	+ .080	8.90309
i_4	− 16 .357	1.21370_n
i_5	− 32 .588	1.51305_n

We can now use these values for reducing the incomplete transits of *Polaris*, 5 *Ursæ Minoris*, and *γ Ceti*.

In cases where the transit is observed over the five threads the arithmetical mean is taken.

Let us compute the reduction of *Polaris* in full.

$\cos \delta = 8.35913;$
$\log \sec \delta = 1.64087.$

log i.	log $\sqrt[3]{\sec. I}$.*	log I.	I.	Time reduced to Mean Thread.
Clamp west.				
1.51305	.00078	3.15471	+ 23ᵐ 47ˢ.9	1ʰ 17ᵐ 21ˢ.9
1.21370	.00020	2.85477	+ 11 55 .8	17 26 .8
8.90309_n		$.54396_n$	− 3 .5	17 21 .5
Clamp east.				
1.21370_n	.00020	2.85477	− 11 55 .8	1 17 7 .2
1.51305_n	.00078	3.15471	− 23 47 .9	1 17 7 .1

Clamp west, mean 1ʰ 17ᵐ 23ˢ.4;
Clamp east, mean 1 17 7 .2.

---

* See table, Art. 174. Page 292.

The value of I used in taking $\sqrt{\sec I}$ from the table is obtained by subtracting the times of transit over threads V and IV respectively from the time over the middle thread. Thus we have from the observation

$$I_v = 23^m\ 41^s; \qquad I_{iv} = 11^m\ 54^s.$$

The quantity marked β or β', in connection with the observations, is the inclination of the axis in terms of one division of the level, uncorrected for inequality of pivots.

From the first level-reading we have

$$\beta = +\ .838;$$
$$^{*}p = +\ .062.$$

Corrected, $\beta = +\ .900;\qquad d = {}^s.174.$

Therefore $b = +\ .157.$

The value of b used for those stars in connection with which the level is not directly read is obtained by interpolating between the observed values. Thus we have—

Star.	β.	β corrected for p.	b.
Polaris, clamp west....	+ .838	.900	+ ^{s}.157
Polaris, clamp east.....	+ .900	.838	.146
β Arietis..............			.167
γ Andromedæ..... ...			.188
α Arietis.............			.209
ξ' Ceti...............			.230
γ Trianguli.	+ 1.512	1.450	.252
5 Ursæ Minoris, s.p....			.252
δ Ceti.................			.251
γ Ceti................	+ 1.500	1.438	.250
σ^2 Arietis..............			.204
47 Cephei.............	+ .962	.900	.157

For computing the error of collimation c we have, from the observed transits of *Polaris*,

Clamp east	$T' = 1^h\ 17^m\ 7^s.2$	$b' = {}^s.146$	$\varphi = 40°\ 36'\ 24''$
Clamp west	$T = 1\ \ 17\ \ 23.4$	$b = +\ .157$	$\delta = 88\ \ 41\ \ 24$
	$T' - T = -\ 16.2$	$b' - b = -\ .011$	$\varphi - \delta = -48\ \ 5\ \ 0$

* Example, Art. 179.

$$\begin{array}{rlrl} \log \tfrac{1}{2}(T' - T) = & 0.90849_n & \log \tfrac{1}{2}(b' - b) = & 7.74036_n \\ \cos \delta = & 8.35913 & \cos(\varphi - \delta) = & 9.82481 \\ \text{sum} = & 9.26762_n & & 7.56517_n \\ \text{Nat. No.} & -.1852 & \text{Nat. No.} & -.0037 \end{array}$$

Therefore $\qquad c = \mp {}^s.1889 \text{ clamp } \left\{ \begin{matrix} \text{west} \\ \text{east} \end{matrix} \right\}.$

In applying the formula of (XVII), the term $\tfrac{1}{2}(T' - T)\,\delta T \cos \delta$ has been disregarded, as in this case it is inappreciable. It is convenient to combine the correction for diurnal aberration with c.

Thus, if we write $\qquad c' = c - {}^s.021 \cos \varphi,$

we have in this case $\qquad c' = + {}^s.173$ clamp east,

$\qquad c' = - {}^s.205$ clamp west.

The last but one of (XVII) will now give us the azimuth constant a.

We have seen that the best result is to be expected when we use the observed transits of two circumpolar stars, one at upper and the other at lower culmination. We therefore determine this constant from 5 *Ursæ Minoris* and 47 *Cephei*.

Referring to the derivation of the formula for a (Art. 186), we have for t and t'

$$t = T + b \cos(\varphi - \delta) \sec \delta + c' \sec \delta;$$
$$t' = T' + b' \cos(\varphi - \delta') \sec \delta' + c' \sec \delta';$$

the term in δT—the rate—being inappreciable.

The computation is then as follows:

5 Ursæ Minoris, s.p.

$$\begin{array}{rlrl} \delta = & 103^\circ\ 47'\ 8'' & \log \sec = & 0.62290_n = \log C \\ \varphi = & 40\ \ 36\ \ 24 & & \\ \varphi - \delta = & -\ 63\ \ 10\ \ 44 & \log \cos = & 9.65438 \\ & & \text{Sum} = & .27728_n = \log B \\ b = & 0^s.252 & \log b = & 9.40140 \\ c' = & +\ .173 & \log c' = & 9.23805 \\ Bb = & -\ .477 & \log Bb = & 9.67868_n \\ Cc' = & -\ .726 & \log Cc' = & 9.86095_n \\ \hline T = & 2^h\ 27^m\ 46^s.85 & & \\ Bb + Cc' = & -\ 1.20 & & \\ t = & 2\ \ 27\ \ 45.65 & & \end{array}$$

47 Cephei.

$$\begin{array}{rlrl} \delta' = & 78^\circ\ 57'\ 18'' & \log \sec = & 0.71765 = \log C \\ \varphi = & 40\ \ 36\ \ 24 & & \\ \varphi - \delta' = & -\ 38\ \ 20\ \ 54 & \log \cos = & 9.89446 \\ & & \text{Sum} = & .61211 = \log B \end{array}$$

$b = \quad 0^s.157$	$\log b = 9.19590$
$c' = + \quad .173$	$\log c' = 9.23805$
$Bb = + \quad .643$	$\log Bb = 9.80801$
$Cc' = + \quad .903$	$\log Cc' = 9.95570$

$T' = 2^h\ 50^m\ 52^s.06$	$\alpha' = 2^h\ 50^m\ 50^s.41$
$Bb + Cc' = \quad +1.55$	$\alpha = 2\ 27\ 40.14$
$t' = 2\ 50\ 53.61$	$\alpha' - \alpha = \quad 23\ 10.27$
Nat tan $\delta' = +5.1231$	$t' - t = \quad 23\ 7.96$
Nat tan $\delta = -4.0758$	$(\alpha' - \alpha) - (t' - t) = \quad +2.31$
$\tan\delta - \tan\delta' = -9.1989$	$\log = 0.96373_n$
	$\cos\varphi = 9.88036$
	log denominator $= 0.84409_n$
	$\log[(\alpha' - \alpha) - (t' - t)] = 0.36361$
$a = -^s.331$	$\log a = 9.51952_n$

We may now compute the clock correction ΔT from the last of formulæ (XVII), using for this purpose the observed transits of the zenith and equatorial stars. We require first the values of the coefficients.

$$A = \frac{\sin(\varphi - \delta)}{\cos\delta}; \quad B = \frac{\cos(\varphi - \delta)}{\cos\delta}; \quad \text{and} \quad C = \frac{1}{\cos\delta}.$$

If the instrument is to be much used at any one place, as in an observatory for determining the local time, it will be very convenient to tabulate these quantities with the argument δ. On pages 220–227 of the U. S. Coast Survey Report for 1880, Schott gives tables of these factors to two decimal places, with the double arguments δ and $z = \varphi - \delta$, by means of which the factors may be found for any latitude and declination within the limits of the table. If such tables are not at hand, a computation with four-place logarithms will give the necessary degree of accuracy. The work may be arranged as follows:

Star β Arietis.		γ Andromedæ.	
$\delta = 20°14'.5$	$\sin(\varphi-\delta) = 9.5416$	$\delta = 41°46'.1$	$\sin(\varphi-\delta) = 8.307_n$
$\varphi = 40\ 36.4$	$\cos\delta = 9.9723$	$\varphi = 40\ 36.4$	$\cos\delta = 9.8726$
$\varphi-\delta = 20\ 21.9$	$\cos(\varphi-\delta) = 9.9720$	$\varphi-\delta = -1\ 9.7$	$\cos(\varphi-\delta) = 9.9999$
$A = + .371$	$\log A = 9.5693$	$A = - .027$	$\log A = 8.434_n$
$B = + .999$	$\log B = 9.9997$	$B = + 1.341$	$\log B = .1273$
$C = + 1.066$	$\log C = .0277$	$C = + 1.341$	$\log C = .1274$

The determination of ΔT is then as follows:

STAR.	A	B	C	Aa	Bb	Cc'	T	a	ΔT	v
β Arietis	+.37	1.00	1.07	−.12	+.17	+.18	$1^h\ 48^m\ 19^s.78$	$1^h\ 48^m\ 15^s.35$	− 4.66	− 8
γ Andromedæ	−.03	1.34	1.34	+.01	.25	.23	1 56 53 .04	1 56 48 .81	− 4.72	− 2
α Arietis	+.33	1.03	1.08	−.11	.22	.19	2 0 44 .02	2 0 39 .54	4.78	+ 4
ξ' Ceti	+.54	.85	1.01	−.18	.20	.17	2 6 57 .00	2 6 52 .35	4.84	+10
γ Trianguli	+.15	1.19	1.20	−.05	.30	.21	2 10 31 .14	2 10 26 .83	4.77	+ 3
δ Ceti	+.65	.76	1.00	−.22	.19	.17	2 33 37 .98	2 33 33 .35	4.77	+ 3
γ Ceti	+.61	.79	1.00	−.20	.20	.17	2 37 23 .12	2 37 18 .54	4.75	+ 1
σ^2 Arietis	+.45	.93	1.03	−.15	.19	.18	2 45 11 .00	2 45 6 .57	4.65	− 9

Mean $\Delta T = -4^s.744 \pm .022$

The column headed v contains the residuals from which the probable error is found by formula (27) or (28). Page 21

Application of Formulæ (XVIII).

These formulæ will not often be used for reducing observations made with an instrument of this class, but for illustration we may apply them to the above observations.

Computation of n. We use the transits of 5 *Ursæ Minoris* and 47 *Cephei.*

$$t = T + c' \sec\delta \quad = 2^h\ 27^m\ 46^s.85 - {}^s.73 \qquad \delta = 103^\circ\ 47'\ \ 8''$$
$$t' = T' + c' \sec\delta' \quad = 2\ \ 50\ \ 52.06 + .90 \qquad \delta' = \ \ 78\ \ 57\ \ 18$$

$$\begin{aligned} \alpha' &= 2^h\ 50^m\ 50^s.41 & \tan\delta &= \quad 5.1231 \\ \alpha &= 2\ \ 27\ \ 40.14 & \tan\delta' &= -\ 4.0758 \\ \alpha' - \alpha &= \quad 23\ \ 10.27 & & \\ t' - t &= \quad 23\ \ \ 7.96 & \tan\delta - \tan\delta' &= +\ 9.1989 \\ (\alpha' - \alpha) - (t' - t) &= \quad\quad + 2.31 & & \end{aligned}$$

Therefore $n = + {}^s.373$.

For β Arietis $b = + .167$. Therefore $m = b \sec\varphi - n \tan\varphi = - {}^s.100$. Then we have,

$$\begin{array}{rrl} \beta \text{ Arietis,} & T = & 1^h\ 48^m\ 19^s.78 \\ & m & \quad\quad - .10 \\ & n \tan\delta & \quad\quad + .14 \\ & c' \sec\delta & \quad\quad + .18 \\ & \alpha & 1\ \ 48\ \ 15.35 \qquad \Delta T = -\ 4^s.65. \end{array}$$

Personal Equation.

188. When the results of transit observations made by different observers are compared, it is found that they differ generally by small but nearly constant quantities. One observer perhaps acquires a habit of noting the transit too early by a fraction of a second, while another will note it uniformly too late. This difference is called the *personal equation*. It is customary to speak of the *relative* and the *absolute* personal equation, the former being the constant difference between the right ascensions, or clock corrections deduced from observations made by two different observers, and the latter the difference between the absolute value of the quantity and that obtained by an observer who notes the time uniformly too early or too late. When results obtained from observations of two different observers are to be compared, as in the determination of longitude, the personal equation should always be determined and the necessary correction applied.

The existence of a large personal equation is not an indication of a poor observer, but perhaps the contrary. Thus the noted observers Bessel and Struve found that in 1814 their relative personal equation was zero; in 1821 it was $0^s.8$, while in 1823 it amounted to an entire second: thus indicating the gradual formation of a fixed habit of observing on the part of both. Also in 1823 the relative personal equation between Bessel and Argelander was $1^s.2$, a surprisingly large quantity.

The personal equation also depends to some extent on the instruments employed and the method of observation. It is generally much smaller when the chronograph is used than when the eye and ear method is employed. Bessel found that when he used a chronometer beating half-seconds he

observed transits $0^s.49$ later than when he employed a clock beating seconds.

There are various methods of determining the personal equation, those most commonly employed being the following:

First Method. Let one observer note the transit of the star over the first two or three threads, and the other observer its transit over the remaining threads. The observed times are reduced to the middle (or mean) thread by means of the equatorial intervals, and the difference of the reduced times will be the relative personal equation.

A considerable number of stars should be observed in this way, each observer leading alternately. Among the various methods used, this is considered one of the most reliable.

Second Method. The two observers may each use a different instrument and determine the clock correction separately, observing the same list of stars. When the instruments which the observers are accustomed to use differ considerably in the arrangement of the threads or in other respects, this method may be superior to the former, as each observer may use his own instrument and make his observations deliberately and in his usual manner.

Third Method. By a personal-equation apparatus. Various mechanical devices have been constructed for measuring both the relative and absolute personal equation. Prof. Hilgard describes two machines of this kind in Appendix 17, Coast Survey Report 1874. An instrument designed by Prof. Eastman has been in use at the Naval Observatory for a number of years, for a description and drawing of which see Appendix I, Washington Observations, 1875. These all consist of a mechanical device for causing an artificial star to pass across a field of view arranged to appear as nearly as may be like that of the transit instrument. The observer notes the time of transit across the threads either by the

chronographic or the eye and ear method, while the machine by an electric arrangement records the time automatically, constant differences between the actual time of transit and that recorded by the machine being eliminated by causing the star to cross the field in both directions. The difference between the automatic record and that of the observer is his absolute personal equation.

Prof. Eastman gives the following examples of the relative personal equation deduced on the same night by this instrument and by method first:

		By Stars.	By Apparatus.
October 25, 1875,	Professor Eastman—Assistant Skinner...	$0^s.251$	$0^s.227$
November 5, 1875,	Professor Eastman—Assistant Paul......	.174	.173
December 6, 1876,	Professor Eastman—Assistant Paul......	.035	.052
December 31, 1877,	Professor Eastman—Assistant Frisby.....	.052	.044
March 13, 1878,	Professor Eastman—Assistant Frisby.....	.052	.054
March 23, 1878,	Professor Eastman—Assistant Paul......	.107	.092

This close agreement between the results obtained by two methods so entirely different must be regarded as exceedingly satisfactory.

The observer's physical and mental condition is sometimes found to exert a marked influence upon his personal equation. It is therefore very desirable that while prosecuting observations where great accuracy is essential he should maintain as far as possible his ordinary habits of mind and body.

In the more accurate longitude work of the Coast Survey the effect of personal equation is eliminated by the observers exchanging stations when the work is about half finished.

Probable Error and Weight of Transit Observations.

189. The probable error of an observed transit consists practically of two parts: *first*, the probable error of the observer in noting the time of the stars passing the threads, independent of his personal equation; and *secondly*, the vari-

ous errors which together form what is known as the *culmination error*. Among these latter are those due to atmospheric displacement, outstanding instrumental errors, irregularities of the clock rate, and changes in the personal equation. The culmination error is not diminished by increasing the number of threads of the reticule.

The first part of the probable error, which for present purposes we may call the personal error, may be determined by comparing together the individual values of the equatorial intervals deduced from a large number of observations, using for the purpose the formula

$$r = .6745\sqrt{\frac{[vv]}{m-1}},$$

m being the whole number of determinations.

Let ε = the probable error of the observed time of an equatorial star over one thread.

Then, since the equatorial interval is the difference of two observed quantities, each of which has the probable error ε, we shall have (Eq. 29)

$$r = \sqrt{\varepsilon^2 + \varepsilon^2},$$

from which $$\varepsilon = \frac{r}{\sqrt{2}} = .6745\sqrt{\frac{[vv]}{2(m-1)}}. \quad . \quad . \quad . \quad . \qquad (310)$$

As the result of the discussion of a large number of observations made with the different instruments of the Coast Survey, Schott gives,* for the larger instruments,

$$\varepsilon = \sqrt{(.063)^2 + (.036)^2 \tan^2 \delta}\,; \quad . \quad . \quad . \quad . \qquad (311)$$

* Coast Survey Report for 1880, p. 236.

and for the smaller instruments,

$$\varepsilon = \sqrt{(.080)^2 + (.063)^2 \tan^2 \delta} \quad . \quad . \quad . \quad . \quad . \quad (312)$$

From these equations the probable error for a star of any declination may be computed, and consequently the weight, by (33). The following table is from the Coast Survey Report, the weight of an equatorial star being unity:

	δ		For large portable transits.			For small portable transits.		
			ε	p	$\sqrt{p}$	ε	p	$\sqrt{p}$
	°	′	s.			s.		
	0		± 0.06	1	1	± 0.08	1	1
	10		.06	1	1	.08	0.98	1
	20		.06	0.98	1	.08	.92	0.96
	30		.07	.91	0.95	.09	.83	.91
	40		.07	.82	.90	.10	.70	.83
	45		.07	.76	.87	.10	.62	.79
	50		.08	.69	.83	.11	.53	.73
	55		.08	.61	.78	.12	.44	.66
	60		.09	.51	.71	.14	.34	.59
	65		.10	.40	.63	.16	.26	.51
	70		.12	.29	.54	.19	.18	.42
	75		.15	.18	.43	25	.10	.32
	80		.21	.09	.30	.37	.05	.22
	85		.42	.02	.15	.72	.01	.11
δ Ursæ Minoris...	86	36	0.61	0.011	0.103	1.1	0.006	0.075
51 Cephei.........	87	14	0.75	0.007	0.084	1.3	0.004	0.061
α Ursæ Minoris...	88	39	1.5	0.002	0.041	2.7	0 001	0.030
λ Ursæ Minoris...	88	56	1.9	0.001	0.033	3.4	0.001	0.024

In the application of the multiplier $\sqrt{p}$ it generally suffices to employ but one significant figure.

Relative Weights of Incomplete Transits.

190. Let ε = the probable error of the transit of an equatorial star over a single thread;

ε_1 = the probable culmination error;

r = the probable error of the transit observed over n threads, both sources of error being considered.

Then $$r^2 = \varepsilon_1^2 + \frac{\varepsilon^2}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (313)$$

Schott concludes, from the examination of 558 individual values of the right ascensions of 36 stars observed at the U. S. Naval Observatory, that for the larger instruments of the Coast Survey $r = 0^s.051$, and for the smaller instruments $r = 0^s.060$. When assigning to ε the values $0^s.063$ and $0^s.080$ from (311) and (312), it is found that $\varepsilon_1 = \pm {}^s.049$ and $\pm {}^s.056$ respectively. Then let

$N =$ the whole number of threads;
$p =$ the weight of an observation over n threads;
Unity $=$ the weight of an observation over all of the N threads.

Then, (33), $$p = \frac{\varepsilon_1^2 + \dfrac{\varepsilon^2}{N}}{\varepsilon_1^2 + \dfrac{\varepsilon^2}{n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (314)$$

Substituting the above values for ε and ε_1, we have—

For the larger instruments $$p = \frac{1 + \dfrac{1.6}{N}}{1 + \dfrac{1.6}{n}}; \quad . \quad . \quad . \quad . \quad . \quad (315)$$

For the smaller instruments $$p = \frac{1 + \dfrac{2.0}{N}}{1 + \dfrac{2.0}{n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (316)$$

Let $N = 25$ in (315) and 9 in (316) respectively; we find the following values of p for the values of n indicated.

n	p	n	p
1	.41	13	.95
2	.59	14	.96
3	.69	15	.96
4	.76	16	.97
5	.81	17	.97
6	.84	18	.98
7	.87	19	.98
8	.89	20	.98
9	.90	21	.99
10	.92	22	.99
11	.93	23	.99
12	.94	24	1.00
		25	1.00

n	p
1	.41
2	.61
3	.73
4	.82
5	.87
6	.92
7	.95
8	.98
9	1.00

It appears, therefore, that the gain in accuracy obtained by increasing the number of hreads soon becomes practically insignificant. Bessel thought that no practical advantage resulted from the use of more than five threads.

Reduction of Transit Observations by Least Squares.

191. When the time is to be determined by a series of observations with the portable transit instrument, the method of least squares may be applied with advantage in case the results are required with extreme accuracy. This will be the case particularly where the time is required for longitude determination, and where the clock correction, the azimuth and collimation constants, and sometimes the rate, are all to be determined from the same series of observations.

An observing list should be prepared beforehand, embracing stars adapted to the determination of these quantities. We have seen that stars which culminate near the zenith are best adapted to the determination of ΔT; also that circum-

polar stars observed at upper and lower culmination are best for the determination of a. One half the stars should be observed in each position of the axis for the purpose of determining c.

It is a very good arrangement to divide the stars into groups of about five or six stars, each group to contain two circumpolar stars, one at upper and one at lower culmination, the remaining three or four stars being near the zenith or between the zenith and equator. It is not advisable to include the close circumpolar stars in such a group.

The instrument having been carefully adjusted, the observations will be conducted as follows:

1st. Read the level.
2d. Observe the first group of five or six stars.
3d. Read the level.
4th. Reverse the instrument.
5th. Read the level.
6th. Observe the second group of five or six stars.
7th. Read the level.

This may be regarded as a complete series, as it contains everything necessary for determining all of the unknown quantities. If considered desirable, a third and fourth group may be observed in the same manner. If there is time between the stars of the group, more level-readings may be taken; but if the mounting is reasonably firm, the level corrections for the individual stars may be interpolated from those at the beginning and end.

If there are no imperfect transits, a knowledge of the equatorial intervals will not be required; otherwise they may be determined from the suitable stars of the series just observed. It must be remembered that in transporting the instrument from one station to another the relative position

of the threads is liable to be disturbed. This difficulty is avoided by the use of the glass reticule, the distances of the lines of which may be determined once for all.

The reduction is then as follows:

Let
$$\begin{aligned} A &= \sin(\varphi - \delta)\sec\delta; \\ B &= \cos(\varphi - \delta)\sec\delta; \\ C &= \sec\delta; \\ \varDelta T_0 &= \text{the clock correction at time } T_0; \\ \delta T &= \text{the hourly rate}; \\ \alpha &= \text{the stars' apparent right ascension.} \end{aligned}$$

We can always infer from our observations a value of $\varDelta T_0$ which will be very near the true one, and as the labor of computation will be diminished by making the numerical values of the unknown quantities as small as possible, we may assume an approximate value of this quantity, and determine a correction to this assumed value.

Let ϑ = the assumed value of the clock correction;
$$\varDelta T_0 = \vartheta + x.$$

Then x is a small unknown correction to ϑ.

Introducing this notation into Mayer's formula, it becomes

$$T + \vartheta + x + \delta T(T - T_0) + Aa + Bb + Cc - {}^s.021\, C\cos\varphi = \alpha.$$

In which x, δT, a, and c may be considered unknown quantities.

Writing $l = T + \vartheta + Bb - {}^s.021\, C\cos\varphi - \alpha$,

viz., the sum of the known quantities, we have

$$Aa + Cc + \delta T(T - T_0) + x + l = 0. \quad . \quad (317)$$

Every observed transit furnishes one equation of this form for determining the four unknown quantities a, c, δT, and x. Four perfect observations would be sufficient. As a much larger number will be taken, the most probable values must be determined by the method of least squares (Art. 21).

If δT is known, the number of unknown quantities will be reduced to three. If in addition c has been determined by some other method, there will only be two.

If there is a suspicion that the azimuth has changed during the progress of the observations, an additional azimuth constant may be introduced as another unknown quantity.

The reduction will be facilitated by tabulating the factors A, B, and C. Such a table has been published by the U. S. Coast Survey, in which A and B are given with the double argument δ and $z = (\varphi - \delta)$. C is of course given with the argument δ.

When many observations are to be reduced at one place, or in the same latitude, a special table is more conveniently computed for the latitude of the place. The only argument will then be δ.

It will be convenient to make the computation of l directly in the book used for recording the transits. The means of the times over the threads being taken, this will be T, which is written below. In case of incomplete transits, the time over the mean thread is computed as already illustrated. α and δ are taken from the Nautical Almanac and written in the same book. The small corrections $B.b$ and $-^s.021 \cos \varphi . C$ are applied directly to T. Subtracting α from the algebraic sum, we have $l - \vartheta$, in which ϑ will be assumed of such value as to make l small. An example follows.

Reduction of Transit Observations made at the Sayre Observatory, 1883, *October* 11.

An observing list was first prepared, of which the following is a specimen:

STAR.	Magnitude.	α	δ	Setting.
μ Aquarii	4.7	$20^h\ 46^m\ 21^s$	$-\ 9°\ 25'.3$	$140°\ 1'.7$
ν Cygni	4.0	20 52 49	40 43.0	89 53.4
σ^2 Ursæ Majoris, s.p.	5.0	21 0 5	112 23.5	18 12.9
ζ Cygni	3.0	21 7 57	29 44.8	100 51.6
τ Cygni	4.0	21 10 7	37 32.8	93 3.6
α Cephei	2.7	21 15 47	62 5.4	68 31.0
ϵ Pegasi	2.3	21 38 26	9 20.3	121 16.1
π^2 Cygni	4.3	21 42 28	48 46.1	81 50.3
79 Draconis	6.3	21 51 25	73 8.9	57 27.5
α Aquarii	3.0	21 59 46	$-$ 0 53.3	131 29.7
32 Ursæ Majoris, s.p.	6.0	22 9 31	114 18.5	16 17.9
π Aquarii	4.7	22 19 18	$+$ 0 47.0	129 49.4

The two groups are intended to be observed one in each position of the axis. The right ascension and declination are taken from the mean values of the Nautical Almanac. The column headed "Setting" gives the setting of the finding circle. In this case the circle reads zero when the telescope is directed to the north point of the horizon, the latitude being $40° 36' 24''$; the circle will read $130° 36' 24''$ when the line of collimation of the telescope lies in the equator. Therefore the setting for any star will be $130° 36'.4 - \delta$.

Below is the copy of the recorded transits of the above stars as observed on the night of October 11, 1883:

Clamp East.

Level.

E.	W.
12.0	9.9
9.2	13.1
12.0	9.9
9.6	13.0
10.70	11.475

μ Aquarii.

I	57.
II	13.9
III	30.
IV	46.7
V	20 47 3.1

$T = 20\ 46\ 30.14 + .02$
$\alpha = 20\ 46\ 24.07$

ν Cygni.

I	14.4
II	36.3
III	57 5
IV	19.
V	20 53 40.4

$T = 20\ 52\ 57.52 + .06$
$\alpha = 20\ 52\ 51.77$

σ^2 Ursæ Majoris, s.p.

V	49.9
IV	31.2
III	14.8
II	--
I	21 1 40.

$T = 21\ 0\ 14.62 - .01$
$\alpha = 9\ 0\ 7.86$

ζ Cygni.

I	28.9
II	48.
III	6.8
IV	25.8
V	21 8 44.1

$T = 21\ 8\ 6.72 + .05$
$\alpha = 21\ 8\ 0.69$

τ Cygni.

I	—
II	56
III	16.1
IV	36.9
V	21 10 57.6

$T = 21\ 10\ 16.36 + .06$
$\alpha = 21\ 10\ 10.56$

α Cephei.

I			46.1
II			21.7
III			55.9
IV			30.9
V	21	17	5.9
$T =$ 21	15	56.10 + .11	
$\alpha =$ 21	15	50.57	

Level.

E.	W.
9.8	13.3
13.7	9.9
9.3	14.1
12.9	10.8
9.5	14.1
12.8	11.2
11.333	12.233

Clamp West.

Level.

E.	W.
10.2	13.5
12.9	11.2
10.4	13.1
12.7	11.4
11.55	12.30

ε Pegasi.

V			3.1
IV			19.2
III			36.
II			52.8
I	21	39	9.1
$T =$ 21	38	36.04 + .09	
$\alpha =$ 21	38	30.00	

π² Cygni.

V			48.9
IV			13.3
III			38.1
II			2.7
I	21	43	27.5
$T =$ 21	42	38.10 + .15	
$\alpha =$ 21	42	31.96	

79 Draconis.

V			44.
IV			38.7
III	21	51	35.8
II			31.5
I			27.8
$T =$ 21	51	35.56 + .35	
$\alpha =$ 21	51	29.26	

Level.

E.	W.
10.2	13.9
12.6	11.2
10.5	13.7
12.8	11.3
11.525	12.525

α Aquarii.

V			23.9
IV			40.
III			56.8
II			12.9
I	22	0	29.
$T =$ 21	59	56.52 + .10	
$\alpha =$ 21	59	50.21	

32 Ursae Majoris, s.p.

I			19.
II			59.5
III	22	9	38.5
IV			18.5
V			57.5
$T =$ 22	9	38.60 − .12	
$\alpha =$ 10	9	32.66	

π Aquarii.

V			55.5
IV			11.9
III			28.2
II			44.1
I	22	20	0.3
$T =$ 22	19	28.00 + .12	
$\alpha =$ 22	19	21.93	

Level.

E.	W.
12.6	11.2
10.1	14.0
12.1	11.8
10.6	13.6
11.35	12.65

The small quantities added to T above include the corrections for level and diurnal aberration; viz., $Bb - {}^s.021\ C$, cos φ. b is computed from the level-readings as already explained, the value of one division of the level being ${}^s.174$, and the correction for inequality of pivots being $\mp .062$ Cl. $\left\{ \begin{matrix} \text{E.} \\ \text{W.} \end{matrix} \right\}$, expressed in terms of one division of the level.

We now take from the tables the values of the coefficients A, B, and C, or, if tables of these quantities are not at hand, we compute them by the formulæ. For illustrating the application of the proper weights to the equations of condition, the value of $\sqrt{p}$ is taken from the table of Art. 189 for the smaller instruments. All these quantities are conveniently tabulated as follows:

STAR.	δ	A	B	C	Level.	b
Clamp East.						
μ Aquarii	$-$ 9° 24′.9	.78	.65	1.01	+.326	$+^s.057$
ν Cygni	40° 43′.6	.00	1.32	1.32	......	.059
σ^2 Ursæ Majoris, s.p.	112° 24′.0	2.49	$-$ .82	$-$ 2.62	......	.061
ζ Cygni	29° 45′.4	.22	1.13	1.15	......	.063
τ Cygni	37° 33′.4	.07	1.26	1.26	......	.065
α Cephei	62° 6′.0	$-$ .78	1.99	2.14	+.388	.068
Clamp West.						
ϵ Pegasi	9° 20′.8	.53	.87	$-$ 1.01	+.437	.076
π^2 Cygni	48° 46′.7	$-$ .22	1.50	$-$ 1.52	......	.087
79 Draconis	73° 9′.5	$-$1.86	2.91	$-$ 3.45	+.562	.098
α Aquarii	$-$ 0° 52′.8	.66	.75	$-$ 1.00	......	.106
32 Ursæ Majoris, s.p.	114° 19′.0	2.33	$-$ .68	+ 2.43	......	.115
π Aquarii	0° 47′.5	.64	.77	$-$ 1.00	+.712	.124

STAR.	Bb	Aberration.	Sum.	$\sqrt{p}$	$l - \vartheta$	l
Clamp East.						
μ Aquarii	$+^s.04$	$-$ .02	+ .02	1.00	$-$ 6^s.09	$-$ .09
ν Cygni	.08	$-$ .02	+ .06	.82	$-$ 5 .81	+ .19
σ^2 Ursæ Majoris, s.p.	$-$ .05	+ .04	$-$ .01	.46	$-$ 6 .75	$-$ .75
ζ Cygni	+ .07	$-$ .02	+ .05	.91	$-$ 6 .08	$-$ .08
τ Cygni	.08	$-$ .02	+ .06	.85	$-$ 5 .86	+ .14
α Cephei	.14	$-$ .03	+ .11	.56	$-$ 5 .64	+ .36
Clamp West.						
ϵ Pegasi	.07	+ .02	+ .09	1.00	$-$ 6 .13	$-$ .13
π^2 Cygni	.13	+ .02	.15	.74	$-$ 6 .29	$-$ .29
79 Draconis	.29	+ .06	.35	.36	$-$ 6 .65	$-$ .65
α Aquarii	.08	+ .02	.10	1.00	$-$ 6 .41	$-$ .41
32 Ursæ Majoris, s.p.	$-$ .08	$-$ .04	$-$ .12	.50	$-$ 5 .82	+ .18
π Aquarii	.10	+ .02	+ .12	1.00	$-$ 6 .19	$-$ .19

Assumed $\vartheta = -6^s$.

The quantity in the column headed $l - \vartheta$ is obtained by adding algebraically to the quantity T of the above observations the sum of the corrections, viz., $Bb - {}^s.021\ C \cos \varphi$, and subtracting from the result α. We now have all the quantities entering into the equations of condition, each of which has the form

$$\sqrt{p}[Aa + Cc + x] = \sqrt{p} \,.\, l.$$

The rate is here inappreciable, and the term $\delta T(T - T_0)$ has accordingly been dropped.

The coefficient c, as will be seen, has its sign changed for *clamp west.*

Our twelve equations, written out in full, will then be as follows:

$$
\begin{array}{rrl}
1. & .78a + 1.01c + 1.00x = & -.09. \\
2. & .00a + 1.08c + \ .82x = & +.16. \\
3. & 1.15a - 1.21c + \ .46x = & -.35. \\
4. & .20a + 1.05c + \ .91x = & -.07. \\
5. & .06a + 1.07c + \ .85x = & +.12. \\
6. & -.44a + 1.20c + \ .56x = & +.20. \\
7. & .53a - 1.01c + 1.00x = & -.13. \\
8. & -.16a - 1.12c + \ .74x = & -.21. \\
9. & -.67a - 1.24c + \ .36x = & -.23. \\
10. & .66a - 1.00c + 1.00x = & -.41. \\
11. & 1.16a + 1.21c + \ .50x = & +.09. \\
12. & .64a - 1.00c + 1.00x = & -.19.
\end{array}
$$

These now have the general form of the equations of condition (36), viz.,

$$a_1x + c_1z + d_1w = n_1,$$

there being in this case the three unknown quantities a, c, and x, corresponding to the x, z, and w of the general form. The term corresponding to y has disappeared here, as we have assumed the rate of the clock to be inappreciable for the short time over which the observations extend.

We have now to form the normal equations (see Eq. 41). In order that no confusion may arise from the difference of notation, the general form of these equations is here given in full, viz.:

$$
\begin{aligned}
[aa]a + [ac]c + [ad]x &= [an]; \\
[ac]a + [cc]c + [cd]x &= [cn]; \\
[ad]a + [cd]c + [dd]x &= [dn].
\end{aligned}
$$

We shall give the solution of these equations in full with the various checks on the accuracy of the computation, as an illustration of the method. Practically, however, this part of the work will generally be more or less abridged by experienced computers when the number of unknown quantities does not exceed that of the above equations.

We shall require, besides the quantities already indicated, the sums of the coefficients of each equation, viz.:

$$
\begin{aligned}
s_1 &= a_1 + c_1 + d_1 - n_1; \\
s_2 &= a_2 + c_2 + d_2 - n_2. \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

Also, we compute the quantities

$$[as],\ [cs],\ [ds],\ [nn],\ [ns].$$

The computation will first be made by the use of Crelle's table.

We therefore prepare the scheme for computation given below, containing 19 columns, 5 for the quantities a, c, d, $-n$, s, etc., which we rewrite for the sake of convenience, and 14 for the squares and products.

a.	*c.*	*d.*	$-n.$	*s.*	*aa.*	*ac.*	*ad.*	$-an.$	*as.*	*cc.*
.78	1.01	1.00	+.09	2.88	.6084	+ .7878	+ .7800	+.0702	+2.2464	1.0201
.00	1.08	.82	−.16	1.74						1.1664
1.15	−1.21	.46	+.35	.75	1.3225	−1.3915	+ .5290	+.4025	+ .8625	1.4641
.20	1.05	.91	+.07	2.23	.0400	+ .2100	+ .1820	+.0140	+ .4460	1.1025
.06	1 07	.85	−.12	1.86	.0036	+ .0642	+ .0510	−.0072	+ .1116	1.1449
−.44	1.20	.56	−.20	1.12	.1936	− .5280	− .2464	+.0880	− .4928	1.4400
.53	−1.01	1.00	+.13	.65	.2809	− .5353	+ .5300	+.0689	+ .3445	1.0201
−.16	−1.12	.74	+.21	− .33	.0256	+ .1792	− .1184	−.0336	+ .0528	1.2544
−.67	−1.24	.36	+.23	−1.32	.4489	+ .8308	− .2412	−.1541	+ .8844	1.5376
.66	−1.00	1 00	+.41	1.07	.4356	− .6600	+ .6600	+.2706	+ .7062	1.0000
1.16	+1.21	.50	−.09	2.78	1.3456	+1.4036	+ .5800	−.1044	+3.2248	1.4641
.64	−1.00	1.00	+.19	.83	.4096	− .6400	+ .6400	+.1216	+ .5312	1.0000
					5.1143	− .2792	+3.3460	+.7365	8 9176 8.9176	14.6142
					[*aa*]	[*ac*]	[*ad*]	− [*an*]	[*as*]	[*cc*]

cd.	$-cn.$	*cs.*	*dd.*	$-dn.$	*ds.*	*nn.*	$-ns.$	*v.*	*vv.*
+ 1.0100	+ .0909	+ 2.9088	1.0000	+.0900	+ 2 8800	.0081	+.2592	+.09	.0081
+ .8856	− .1728	+ 1.8792	.6724	−.1312	+ 1.4268	.0256	−.2784	−.07	49
− .55'6	− .4235	− .9075	.2116	+.1610	+ .3450	.1225	+.2625	+.04	16
+ .9555	+ .0735	+ 2.3415	.8281	+.0637	+ 2.0293	.0049	+.1561	+.13	169
+ .9095	− .1284	+ 1 9902	.7225	−.1020	+ 1.5810	.0144	−.2232	−.04	16
+ .6720	− .2400	+ 1.3440	.3136	−.1120	+ .6272	.0400	−.2240	−.04	16
− 1.0100	− .1313	− .6565	1.0000	+.1300	+ .6500	.0169	+.0845	−.15	225
− .8288	− .2352	+ .3696	.5476	+.1554	− .2442	.0441	−.0693	−.00	
− .4464	− .2852	+ 1.6368	.1296	+.0828	− .4752	.0529	−.3036	+.07	49
− 1.0000	− .4100	− 1.0700	1.0000	+.4100	+ 1.0700	.1681	+.4387	+.12	144
+ .6050	− .1089	+ 3.3638	.2500	−.0450	+ 1.3900	.0081	−.2502	−.05	25
− 1.0000	− .1900	− .8300	1.0000	+.1900	+ .8300	.0361	+.1577	−.09	81
+ .1958	−2 1609	12.3699 12.3699	7.6754	+.8927	12.1099 12.1099	+.5417	+.0100		
[*cd*]	− [*cn*]	[*cs*]	[*dd*]	− [*dn*]	[*ds*]	[*nn*]	− [*ns*]		

$[vv] = .0871.$

The agreement of the values of $[as]$, $[cs]$, and $[ds]$ proves the accuracy of this part of the computation.

The normal equations are then

$$5.1143a - \ .2792c + 3.3460x = -\ .7365\,;$$
$$-\ .2792a + 14.6142c + \ .1958x = \ 2.1609\,;$$
$$3.3460a + \ .1958c + 7.6754x = -\ .8927.$$

These equations are now to be solved, following the method and notation explained in Art. 28. We shall therefore require the following auxiliary coefficients, viz.,

$$[cc\,1],\ [cd\,1],\ [cn\,1],\ [cs\,1],\ [dd\,1],\ [dn\,1],\ [ds\,1],\ [nn\,1],\ [ns\,1],$$
$$[dd\,2],\ [dn\,2],\ [ds\,2],\ [nn\,2],\ [ns\,2];$$

$[ds\,1]$, $[ns\,1]$, etc., being computed for checks on the accuracy of the work.

The computation will then be made according to the following scheme:

a.	*c.*	*x.*	*n.*	*s.*	Proof.	
$[aa]$ 5.1143 log .70879	$[ac]$ − .2792 log 9.44592$_n$	$[ad]$ 3.3460 log .52453	$[an]$ − .7365 log 9.86717$_n$	$[as]$ 8.9176 log .95025	8.9176	(1)
log $\frac{[ac]}{[aa]}$ 8.73713$_n$	$[cc]$ 14.6142 $\frac{[ac]}{[aa]}[ac]$ .0152	$[cd]$ .1958 $\frac{[ac]}{[aa]}[ad]$ −.1827	$[cn]$ + 2.1609 $\frac{[ac]}{[aa]}[an]$ + .0402	$[cs]$ 12.3699 $\frac{[ac]}{[aa]}[as]$ .4868		
	$[cc\,1]$ 14.5990 log 1.16432	$[cd\,1]$ .3785 log 9.57807	$[cn\,1]$ + 2.1207 log .32648	$[cs\,1]$ 12.8567 log 1.10913	12.8568	(2)
log $\frac{[ad]}{[aa]}$ 9.81574		$[dd]$ 7.6754 $\frac{[ad]}{[aa]}[ad]$ 2.1891	$[dn]$ − .8927 $\frac{[ad]}{[aa]}[an]$ − .4818	$[ds]$ 12.1099 $\frac{[ad]}{[aa]}[as]$ 5.8343		
log $\frac{[cd\,1]}{[cc\,1]}$ 8.41375		$[dd\,1]$ 5.4863 $\frac{[cd\,1]}{[cc\,1]}[cc1]$ .0098	$[dn\,1]$ − .4109 $\frac{[cd\,1]}{[cc\,1]}[cn\,1]$ .0550	$[ds\,1]$ 6.2756 $\frac{[cd\,1]}{[cc\,1]}[cs\,1]$ .3334	6.2757	(3)
		$[dd\,2]$ 5.4765 log .73850	$[dn\,2]$ − .4659 log 9.66838$_n$	$[ds\,2]$ 5.9422 log .77395	5.9424	(4)
			log $x =$ 8.92988$_n$ $x = -$ .08509			
log $\frac{[an]}{[aa]}$ 9.15838$_n$			$[nn]$ .5417 $\frac{[an]}{[aa]}[an]$ .1061	$[ns]$ − .0100 $\frac{[an]}{[aa]}[as]$ − 1.2842		
log $\frac{[cn\,1]}{[cc\,1]}$ 9.16216			$[nn\,1]$ .4356 $\frac{[cn\,1]}{[cc\,1]}[cn\,1]$ .3081	$[ns\,1]$ 1.2742 $\frac{[cn\,1]}{[cc\,1]}[cs\,1]$ 1.8676	1.2742	(5)
log $\frac{[dn\,2]}{[dd\,2]}$ 8.92988$_n$			$[nn\,2]$. .1275 $\frac{[dn\,2]}{[dd\,2]}[dn\,2]$ .0397	$[ns\,2]$ − .5934 $\frac{[dn\,2]}{[dd\,2]}[ds2]$ −.5056	− .5934	(6)
			$[nn\,3]$ .0878	$[ns\,3]$ −.0878	$[vv] =$ .0871	(7)

The accuracy of the work at different stages of progress is shown by the manner in which the proof-equations are satisfied. Those referred to by the numbers in the last column above are as follows:

$$
\begin{aligned}
&(1)\quad [as\ ] = [aa\ ] + [ac\ ] + [ad\ ] - [an];\\
&(2)\quad [cs\ 1] = [cc\ 1] + [cd\ 1] - [cn\ 1];\\
&(3)\quad [ds\ 1] = [dd\ 1] + [cd\ 1] - [dn\ 1];\\
&(4)\quad [ds\ 2] = [dd\ 2] - [dn\ 2]:\\
&(5)\quad [ns\ 1] = [cn\ 1] + [dn\ 1] - [nn\ 1];\\
&(6)\quad [ns\ 2] = [dn\ 2] - [nn\ 2];\\
&(7)\quad [ns\ 3] = -[nn\ 3] = [vv].
\end{aligned}
$$

We now determine c and a by the equations

$$
\begin{aligned}
&[cc\ 1]c + [cd\ 1]x = [cn\ 1];\\
&[aa]a + [ac]c + [ad]x = [an].
\end{aligned}
$$

$$
\begin{aligned}
[cn\ 1] &= 2.1207 & [an] &= -\ .7365\\
[cd\ 1]x &= .0322 & -[ad]x &= +\ .2847\\
c &= +\frac{2.1529}{14.5990} & -[ac]\ c &= +\ .0412\\
 & & a &= -\frac{.4106}{5.1143}\\
c &= +\ .1475 & a &= -\ .0803
\end{aligned}
$$

The Weights and Probable Errors.

The weights of a, c, and x will be given by formulæ (76), viz.:

$$
\begin{aligned}
p_x &= [dd\ 2];\\
p_c &= [cc\ 1]\frac{[dd\ 2]}{[dd\ 1]};\\
p_a &= [aa]\frac{[cc\ 1]}{[cc]}\frac{[dd\ 2]}{[dd\ 1]_a}.
\end{aligned}
$$

In which $$[dd\ 1]_a = [dd] - \frac{[cd]}{[cc]}[cd].$$

Therefore $p_x = 5.476;$

$$
\begin{aligned}
\log [cc\ 1] &= 1.16432\\
\log [dd\ 2] &= .73850\\
\log \frac{1}{[dd\ 1]} &= 9.26072\\
\log p_c &= 1.16354
\end{aligned}
$$

$p_c = 14.573;$

$$\log [cd]^2 = 8.58362$$
$$\log \frac{1}{[cc]} = 8.83522$$
$$\text{Nat. No. } .0026 \qquad 7.41884$$
$$[dd] = 7.6754$$
$$[dd\,1]_a = 7.6728 \qquad \log \frac{1}{[dd\,1]_a} = 9.11504$$
$$\log \frac{1}{[cc]} = 8.83522$$
$$\log [aa] = .70879$$
$$\log [cc\,1] = 1.16432$$
$$\log [dd\,2] = .73850$$
$$p_a = 3.646. \qquad \log p_a = .56187$$

The mean error of a single observation of weight unity is—see equation (88)—

$$\varepsilon = \sqrt{\frac{[vv]}{m - \mu}}.$$

In this case $m = 12$; $\mu = 3$; $[vv] = .0871$. Therefore $\varepsilon = .098$.

$$*r_x = \tfrac{2}{3}\varepsilon_x = .028; \qquad \dagger\varepsilon_x = \frac{\varepsilon}{\sqrt{p_x}} = .042;$$
$$r_c = \tfrac{2}{3}\varepsilon_c = .017; \qquad \varepsilon_c = \frac{\varepsilon}{\sqrt{p_c}} = .026;$$
$$r_a = \tfrac{2}{3}\varepsilon_a = .034; \qquad \varepsilon_a = \frac{\varepsilon}{\sqrt{p_a}} = .051.$$

We now have $\Delta T = \vartheta + x$. Therefore

$$\Delta T = -\,6^s.085 \pm .028;$$
$$c = +\ \ .147 \pm .017;$$
$$a = -\ \ .080 \pm .034.$$

Formation of the Normal Equations by a Table of Squares.

We have seen in Art. 26 that all of the multiplications necessary for deriving the normal equations from the equations of condition can be performed by

---

* See equations (27). † See equations (89).

means of a table of squares, with little, if any, more labor than by the use of Crelle's table. For the purpose of illustrating the method it will be applied to the present example.

By referring to the formulæ and explanations of Art. 26 the details of the computation which follow will be sufficiently clear.

$(a+c)$.	$a+d$.	$a-n$.	$c+d$.	$c-n$.	$d-n$.	aa.	cc.	dd.
1.79	1.78	.87	2.01	1.10	1.09	.6084	1.0201	1.0000
1.08	.82	− .16	1.90	.92	.66		1.1664	.6724
− .06	1.61	1.50	− .75	− .86	.81	1.3225	1.4641	.2116
1.25	1.11	.27	1.96	1.12	.98	.0400	1.1025	.8281
1.13	.91	− .06	1.92	.95	.73	.0036	1.1449	.7225
.76	.12	− .64	1.76	1.00	.36	.1936	1.4400	.3136
− .48	1.53	.66	− .01	− .88	1.13	.2809	1.0201	1.0000
−1.28	.58	.05	− .38	− .91	.95	.0256	1.2544	.5476
−1.91	− .31	− .44	− .88	−1.01	.59	.4489	1.5376	.1296
− .34	1.66	1.07	0	− .59	1.41	.4356	1.0000	1.0000
+2.37	1.66	1.07	1.71	+1.12	.41	1.3456	1.4641	.2500
− .36	1.64	.83	0	− .81	1.19	.4096	1.0000	1.0000
						5.1143 [aa]	14.6142 [cc]	7.6754 [dd]

nn.	ss.	$(a+c)^2$.	$(a+d)^2$.	$(a-n)^2$.	$(c+d)^2$.	$(c-n)^2$.	$(d-n)^2$.
.0081	8.2944	3.2041	3.1684	.7569	4.0401	1.2100	1.1881
.0256	3.0276	1.1664	.6724	.0256	3.6100	.8464	.4356
.1225	.5625	.0036	2.5921	2.2500	.5625	.7396	.6561
.0049	4.9729	1.5625	1.2321	.0729	3.8416	1.2544	.9604
.0144	3.4596	1.2769	.8281	.0036	3.6864	.9025	.5329
.0400	1.2544	.5776	.0144	.4096	3.0976	1.0000	.1296
.0169	.4225	.2304	2.3409	.4356	.0001	.7744	1.2769
.0441	.1089	1.6384	.3364	.0025	.1444	.8281	.9025
.0529	1.7424	3.6481	.0961	.1936	.7744	1.0201	.3481
.1681	1.1449	.1156	2.7556	1.1449		.3481	1.9881
.0081	7.7284	5.6169	2.7556	1.1449	2.9241	1.2544	.1681
.0361	.6889	.1296	2.6896	.6889		.6561	1.4161
.5417	33.4074	19.1701	19.4817	7.1290	22.6812	10.8341	10.0025
		19.7285	12.7897	5.6560	22.2896	15.1559	8.2171
		− .5584	6.6920	1.4730	.3916	−4.3218	1.7854
		− .2792	3.3460	.7365	.1958	−2.1609	.8927
[nn]	[ss]	[ac]	[ad]	− [an]	[cd]	−[cn]	− [dn]

The proof-formula becomes in this case

$$[ss] + 2\{[aa] + [cc] + [dd] + [nn]\} = [(a+c)^2] + [(a+d)^2] + [(a-n)^2]$$
$$+ [(c+d)^2] + [(c-n)^2] + [(d-n)^2],$$

which is completely verified, as may be seen by substituting the above values. Of course the resulting normal equations are the same as those obtained before.

Correction for Flexure.

192. The second form of transit instrument, that in which the eye-piece is at one end of the axis (see Fig. 28), requires a special correction for flexure of the horizontal axis. The amount of this flexure or bending is assumed to be the same in all positions of the telescope, as it will be if the material of which the axis is composed is homogeneous. The effect will be to bring the reflecting prism lower down than it would be otherwise without changing the direction of the reflecting surface. When the eye-piece is east this will cause the star to reach the collimation axis too late by a small quantity, which is a maximum in the zenith and nothing in the horizon. Suppose WE to represent the rotation axis bent as shown in the figure, CO being the collimation axis of the telescope. Let E be the eye end of the axis. The effect on the observed time of a star's transit will evidently be the same as that produced by elevating the end marked E, and when the proper coefficient is found it may be combined with the level correction.

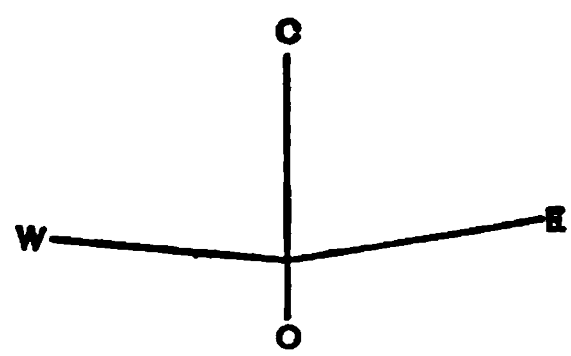

Fig. 36.

Let $f =$ the coefficient of flexure.

f will be the maximum displacement of the transit thread, and will be the value of this displacement when the telescope is directed to the zenith.

The clamp being on the end of the axis opposite the eye-piece, we must add to Mayer's formula the term

$$\mp f \frac{\cos(\varphi - \delta)}{\cos \delta} \begin{Bmatrix} \text{clamp west} \\ \text{clamp east} \end{Bmatrix} \quad . \quad . \quad . \quad . \quad (318)$$

If we write $(\varphi - \delta) = z$, the terms of Mayer's formula, which give the correction of the observed time of a star's transit for collimation, flexure, and inequality of pivots, may be written as follows:

$$(p \cos z - f \cos z + c) \sec \delta ; \quad . \quad . \quad . \quad (319)$$

in which p is determined by (297) or $(297)_1$, and which we see is involved in the same manner as f.

These instruments are generally provided with micrometers, which may be used for determining f and c at the same time, as follows:

In order to make a satisfactory determination, and at the same time to test the accuracy of the assumed law of change expressed by the formula $f \cos z$, a collimating telescope is necessary, mounted in a frame in such a manner that it may be placed vertically over the transit telescope and at different zenith distances from zero to 90°. The collimation error is then measured, as explained in Articles 182–184, with the telescope pointed at various zenith distances. This measured value will include the term $f \cos z$, which will be zero when $z = 90°$, and a maximum when $z = 0$. It will therefore be possible to separate c from f.

It will be advisable to make a considerable number of measurements, from which c and f can then be derived by the method of least squares. If the resulting values satisfy the equations within the limit of the probable error of measurement, the assumed law of change expressed by the formula $f \cos z$ will be verified.

In some cases there is found to be a correction required depending on the temperature. This may be detected by making the measurements for collimation and flexure at different temperatures. If then different values are found varying with the temperature according to any law, the necessary correction may be determined.

In Vol. XXXVII, *Memoirs Royal Astronomical Society*, Captain Clarke, R.E., gives an example of the investigation of the flexure coefficient with an apparatus of the kind just described. In addition to the movable collimator, another was used which was fixed in the horizon. The collimation measured on this was free from the effect of flexure, so that by taking the difference between the quantity ($f \cos z + c$), measured at a zenith distance z by means of the movable collimator, and the quantity c, measured at the same time with the fixed collimator, a direct measurement of the quantity $f \cos z$ was obtained. Twelve measurements made at zenith distances from 0° to 55° gave the following results:

z	Difference.	v	z	Difference.	v	z	Difference.	v
0°	2.80	+ 22	20°	2.72	+ 09	40°	2.46	− 15
5	2.68	+ 33	25	2.98	− 24	45	1.98	+ 15
10	3.11	− 13	30	2.40	+ 22	50	2.02	− 08
15	3.04	− 12	35	2.90	− 43	55	1 69	+ 04

The column headed z gives the zenith distance of the upper collimator; the next column gives the difference between the collimation determined on the upper and lower collimators; and the column headed v gives the residuals.

Referring to equation (319), we see that the quantity called "difference" is equal to $(f - p) \cos z$. From the twelve measured values of this quantity it was found that

$(f - p) = 3.021 \pm .050$ expressed in divisions of the micrometer.

From level-readings,

$p = .779 \pm .026$ expressed in divisions of the micrometer;

therefore $f = 3.800$.

One division of the micrometer $= 0''.8345$;

therefore $f = 3''.171 = 0^s.211$.

193. The use of such an apparatus as we have described will not generally be practicable in the field. The coefficient f may then be determined from the observed transits by adding to the equations of condition (317) the term

$$\mp f \frac{\cos(\varphi - \delta)}{\cos \delta}.$$

The complete equation will then be

$$Aa + Bf + Cc + \delta T(T - T_0) + x + l = 0. \quad . \quad (320)$$

$a, f, c, \delta T$, and x being unknown quantities.

If δT is known, as it ordinarily will be, the number of unknown quantities will be four.

The Transit Instrument out of the Meridian.

194. Equations (275) and (281) are strictly general, and are applicable to the reduction of transits with the instrument in any position whatever. We have seen that when the instrument is so near the meridian that the squares and higher powers of a, b, m, and n may be neglected* these formulæ become very simple. Bessel, Hansen, and others have given more general methods of solving the equations intended for use in those cases where the observer in the field cannot afford the time for adjusting his instrument accurately in the meridian. When, however, the observer is provided with a good list of stars reduced to apparent place, like that given

* That is, we may write a, b, m, and n for sin a, sin b, etc., and unity for cos a, cos b, etc.

in the American Ephemeris, this adjustment is made so readily, and the labor of reduction is so much less than with the more general methods, that the latter have not found much favor, especially in this country. Therefore, however interesting some of these may be from a mathematical point of view, we shall not give their development here.

Transits of the Sun, Moon, and Planets.

195. In the field, transits of the moon will be observed for the determination of longitude when no better method is available. The sun and occasionally a planet will be observed for time.

In case of the sun and moon the method of observing is to note the instant when the limb is tangent to the thread. With the sun the transit of both limbs may be observed; with the moon this will not be practicable except when the transit is observed very near the instant of full moon. In observing a planet, the transits of each limb may be observed alternately, or when a chronograph is used both limbs may be observed, as in case of the sun. With any of these bodies, when both limbs are observed, the time of transit of the centre will be the mean of that of the two limbs. It may, however, be desirable to reduce the limbs separately for the purpose of comparison.

When the moon's limb is observed on a side thread, the hour-angle is affected by parallax: the time required to pass from the thread to the meridian is affected by the moon's motion in right ascension. The reduction is as follows:

Let δ' and t' be the apparent declination and east hour-angle of the moon's limb when observed on a side thread;

δ and t, the geocentric declination and hour-angle;

z and z', the geocentric and apparent zenith distance.

We can reduce the observation by either of the equations (282), (283), or (284). Taking the latter, viz., Mayer's formula, we have

$$t' = a\frac{\sin(\varphi - \delta')}{\cos\delta'} + b\frac{\cos(\varphi - \delta')}{\cos\delta'} + \frac{c' + i}{\cos\delta'}; \quad (321)$$

i being the equatorial interval of the thread.

Having t', t may be determined as follows:

In Fig. 37, let P be the pole, Z the zenith, O the geocentric place of the moon at the instant of observation, O' the apparent place.

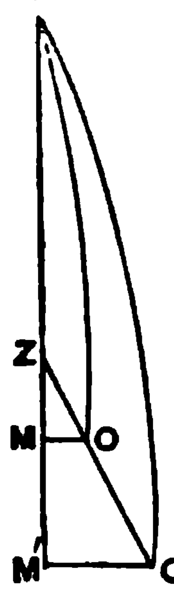

Fig. 37.

$$\text{Angle } MPO = t; \qquad ZO = z;$$
$$MPO' = t'; \qquad ZO' = z'.$$

From the triangles MZO and $M'ZO'$,

$$\sin MZO = \frac{\sin MO}{\sin z} = \frac{\sin M'O'}{\sin z'}. \quad . \quad (322)$$

$$\left.\begin{array}{ll}\text{From triangle } MPO, & \sin MO = \sin t \cos\delta; \\ \text{From triangle } M'PO', & \sin M'O' = \sin t' \cos\delta'.\end{array}\right\} \quad (323)$$

Substituting these values in (322), we have

$$\frac{\sin t \cos\delta}{\sin z} = \frac{\sin t' \cos\delta'}{\sin z'}.$$

As t is small,

$$t = t'\frac{\cos\delta'}{\cos\delta} \cdot \frac{\sin z}{\sin z'}, \quad . \quad . \quad . \quad (324)$$

the required value of t in terms of t'.

Let λ = the increase of the moon's right ascension in one

sidereal second; then t being expressed in seconds, the time required for the moon to pass over this interval will be

$$\frac{t}{1 - \lambda}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (325)$$

$1 - \lambda$ representing the velocity with which the moon approaches the meridian.

There remains the correction for the moon's semidiameter.

Let S = the geocentric semidiameter of the moon at the time of transit, taken from the ephemeris;
S' = the hour-angle of the centre when the limb is on the meridian.

Then, from Fig. 38,

$$\sin S' = \frac{\sin S}{\cos \delta};$$

Writing S and S' for their sines and dividing by 15 to reduce to time,

$$S' = \frac{S}{15 \cos \delta}.$$

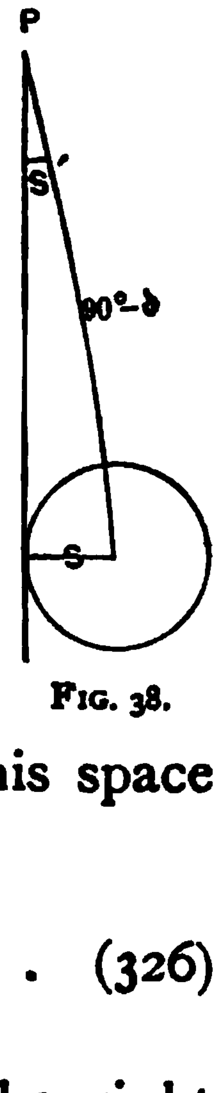

Fig. 38.

The time required for the moon to pass over this space will be

$$\frac{S'}{1 - \lambda} = \frac{S}{15(1 - \lambda) \cos \delta}. \quad . \quad . \quad . \quad . \quad . \qquad (326)$$

From (321), (324), (325), and (326), we have for the right ascension of the moon's centre when the limb is observed on any thread of the transit instrument,

$$\alpha = T + \Delta T + \frac{1}{1 - \lambda} \cdot \frac{\cos \delta'}{\cos \delta} \cdot \frac{\sin z}{\sin z'} \left(a' \frac{\sin(\phi - \delta')}{\cos \delta'} + b \frac{\cos(\phi - \delta')}{\cos \delta'} + \frac{c' + i}{\cos \delta'} \right) \pm \frac{S}{15(1 - \lambda) \cos \delta}. \qquad (327)$$

The geocentric declination, δ, and the equatorial horizontal parallax, π, are taken from the ephemeris. Then from $(\mathrm{XI})_1$, Art. 85, we have with sufficient accuracy for this purpose

$$\delta' = \delta - \pi\rho \sin(\varphi' - \delta'); \quad . \quad . \quad . \quad . \quad (328)$$

where generally δ may be substituted for δ', and φ for φ', in the second member.

Then p being the parallax in zenith distance, we have

$$z' = z + p,$$

and the factor $\frac{\sin z}{\sin z'}$ in equation (327) becomes

$$\frac{\sin z}{\sin z'} = \frac{\sin z}{\sin z \cos p + \cos z \sin p} = \cos p - \cot z \sin p$$

approximately. And from $(\mathrm{VII})_1$, Art. 82 with sufficient accuracy for this purpose,

$$\frac{\sin z}{\sin z'} = 1 - \rho \sin \pi \cos(\varphi' - \delta).$$

If then we write
$$\left.\begin{aligned} A_1 &= 1 - \rho \sin \pi \cos(\varphi' - \delta), \\ B_1 &= \frac{1}{1 - \lambda}, \\ F &= A_1 B_1 \sec \delta, \end{aligned}\right\} \quad . \quad (329)$$

A_1 may be tabulated with the argument $\log \rho \sin \pi \cos(\varphi' - \delta)$ as in table XIII of Bessel's *Tabulæ Regiomontanæ*; B_1 may be tabulated with the argument $\Delta\alpha$ = moon's change in right ascension in one minute, $\Delta\alpha$ being given in the ephemeris. The term $\frac{S}{15(1 - \lambda)\cos\delta}$ may be taken from the table of "Moon Culminations" of the ephemeris where it is given under the heading "Sidereal time of semidiameter passing

the meridian." The complete formulæ for the moon's right ascension are then as follows:

$$\left.\begin{aligned}
\delta' &= \delta - \pi\rho \sin (\varphi' - \delta); \\
A_1 &= 1 - \rho \sin \pi \cos (\varphi' - \delta); \\
B_1 &= \frac{1}{1 - \lambda}; \\
F &= A_1 B_1 \sec \delta; \\
\alpha &= T + \Delta T + iF + \left(a\frac{\sin (\phi - \delta')}{\cos \delta'} + b\frac{\cos (\phi - \delta')}{\cos \delta'} + \frac{c'}{\cos \delta'}\right) F \cos \delta \pm \frac{S}{15(1 - \lambda) \cos \delta}.
\end{aligned}\right\} \text{(XIX)}$$

The use which will be made of this value of α in the determination of longitude will be explained hereafter. A series of stars will be observed in connection with the moon for determining the clock correction ΔT and the constants a and c. Sometimes the clock correction is made to depend exclusively on about four stars whose declination is nearly the same as that of the moon; two of these precede the moon and two follow.

Correction to the Moon's Defective Limb.

196. The transit of both limbs of the moon can only be observed when the culmination occurs very near the time of full moon. If one limb is defective it may still be used if it is sharply defined, and a correction applied for defective illumination.

For this purpose we may regard the moon as a sphere, and we may consider the rays of light from the sun to the moon as parallel to those from the sun to the earth. The curve of contact of the surface of the moon with the cone of rays tangent to its surface will separate the light from the dark part of the moon. When the defective limb is observed, the point whose contact with the thread of the reticule is noted is a point on this curve; and instead of the semidiameter S,

we shall require for this limb the perpendicular from the centre of the disk upon the hour-circle, of which the transit thread may be regarded as forming a small arc. Thus aa' being the position of the thread at the instant of the observed transit of the defective limb L, we shall require the distance $CL = S_1$ instead of S. Fig. 40 may be regarded as a section formed by the plane passing through the rotation and collimation axes of the instrument, and Fig. 39 a section formed by the plane perpendicular to the collimation axis.

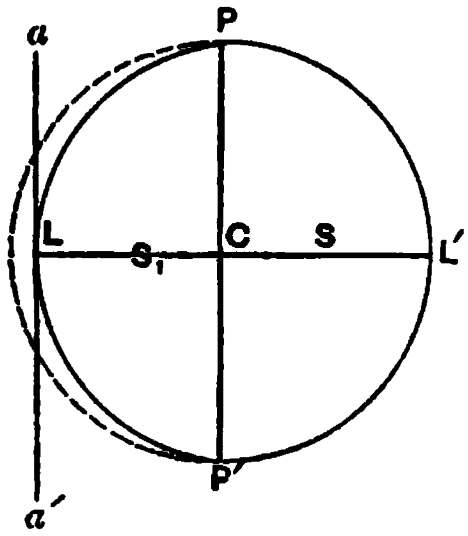

FIG. 39.

E is the point on the earth's surface from which the observation is made.

$E\Sigma$ and $K\Sigma$ are the projections on the plane of the instrument of rays of light coming from the sun. These lines are practically parallel.

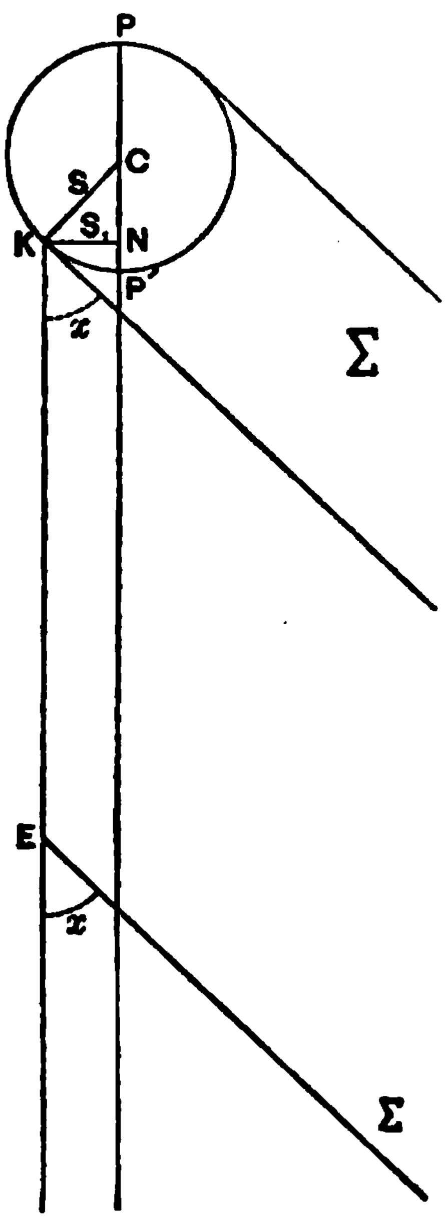

FIG. 40.

Let x = the angle formed with the plane of the meridian by the line drawn from the sun to the moon.

This will be practically the same angle as that formed by lines joining the sun and earth.

CK will be perpendicular to this line. Also, KN is perpendicular to the plane of the meridian. Therefore

$$S_1 = S \cos x. \quad . \quad . \quad . \quad (330)$$

x is now the angle which a line drawn from the sun to the earth forms with the lower branch of the meridian.

Let $\alpha =$ the moon's right ascension at the time of culmination;

$\alpha' =$ the sun's right ascension;

$\alpha' - \alpha =$ angle formed by the hour-circles drawn through the moon and sun;

$180° - (\alpha' - \alpha) =$ angle formed by sun's hour-circle with the lower branch of the meridian.

$\delta' =$ sun's declination.

In Fig. 41, E is the earth, P the pole of the heavens, and S the projection of the sun on the celestial sphere. PR is the lower branch of the meridian. SR is the arc of a great circle perpendicular to the meridian.

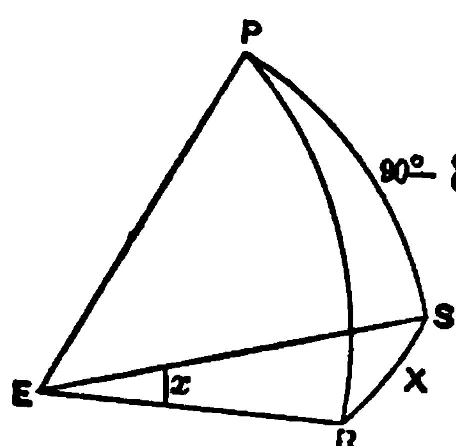

FIG 41.

Therefore $SER = x =$ arc SR.

The right-angle triangle SPR therefore gives

$$\sin x = \cos \delta' \sin (\alpha' - \alpha) \quad . \quad . \quad . \quad . \quad (331)$$

(330) and (331) therefore give the required value of S', and the correction to be applied will be of the same form as in case of S, viz., $\pm \dfrac{S'}{15\,(1 - \lambda) \cos \delta} \left\{ \begin{matrix} + \\ - \end{matrix} \right\}$ when $\left\{ \begin{matrix} \text{first} \\ \text{second} \end{matrix} \right\}$ limb is defective.

Example. 1883. October 15, the moon was observed with the portable transit instrument of the Sayre observatory as follows:

Cl. east.	First Limb.		Second Limb.		Level. E.	Level. W.
	I	$21^s.2$	I	$43^s.1$		
	II	38 .8	II	0 .1	12.9	12.9
	III	55 .1	III	16 .9	11.5	14.2
	IV	12 .5	IV	34	12.9	12.9
	V	$1^h\ 16^m\ 29^s$	V	$1^h\ 18^m\ 50^s.8$	11.3	14.4
	$T =$	1 15 55 .32		1 18 16 .98	12.15	13.60

From the table of moon culminations (page 379 of the ephemeris) we find, for the time of the moon's transit at Bethlehem:

Apparent declination	$= \delta =$	$9^\circ\ 14'\ 18''$
Equatorial horizontal parallax	$= \pi =$	$3681''$
	λ	$.0425$
Sidereal time of semidiameter passing the meridian	$=$	$70^s.76$
We also have	$\varphi' =$	$40^\circ\ 25'\ 2''$
	$\log \rho =$	9.99939
Correction for inequality of pivots	$= p = -$	$.062$

The computation by formulæ (XIX), Art. 195, is now as follows:

$$\varphi' - \delta = 31^\circ\ 10'\ 44'' \qquad \sin(\varphi' - \delta) = 9.7141 \qquad \cos(\varphi' - \delta) = 9.9323$$

$$\log \pi = 3.5660 \qquad \sin \pi = 8.2515$$

$$\log \rho = 9.9994 \qquad \log \rho = 9.9994$$

$$\text{Sum} = 3.2795 \qquad \text{Sum} = 8.1832$$

$$\text{Nat No. } 1903'' \qquad .01525$$

$$\delta' = 8^\circ\ 42'\ 35'' \qquad A_1 \quad .98475$$

$$\begin{aligned}
1 - \lambda &= 0.9575 \\
\log(1 - \lambda) &= 9.9811 \\
\log B_1 &= .0189 \\
\cos \delta' &= 9.9950 \\
\log F &= .0179 \\
\log F \cos \delta' &= .0129 \qquad F \cos \delta' = 1.030
\end{aligned}$$

The above level-readings in connection with p give $b = + ^s.115$.
We have derived from transits of stars $c' = + .154$;
$a = - .065$;
$\Delta T = - 5^s.47$.

We now apply the last of formulæ (XIX):

$$\begin{aligned}
a \frac{\sin(\varphi - \delta')}{\cos \delta'} &= - ^s.035 \\
b \frac{\cos(\varphi - \delta')}{\cos \delta'} &= + .099 \\
\frac{c'}{\cos \delta'} &= + .156 \\
\text{sum} &= + .220 \\
(\text{Sum})\ F \cos \delta' &= + ^s.227
\end{aligned}$$

	First Limb.	Second Limb.
$T =$	$1^h\ 15^m\ 55^s.32$	$1^h\ 18^m\ 16^s.98$
$\varDelta T =$	$-\ 5.47$	$-\ 5.47$
Corrections	$+\ \ .23$	$+\ \ .23$
Right ascension of limb	$1^h\ 15^m\ 50^s.08$	$1^h\ 18^m\ 11^s.74$

The right ascension of the centre will be obtained from either of these by applying the correction for semidiameter, which is the same as the *sidereal time of the semidiameter passing the meridian.* The illumination of the second limb, however, was defective, and therefore the correction given by formulæ (330) and (331) should be applied.

From the ephemeris we have

$$\begin{aligned} &\text{Sun's right ascension} &&= \alpha' = 13^h\ 23^m\ 10^s \\ &\text{Sun's declination} &&= \delta' = -\ 8^\circ\ 45'\ 18'' \\ &\text{Moon's right ascension} &&= \alpha = \ \ 1^h\ 17^m\ \ 1^s \end{aligned}$$

Applying formula (331),
$$\begin{aligned} \alpha' - \alpha &= 12^h\ \ 6^m\ \ 9^s \\ &= 181^\circ\ 32'\ 15'' \end{aligned}$$

$$\begin{aligned} \sin(\alpha' - \alpha) &= 8.4286 \\ \cos\delta' &= 9.9949 \\ \sin x &= 8.4235 \\ \cos x &= 9.99985 \\ \log 70^s.76 &= 1.84979 \end{aligned}$$

Corrected value $= 70^s.74$ $\qquad \log = 1.84964$

Therefore

Right ascension moon's centre from observation of first limb $= 1^h\ 17^m\ 0^s.84$.

Right ascension moon's centre from observation of second limb $= 1\ \ 17\ \ 1\ .00$.

Transits of the Sun and Planets.

197. Formulæ (XIX) derived for the moon apply equally to the sun and planets. As, however, the parallax in these cases will always be small, we can write without appreciable error $z = z'$ and $\delta = \delta'$.

Then $\qquad A_1 = 1; \qquad B_1 = \dfrac{1}{1 - \lambda}; \qquad F = B_1 \sec\delta;$

$$\alpha = T + \Delta T + iB_1 \sec\delta + \left(a\frac{\sin(\phi-\delta)}{\cos\delta} + \frac{b\cos(\phi-\delta)}{\cos\delta} + \frac{c}{\cos\delta}\right)B_1 \pm \frac{S}{15(1-\lambda)\cos\delta}. \quad (332)$$

The last term can be taken directly from the ephemeris, where it is given under the heading "Sidereal time of semidiameter passing the meridian." The object of such an observation will be to determine the clock correction ΔT.

If the sun is observed with a mean time chronometer, the rate of which is small, λ may be neglected, as then the motion of the sun will practically correspond with that of the chronometer. If the chronometer has a large rate on *apparent* time, this rate may be placed equal to λ, $+$ when the chronometer is gaining, $-$ when losing.

Let $E =$ the equation of time for the instant of transit;

$S'' =$ the mean time of semidiameter passing the meridian;

$T =$ chronometer time of observation reduced to middle (or mean) thread;

$\Delta T =$ the chronometer correction on mean time.

Then $12^h + E =$ mean time of sun's transit.

Therefore

$$12^h + E = T + \Delta T + a\frac{\sin(\varphi-\delta)}{\cos\delta} + b\frac{\cos(\varphi-\delta)}{\cos\delta} + \frac{c}{\cos\delta} \pm S''. \tag{333}$$

S'' is $+$ for preceding limb, and $-$ for following limb; when both are observed it vanishes from the mean. ΔT will then be given by (333).

The Transit Instrument in the Prime Vertical.

198. The transit may be employed for determining the instant of a star's passing the prime vertical, in a manner similar to that already explained for determining its passage over the meridian. Such observations furnish a very accurate method of determining the latitude of the place of observa-

tion, or, in a fixed observatory where the latitude is known, for determining the declinations of the stars observed. The practical application of the transit to these purposes is due to Bessel, although a prime vertical transit was used by Roemer more than a hundred years earlier.

This method of determining latitude has been considerably used by the astronomers of Europe, and to a less extent in America. It is now almost entirely superseded by the use of the zenith telescope, so that a complete presentation of the theory is relatively much less important now than it was thirty or forty years ago.

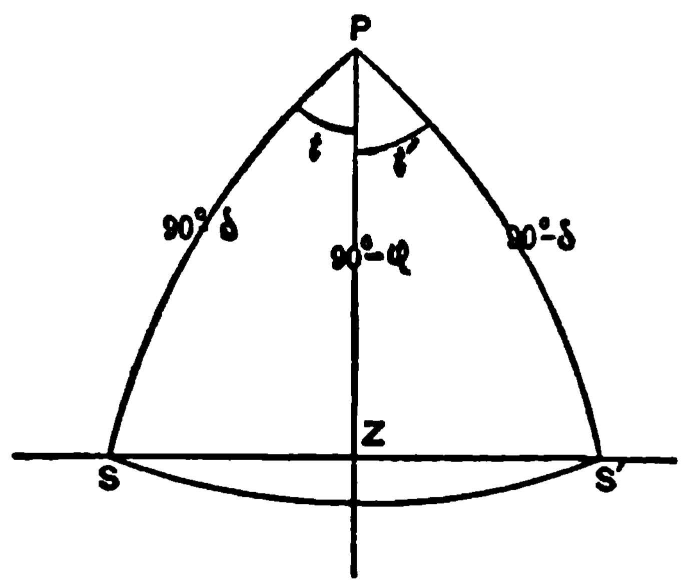

FIG. 42.

The principle is as follows: Let P be the pole, Z the zenith, and S a star which crosses the prime vertical at S and S'. Suppose the instant of the star's passing the prime vertical to be observed with a transit instrument perfectly adjusted in this plane; then if the rate of the clock is known, the difference between the two times of transit will be the angle SPS', one half of which is equal to $SPZ = t$. Then from the right-angle triangle SPZ or $S'PZ$ we have

$$\tan \varphi = \tan \delta \sec t, \quad . \quad . \quad . \quad . \quad . \quad (334)$$

from which either φ or δ may be determined when the other is known. In the field it will of course be φ which is to be determined.

The process is then analogous to that employed with the instrument mounted in the meridian; viz., the adjustments are made as accurately as may be, and the corrections to the final result determined for outstanding deviations. As we shall see, the value of the method consists largely in the

facility with which the effect of instrumental errors may be eliminated. It is evident that only those stars can be observed on the prime vertical which culminate between the equator and the zenith, that is, whose declinations are between 0 and φ.

Adjustments.

199. It is only necessary to explain the method of placing the instrument in the prime vertical, all the remaining adjustments being the same as when the instrument is in the meridian. For this purpose a star is selected whose declination is small, and the clock time computed when the star will be on the prime vertical. Triangle *PSZ* of Fig. 42 gives

$$\cos t = \frac{\tan \delta}{\tan \varphi} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (335)$$

The clock time of the star's passing the prime vertical will then be

$$\alpha \pm t - \varDelta T \begin{Bmatrix} \text{west} \\ \text{east} \end{Bmatrix} . \quad . \quad . \quad . \quad . \quad . \quad (336)$$

When the clock time is that given by this formula, the middle thread of the reticule must be brought on the star by the fine-motion azimuth screw.

It will be observed that a knowledge of the latitude is necessary for computing t, but from (335) it appears that when a star is chosen whose declination is nearly 0, a small error in the assumed value of φ may exist without materially affecting the value of t. The adjustment should be tested by stars both east and west of the meridian, as an error in the assumed value of φ will affect the computed times for east and west stars with opposite signs.

Some instruments are provided with azimuth circles like that shown in Fig. 28, in which case the simplest method of proceeding will be to first adjust the instrument in the plane of the meridian and then turn it in azimuth 90° by the circle.

Method of Observing.

200. A list of stars to be observed should first be prepared, for which the time of passing the prime vertical, both east and west, must be computed, also the zenith distance or setting of the finding circle. Formulæ (335) and (336) give the required time. The zenith distance is given by

$$\cos z = \frac{\sin \delta}{\sin \varphi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (337)$$

If the star is near the zenith, the time required to pass the thread intervals will be comparatively large, so that it will be convenient to compute approximately the time of passing the first thread.

Let i = the equatorial interval of the first thread;
I = the corresponding star interval.

Then $I = \dfrac{i}{\sin \varphi \cos \delta \sin t} = \dfrac{i}{\sin \varphi \sin z}$ approximately. (338)

The proof of this formula will be given hereafter. I will be subtracted from the time given by (336) for a star either east or west.

As the star moves obliquely across the field, it will be necessary to change the zenith distance of the telescope for every thread in order to have the transits take place between the two horizontal threads.

Mathematical Theory.

201. The equations (275) and (281) apply to the transit instrument in any position whatever, and consequently may be used in this case. It will perhaps be better to derive the formulæ directly.

Let us consider the point where the north end of the axis produced pierces the celestial sphere. This we shall call the north end of axis.

Let this point be referred to a system of rectangular axes, the horizon being the plane of xy, the positive axis of x being directed north, the positive axis of y east, and the positive axis of z to the zenith.

Let a = the azimuth of the north end of axis, reckoned from the north point towards the east;
b = the altitude.

Then $x = \cos b \cos a$; $\quad y = \cos b \sin a$; $\quad z = \sin b$. (339)

In the second system let the equator be the plane of xy, the positive axis of z being parallel to the earth's axis, the positive axis of x being directed to the point where the lower branch of the meridian intersects the equator, and the axis of y coinciding with that in the first system.

Let n and $180° + m$ = the declination and hour-angle of the north end of axis.

Then $x' = \cos n \cos m$; $\quad y' = \cos n \sin m$; $\quad z' = \sin n$. (340)

The formulæ for transformation of co-ordinates give

$$\left.\begin{aligned} \cos n \cos m &= \cos b \cos a \sin \varphi - \sin b \cos \varphi; \\ \cos n \sin m &= \cos b \sin a; \\ \sin n &= \cos b \cos a \cos \varphi + \sin b \sin \varphi. \end{aligned}\right\} \quad (341)$$

If the instrument is carefully levelled and adjusted in the prime vertical, we way write

$$\cos b = 1; \qquad \cos a = 1; \qquad \sin b = b; \qquad \sin a = a;$$

when the above equations may be written

$$\left.\begin{aligned} \cos n \cos m &= \sin(\varphi - b); \\ \cos n \sin m &= a; \\ \sin n &= \cos(\varphi - b). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (342)$$

We shall find these formulæ useful in subsequent transformations.

202. Let $90° + c$ = the angle between the clamp end of the rotation axis and the object end of the collimation axis;

t and δ = the hour-angle and declination of a star observed on the middle thread.

Let the star be referred to a system of rectangular axes, the equator being the plane of xy, the axis of x being in the direction of the rotation axis, positive towards the north.

Then the angle formed by the radius vector with the plane of xy will be δ, and the angle between the projection of the radius on the plane of xy and the axis of x will be

$$180° + (t - m).$$

$$x = -\cos\delta\cos(t-m); \quad y = -\cos\delta\sin(t-m); \quad z = \sin\delta. \quad (343)$$

In the second system, let the axis of x coincide with the rotation axis, the axis of y coinciding with that of the former system. Then the position of the instrument being *clamp north*, $- c$ will be the angle formed by the radius vector and

the plane of yz. Let δ_1 be the angle formed with the axis of y by the projection of the radius vector on the plane of yz. Then

$$x' = -\sin c; \quad y' = \cos c \cos \delta_1; \quad z' = \cos c \sin \delta_1. \quad (344)$$

The angles between the axis of x and x' being n, we have

$$x' = x \cos n + z \sin n; \quad y' = y; \quad z' = -x \sin n + z \cos n. \quad (345)$$

We therefore have

$$\left.\begin{aligned} \sin c &= \cos \delta \cos (t - m) \cos n - \sin \delta \sin n; \\ \cos c \cos \delta_1 &= -\cos \delta \sin (t - m); \\ \cos c \sin \delta_1 &= \cos \delta \cos (t - m) \sin n + \sin \delta \cos n. \end{aligned}\right\} \quad (346)$$

Equations (341) and (346) express in the most general form the relations between the quantities which determine the position of the instrument and the quantities φ, δ, and t.

203. The adjustments may always be made accurately enough so that the first of (346) may be written

$$c = \cos \delta \frac{\cos (t - m)}{\cos m} \sin (\varphi - b) - \sin \delta \cos (\varphi - b); \quad (347)$$

where the values of sin n and cos n given by (342) have been substituted.

Let
$$\begin{aligned} h \sin \varphi' &= \sin \delta; \\ h \cos \varphi' &= \frac{\cos \delta \cos (t - m)}{\cos m} \quad . \quad . \quad . \quad . \quad (348) \end{aligned}$$

Then (347) becomes $\quad c = h \sin (\varphi - \varphi' - b)$.

From the first of (348), $h = \frac{\sin\delta}{\sin\varphi'}$, and therefore when δ is not too small we may write

$$\sin(\varphi - \varphi' - b) = \varphi - \varphi' - b = \frac{c \sin\varphi'}{\sin\delta},$$

or

$$\varphi = \varphi' + b + \frac{c\sin\varphi'}{\sin\delta}. \quad . \quad . \quad . \quad . \quad (349)$$

Dividing the first of (348) by the second, we obtain

$$\tan\varphi' = \tan\delta\sec(t - m)\cos m. \quad . \quad . \quad . \quad (350)$$

When c, m, and b are known quantities, (349) and (350) will give the latitude, as δ is the known declination of the star, and t is obtained by observation.

204. b is determined as in previous discussions by the striding-level. This should be done with care, as we see from (349) that an error in b will affect the latitude by its full amount. t and m are determined as follows:

Let t' and t = hour-angles of the star at east and west transit respectively;
T' and T = observed clock times at east and west transit respectively;
$\varDelta T'$ and $\varDelta T$ = corresponding clock corrections;
2ϑ = elapsed time between east and west observation;
α = star's right ascension = side real time of culmination.

Then

$$\begin{aligned} t' &= T' + \varDelta T' - \alpha; \\ t &= T + \varDelta T - \alpha; \\ \vartheta &= \tfrac{1}{2}[(T' + \varDelta T') - (T + \varDelta T)]; \\ m &= \tfrac{1}{2}[(T + \varDelta T) + (T' + \varDelta T')] - \alpha. \quad . \quad (351) \end{aligned}$$

Therefore $\vartheta = t - m = -(t' - m)$.

For determining ϑ we see that the clock rate must be known, but neither the clock correction nor the star's right ascension is required. For determining m a knowledge of both these quantities will be essential.

With the portable instrument c may most readily be determined by observation in the meridian, as already explained,* but on account of the facility with which an error in this quantity may be eliminated its exact determination is not very important.

Effects of Errors in the Data.

205. Let us now investigate the effect upon the latitude of uncorrected errors in the quantities b, c, δ, and $\vartheta = t - m$.

Suppose the same star observed both east and west on two different nights, first with the instrument in the position *clamp north;* second, *clamp south.*

Let b and b' = the inclination given by the level for *clamp north* and *south;*

p = the (unknown) correction for inequality of pivots;

c = collimation constant, + for *clamp north;*

q = the unknown error in determining c.

Then $(b + p)$ and $(b' - p)$ = the true inclination of axis for clamp north and south respectively;

$c + q$ = true value of collimation constant.

* See equation (305).

Let φ' and φ'' = the latitude given by (350) from transits of the same star *clamp north* and *south* respectively.

Then (349) gives

$$\varphi = \varphi' + b + p + (c + q)\frac{\sin \varphi'}{\sin \delta} \text{ clamp north;}$$

$$\varphi = \varphi'' + b' - p - (c + q)\frac{\sin \varphi''}{\sin \delta} \text{ clamp south.}$$

The mean is

$$\varphi = \tfrac{1}{2}[\varphi' + \varphi'' + b' + b''] + (c + q)\frac{\sin \varphi' - \sin \varphi''}{2 \sin \delta}. \quad (352)$$

Unless the errors of adjustment are very large the last term of this equation will be inappreciable, so that practically constant errors of collimation and level are eliminated by combining observations on the same star in different positions of the axis.

Errors in ϑ may result either from errors in the clock rate or they may be simply the unavoidable errors of observation. To ascertain their effect upon φ we differentiate (350) with respect to φ and ϑ, by which means we derive

$$d\varphi = \tfrac{1}{2} \sin 2\varphi \tan \vartheta \, d\vartheta \text{ (nearly)}. \quad . \quad . \quad . \quad (353)$$

From this equation it appears that an error in ϑ will produce the less effect upon φ the smaller ϑ is. Also, that the algebraic sign when the star is east is the opposite of that when it is west. Therefore

The effect of a small error in ϑ will be eliminated by observing the star both east and west of the meridian.

Differentiating (350) with respect to φ and δ, we find

$$d\varphi = \frac{\sin 2\varphi}{\sin 2\delta} d\delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (354)$$

As the declination cannot be greater than φ, we see that when φ is less than 45° an error in δ will produce a larger error in φ. For φ greater than 45°, $d\varphi < d\delta$ for all stars whose δ is between φ and $90° - \varphi$. In any case the effect upon φ will be less the nearer the star is to the zenith.

The best result will therefore be obtained by observing at both the east and west transit a star which culminates near the zenith and in both positions of the axis. The observations may be made on the same star on two different nights, the clamp being north in one case and south in the other. Or they may all be made on the same night if the star passes quite near the zenith, as follows: *First*, observe the east transit over the first half of the threads of the reticule; *second*, reverse the instrument and observe the transit over the same threads, now in the reverse position; *third*, observe the west transit over the same threads; then, *fourth*, reverse the instrument again and finish the observation of the west transit over the threads, now in the same position as at first. This method is due to Struve. It will not generally be followed in the field owing to the danger of disturbing the instrument in reversing so frequently.

Reduction to the Middle or Mean Thread.

206. In formula (349), c is the error of collimation of the middle or mean thread. In reducing the observations over a side thread we may replace c by $c + i$ (i being the equatorial interval of the thread), and reduce each thread separately. It will, however, be simpler to first reduce all observations to the times over the middle or mean threads. This process is less simple than in case of meridian observations, since the mean of the times over the several threads will not in this case be the time over the mean thread.

The reduction may be made in either of two ways: *first*,

by reducing each thread separately to the middle (or mean) thread; *second*, by applying a correction to the mean of the times over the different threads to reduce it to the time over the mean thread.

First. The thread intervals should be determined by meridian transits as already explained.*

Let $i =$ the equatorial interval of any thread from the middle thread;
$I =$ the corresponding star interval;
$t =$ the hour-angle of the star when on the middle (or mean) thread;
$t - I =$ the hour-angle when on the side thread;
$c + i$ may be regarded as the collimation error of the side thread.

Then, from the first of (346),

$$\sin(c + i) = -\sin n \sin \delta + \cos n \cos \delta \cos(t - I - m);$$
$$\sin c = -\sin n \sin \delta + \cos n \cos \delta \cos(t - m).$$

Subtracting, we readily find

$$2 \cos(\tfrac{1}{2}i + c) \sin \tfrac{1}{2}i = \cos n \cos \delta 2 \sin (t - m - \tfrac{1}{2}I) \sin \tfrac{1}{2}I.$$

Since c will be very small, the first term of this may be written $\sin i$ without appreciable error. Then

$$2 \sin \tfrac{1}{2}I = \frac{\sin i}{\cos n \cos \delta \sin(t - m - \tfrac{1}{2}I)}. \quad (355)$$

From (342) we may write $\cos n = \sin(\varphi - b)$. Also, $(t - m) = \vartheta$. $\sin i$ may be written i.

$$2 \sin \tfrac{1}{2}I = I(1 - \tfrac{1}{24}I^2) = I(\cos I)^{\frac{1}{12}}.$$

---

* Art. 174.

Therefore (355) may be written without appreciable error,

$$I = \frac{i}{\sin(\varphi - b)\cos\delta\sin(\vartheta - \tfrac{1}{2}I)(\cos I)^{\frac{1}{2}}}; \quad (356)$$

and with accuracy sufficient for most cases,

$$I = \frac{i}{\sin\varphi\cos\delta\sin(\vartheta - \tfrac{1}{2}I)}. \quad . \quad . \quad . \quad (357)$$

Log $(\cos I)^{\frac{1}{2}}$ might be tabulated, but it will be required so rarely that it will hardly repay the labor. The value of I required in the second member of the above formulæ may be found directly from the observations themselves, by taking the difference of the observed time over the side thread and middle thread.

Care must be taken to give the proper algebraic signs to i, I, and ϑ,—i and I being plus for north threads and minus for south ones; ϑ, plus for west, minus for east transits.

207. *Second.* This method of reduction is due to Bessel, and is more convenient when many stars are to be reduced. Resuming the first of (346), and writing $c + i$ instead of sin c and $t - I$ for t,

$$c + i = -\sin n\sin\delta + \cos n\cos\delta\cos(t - I - m). \quad (358)$$

Such an equation is given by each thread observed. If μ threads are observed, the mean of the resulting equations will be

$$c + i_0 = -\sin n\sin\delta + \cos n\cos\delta\frac{1}{\mu}\Sigma\cos(t - m), \quad (359)$$

where i_0 is the mean of the equatorial intervals, Σ is the summation sign, t represents the hour-angle corresponding to any thread.

Let $T =$ the arithmetical mean of the times observed on the individual threads (supposed corrected for clock error and rate);

$T - I =$ the time over any thread.

Then
$$(t - m) = (T - \alpha - m) - I,$$

and

$$\frac{1}{u}\Sigma \cos (t - m) = \cos (T - \alpha - m) \frac{1}{\mu} \Sigma \cos I$$
$$+ \sin (T - \alpha - m) \frac{1}{\mu} \Sigma \sin I. \quad (360)$$

Now let
$$\left.\begin{aligned} k \cos \varkappa &= \frac{1}{\mu} \Sigma \cos I; \\ k \sin \varkappa &= \frac{1}{\mu} \Sigma \sin I. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (361)$$

Then
$$\frac{1}{\mu} \Sigma \cos (t - m) = k \cos (T - \alpha - \varkappa - m). \quad (362)$$

(359) then becomes

$$c + i_{\circ} = - \sin n \sin \delta + k \cos n \cos \delta \cos (T - \alpha - \varkappa - m). (363)$$

Now let
$$\left.\begin{aligned} \gamma \cos \delta_1 &= k \cos \delta ; \\ \gamma \sin \delta_1 &= \sin \delta . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (364)$$

Then (363) becomes

$$\frac{c + i_{\circ}}{\gamma} = - \sin n \sin \delta_1 + \cos n \cos \delta_1 \cos (T - \alpha - \varkappa - m). (365)$$

Thus, by computing the auxiliary quantities γ, δ_1, and $\varkappa$, the form of the equation for the mean of the threads is the same as that for the middle thread.

Practically γ will seldom differ appreciably from unity.

δ_1 and x may very readily be computed by the aid of tables A and B, page 365. These tables are computed as follows:

Since $\Sigma I = 0$ (T being the mean of the observed times, and I the difference between T and the time on any thread), (361) may be written

$$\left.\begin{aligned} k \cos x &= 1 - \frac{2}{\mu}\Sigma \sin^2 \tfrac{1}{2}I; \\ k \sin x &= \quad - \frac{1}{\mu}\Sigma(I - \sin I). \end{aligned}\right\} \quad . \quad . \quad . \quad (366)$$

From these it appears that $k \sin x$ is of the order I^3, and that $k \cos x$ only differs from unity by a quantity of the order I^2. There will then be no appreciable error in writing

$$\left.\begin{aligned} k &= 1 - \frac{2}{\mu}\Sigma \sin^2 \tfrac{1}{2}I; \\ x &= \quad - \frac{1}{\mu}\Sigma(I - \sin I). \end{aligned}\right\} \quad . \quad . \quad . \quad (367)$$

And since, from (364), we have

$$\tan \delta_1 = \frac{1}{k} \tan \delta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (368)$$

the method of Art. 74 for expanding a function of this form gives

$$\delta_1 = \delta + \left(\frac{1 - k}{1 + k}\right)\frac{\sin 2\delta}{\sin 1''} + \frac{1}{2}\left(\frac{1 - k}{1 + k}\right)^2 \frac{\sin 4\delta}{\sin 1''}. \quad . \quad (369)$$

This becomes, by substituting for k its value,

$$\delta_1 = \delta + \frac{\dfrac{1}{\mu}\Sigma\dfrac{\sin^2 \frac{1}{2}I}{\sin 1''}}{1 - \dfrac{1}{\mu}\Sigma \sin^2 \frac{1}{2}I} \sin 2\delta. \quad . \quad . \quad . \quad . \quad (370)$$

For computing δ_1, table A, page 365, gives the value of $\dfrac{\sin^2 \frac{1}{2}I}{\sin 1''}$; the argument being the difference between each ob-

served time respectively and the mean of all, expressed in minutes and seconds of time for convenience. The arithmetical mean of these quantities will be the numerator of the coefficient of $\sin 2\delta$ in (370). The denominator differs very little from unity. When desirable, this small difference may be corrected by table B, the argument of which is the numerator, viz., $\frac{1}{\mu}\Sigma\frac{\sin^2\frac{1}{2}I}{\sin 1''}$.

The fourth column of table A gives the quantity $(I - \sin I)_0$, the arithmetical mean of these quantities being equal to κ.

If γ is required, we readily find, from (364),

$$\gamma = \frac{1 - (1 - k)\cos^2\delta}{\cos(\delta_1 - \delta)}.$$

The denominator does not differ appreciably from unity, and

$$1 - k = \frac{2}{\mu}\Sigma\sin^2\tfrac{1}{2}I.$$

Therefore $$\gamma = 1 - \frac{2}{\mu}\cos^2\delta\,\Sigma\sin^2\tfrac{1}{2}I. \quad . \quad . \quad . \quad . \quad (371)$$

Since this only appears as the divisor of the small quantity $c + i_0$, it will very rarely be required.

The quantity i_0 will vanish when the star is observed over all of the threads, and the equatorial intervals reckoned from the mean of the threads.

Having shown how our fundamental equation which applies to the time over the middle thread may be reduced to a like form when the time is the mean of the times over the different threads—see equation (365)—we may now solve this equation for φ as before.

Formulæ (349) and (350) will then have the form

$$\left.\begin{aligned} \tan\varphi' &= \tan\delta_1 \sec(T - \alpha - \kappa)\cos m; \\ \varphi &= \varphi' + b + (c + i_0)\frac{\sin\varphi'}{\sin\delta_1}. \end{aligned}\right\} \quad . \quad (372)$$

208. *Formulæ for Latitude by Prime Vertical Transits.*

Preliminary Computation.

$$\left.\begin{aligned}
\cos z &= \frac{\sin \delta}{\sin \varphi};\\
\cos t &= \frac{\tan \delta}{\tan \varphi};\\
I &= \frac{i}{\sin \varphi \sin z};
\end{aligned}\right\} \quad \text{(XX)}$$

Clock time of passing first thread

$$= \alpha \pm t - \Delta T - I \begin{Bmatrix} \text{west} \\ \text{east} \end{Bmatrix}.$$

Reduction to Middle or Mean Thread.

$$\left.\begin{aligned}
I &= \frac{i}{\sin (\varphi - b) \cos \delta \sin (\vartheta - \frac{1}{2}I)};\\
\vartheta &= \tfrac{1}{2} [(T' + \Delta T') - (T + \Delta T)];\\
m &= \tfrac{1}{2} [T' + \Delta T' + T + \Delta T] - \alpha;\\
\tan \varphi' &= \tan \delta \sec \vartheta \cos m;\\
\varphi &= \varphi' + b + c \frac{\sin \varphi'}{\sin \delta}.
\end{aligned}\right\} \quad \text{(XXa)}$$

Bessel's Method of Reduction.

$$\left.\begin{aligned}
\varkappa &= -\frac{1}{\mu} \Sigma (I - \sin I);\\
\delta_1 &= \delta + \frac{\dfrac{1}{\mu} \Sigma \dfrac{\sin^2 \frac{1}{2}I}{\sin 1''}}{1 - \dfrac{1}{\mu} \Sigma \sin^2 \frac{1}{2} I} \sin 2\delta;\\
\tan \varphi' &= \tan \delta_1 \sec (T - \alpha - \varkappa) \cos m;\\
\varphi &= \varphi' + b + (c + i_0) \frac{\sin \varphi'}{\sin \delta_1}.
\end{aligned}\right\} \quad \text{(XXb)}$$

TABLE A.

For reducing transits over several threads to a common instant.

I	$\frac{\sin^2 \frac{1}{2}I}{\sin 1''}$	D	κ
$0^m\ 00^s$	$0''.00$	.03	$''.00$
10	.03	.08	.00
20	.11	.14	.00
30	.25	.19	.00
40	.44	.24	.00
50	0 .68	.30	.00
$1^m\ 00^s$	0 .98	.36	.00
10	1 .34	.41	.00
20	1 .75	.46	.01
30	2 .21	.52	.01
40	2 .73	.57	.01
50	3 .30	.63	.02
$2^m\ 00^s$	3 .93	.68	.02
10	4 .61	.74	.03
20	5 .35	.79	.04
30	6 .14	.84	.04
40	6 .98	.90	.05
50	7 .88	.96	.06
$3^m\ 00^s$	8 .84	1.00	.08
10	9 .84	1.07	.09
20	10 .91	1.12	.11
30	12 .03	1.17	.12
40	13 .20	1.23	.14
50	14 .43	1.28	.16
$4^m\ 00^s$	15 .71	1.33	.18
10	17 .04	1.39	.21
20	18 .43	1.45	.23
30	19 .88	1.50	.26
40	21 .38	1.55	.29
50	22 .93	1.61	.32
$5^m\ 00^s$	24 .54	1.67	.36
10	26 .21	1.71	.40
20	27 .92	1.78	.44
30	29 .70	1.82	.48
40	31 .52	1.88	.52
50	33 .40		.57

I	$\frac{\sin^2 \frac{1}{2}I}{\sin 1''}$	D	κ
		1.94	
$6^m\ 00^s$	$35''.34$	1.99	$''.62$
10	37 .33	2.05	.67
20	39 .38	2.10	.73
30	41 .48	2.15	.79
40	43 .63	2.21	.85
50	45 .84	2.26	.91
$7^m\ 00^s$	48 .10	2.32	.98
10	50 .42	2.37	1.05
20	52 .79	2.43	1.12
30	55 .22	2.48	1.20
40	57 .70	2.54	1.28
50	60 .24	2.59	1.37
$8^m\ 00^s$	62 .83	2.64	1.46
10	65 .47	2.70	1.55
20	68 .17	2.75	1.65
30	70 .92	2.81	1.75
40	73 .73	2.86	1.86
50	76 .59	2.92	1.97
$9^m\ 00^s$	79 .51	2.97	2.08
10	82 .48	3.03	2.20
20	85 .51	3.08	2.32
30	88 .59	3.14	2.45
40	91 .73	3.19	2.58
50	94 .92	3.24	2.72
$10^m\ 00^s$	98 .16	3.30	2.86
10	101 .46	3.35	3.00
20	104 .81	3.41	3.15
30	108 .22	3.46	3.30
40	111 .68	3.52	3.46
50	115 .20	3.57	3.63
$11^m\ 00^s$	118 .77	3.62	3.80
10	122 .39	3.68	3.98
20	126 .07	3.74	4.16
30	129 .81	3.79	4.34
40	133 .60	3.84	4.53
50	137 .44		4.73

TABLE B.

For correcting the coefficient of sin 2δ.

$\frac{1}{\mu} \Sigma \frac{\sin^2 \frac{1}{2}I}{\sin 1''}$	Correction.
10''	+ ''.000
20	.002
30	.004
40	.008
50	.012
60	.017
70	.024
80	.031
90	.039
100	.048
110	.059
120	.070
130	.082
140	.095
150	.109
160	.124
170	.140
180	.157
190	.175
200	.194

209. As an example of the determination of latitude by this method, the following observations have been selected from Pierce's Memoir on the Latitude of Cambridge, Mass. (*Memoirs of American Academy of Sciences*, vol. ii. p. 183):

Star.	Date. 1884.	Clamp.	Transit.	Times of Transit over Threads. t_1	t_2	t_3	t_4	t_5	t_6	t_7	Error of Level. N. end high.
α Lyræ	Dec. 23	S.	E.	$16^h\ 38^m\ 5^s.8$	$36^m\ 57^s.2$	$35^m\ 49^s.0$	$34^m\ 42^s.0$	$33^m\ 34^s.5$	$32^m\ 28^s.2$	$31^m\ 22^s.5$	+ ″.41
"	"	S.	W.	20 21 45 .0	22 54 .0	24 1 .5	25 9 .0	26 16 .2	27 22 .5	28 28 .1	− .02
α Lyræ	Dec. 29	N.	E.	16 31 22 .3	32 28 .5	33 34 .3	34 41 .2	35 47 .1	36 55 .2	37 5 .5	− 1″.25
"	"	N.	W.	20 28 28 .3	27 22 .5	26 18 .4		24 2 .0			− 1. .32
β Persei	Dec. 25	N.	E.	1 26 29 .0	27 57 .0	29 25 .0	30 55 .5	32 27 .8	34 1 .5	35 36 .5	+ .15
"	"	N.	W.	4 26 17 .8	24 49 .5	23 21 .4	21 50 .8	20 19 .0	18 45 .5	17 10 .0	+ .22
β Persei	Dec. 26	S.	E.	1 35 36 .5	34 0 .5	32 27 .5	30 55 .6	29 24 .5	27 56 .5	26 28 .5	+ .87
"	"	S.	W.	4 17 11 .0	18 46 .0	20 19 .6	21 51 .0				+ .87

The equatorial intervals of the threads from the middle thread are

$i_1 = 51^s.11$; $i_2 = 33^s.98$; $i_3 = 17^s.02$; $i_4 = 0^s.00$; $i_5 = 17^s.10$;
$i_6 = 34^s.14$; $i_7 = 51^s.16$.

The clock correction and rate:

Date.	Sidereal Time.	ΔT. Clock slow.	Daily Rate.
Dec. 20	$0^h\ 30^m$	$+ 1^m\ 46^s.83$	
24	2 0	+ 1 48 .74	$-\ {}^s.46$
25	2 15	+ 1 49 .55	− .81
26	2 45	+ 1 47 .78	+ 1 .77
29	0 30	+ 1 48 .13	− .12
Jan. 2	5 15	+ 1 51 .10	− .71

Apparent places of the stars observed:

α Lyræ, December 23d, $\alpha = 18^h\ 31^m\ 40^s.32$; $\delta = 38°\ 38'\ 39''.76$.
α Lyræ, December 29th, $\alpha = 18\ \ 31\ \ 40.36$; $\delta = 38\ \ 38\ \ 38\ .08$.
β Persei, December 25th, $\delta = 40\ \ 21\ \ 25\ .83$.
β Persei, December 26th, $\delta = 40\ \ 21\ \ 25\ .86$.

The collimation error c is assumed equal to zero. Assumed $\varphi = 42°\ 22'\ 48''$. We shall first compute the latitude by formulæ (XXa). The transits over the several threads must first be reduced to the middle thread by the formula

$$I = \frac{i}{\sin(\varphi - b)\cos\delta\sin(\vartheta - \frac{1}{2}I)}.$$

The complete reduction is given for the observations of α Lyræ, December 23d, in order to illustrate the process.

Observed Times.	ϑ.	Observed I.	$\frac{1}{2}I$.	$\vartheta - \frac{1}{2}I$.
16h 38m 5s.8		− 3m 23s.8	− 25′ 28″	− 28° 22′ 55″
36 57 .2		− 2 15 .2	− 16 54	− 28 31 29
35 49 .0		− 1 7 .0	− 8 23	− 28 40 0
34 42 .0				
33 34 .5		+ 1 7 .5	+ 8 26	− 28 56 49
32 28 .2		+ 2 13 .8	+ 16 43	− 29 5 6
31 22 .5	3h 50m 27s.0	+ 3 19 .5	+ 24 56	− 29 13 19
	1 55 13 .5			
20 21 45 .0	28° 48′ 23″.	+ 3 24 .0	+ 25 30	+ 28 22 53
22 54 .0		+ 2 15 .0	+ 16 52	28 31 31
24 1 .5		+ 1 7 .5	+ 8 26	28 39 57
25 9 .0				
26 16 .2		− 1 7 .2	− 8 24	28 56 47
27 22 .5		− 2 13 .5	− 16 41	29 5 4
28 28 .1		− 3 19 .1	− 24 53	+ 29 13 16

sin ($\vartheta - \frac{1}{2}I$).	log Denominator.	log I.	I.	Reduced Time.
9.67701	9.39837	2.31014	− 204s.2	16h 34m 41s.6
9.67901	9.40037	2.13085	− 135 .2	42 .0
9.68098	9.40234	1.82862	− 67 .4	41 .6
				42 .0
9.68485	9.40621	1.82679	+ 67 .1	41 .6
9.68673	9.40809	2.12517	+ 133 .4	41 .6
9.68859	9.40995	2.29898	+ 199 .1	16 34 41 .6
			$T =$	16 34 41 .71
9.67700	9.39836	2.31015	+ 204 .2	20 25 9 .2
9.67902	9.40038	2.13084	+ 135 .2	9 .2
9.68097	9.40233	1.82863	+ 67 .4	8 .9
				9 .0
9.68484	9.40620	1.82680	− 67 .1	9 .1
9.68673	9.40809	2.12517	− 133 .4	9 .1
9.68858	9.40994	2.29899	− 199 .1	20 25 9 .0
			$T' =$	20 25 9 .07

In the above the quantity ϑ is computed from the second of (XXa), using for T' and T the time over the middle thread, and neglecting the rate, which will be less than the probable error of the observation. The "observed I" is found by subtracting the observed time over each thread from the time over the middle thread. The quantities headed "log denominator" are computed

by writing the quantity log ($\sin\varphi\cos\delta$) on the lower edge of a slip of paper and adding it in succession to each of the quantities in the previous column. b is neglected in the quantity $\sin(\varphi - b)$. The quantities $\log i_1$, $\log i_2$, etc., are then written in order on the lower edge of another slip of paper and the "log denominator" subtracted, giving $\log I$. It would be sufficient to compute the intervals I for one transit only, as they are the same for both; but in a case like the above it is well to compute both as a check on the work. In the above, four-figure logarithms would have been sufficiently accurate.

In the same manner the other observations are reduced, the quantities T and T' being those given in the following computation:

Latitude from α Lyræ.

CLAMP SOUTH.

Dec. 23.	$T' = 20^h\ 25^m\ 9^s.07$	
	$\Delta T' \quad +\ 1\ 48.73$	$\tan\delta = 9.9028502$
	$T' + \Delta T' = 20\ 26\ 57.80$	$\sec\vartheta = .0573745$
	$(T' + \Delta T') - (T + \Delta T) = 3\ 50\ 27.53$	$\cos m = 00$
	$\vartheta = 1\ 55\ 13.765$	$\tan\varphi' = 9.9602247$
	$= 28°\ 48'\ 26''.5$	
		$\varphi' = 42°\ 22'\ 47''.68$
	$T = 16^h\ 34^m\ 41^s.71$	
	$\Delta T = +\ 1\ 48.56$	
	$T + \Delta T = 16\ 36\ 30.27$	
	$\tfrac{1}{2}(T + \Delta T + T' + \Delta T') = 18\ 31\ 44.035$	
	$\alpha = 18\ 31\ 40.32$	
	$m = +\ 3.715$	
	$= 55''.7$	

CLAMP NORTH.

Dec. 29.	$T' = 20^h\ 25^m\ 10^s.12$	$\tan\delta = 9.9028429$
	$\Delta T' = 1\ 48.72$	$\sec\vartheta = 0573924$
	$T' + \Delta T' = 20\ 26\ 58.84$	$\cos m = 0$
	$(T' + \Delta T') - (T + \Delta T) = 3\ 50\ 29.58$	$\tan\varphi_1' = 9.9602353$
	$\vartheta = 1\ 55\ 14.79$	
	$= 28°\ 48'\ 41''.9$	$\varphi_1' = 42°\ 22'\ 50''.19$
	$T = 16^h\ 34^m\ 40^s.66$	
	$\Delta T = +\ 1\ 48.60$	mean $\varphi' = 42°\ 22'\ 48''.935$
	$T + \Delta T = 16\ 36\ 29.26$	mean $b = -.545$
	$\tfrac{1}{2}(T + \Delta T + T' + \Delta T' = 18\ 31\ 44.05$	
	$\alpha = 18\ 31\ 40.36$	$\varphi = 42°\ 22'\ 48''.39$
	$m = +\ 3.69$	
	$= +\ 55''.3$	

In a manner precisely similar, from the observations of β *Persei* on December 25th and 26th we find—

$$\begin{array}{lrrrr}
\text{Dec. 25, } \varphi' = & 42^\circ & 22' & 48''.50 \\
\text{Dec. 26, } \varphi' = & 42 & 22 & 48\ .56 \\
\text{Mean} & 42 & 22 & 48\ .53 \\
\text{Mean of the four level-readings} & & & +\ .53 \\
\varphi = & 42 & 22 & 49\ .06
\end{array}$$

The mean of these two determinations from α *Lyræ* and β *Persei* is therefore

$$\varphi = 42^\circ\ 22'\ 48''.73.$$

The value given in the memoir from which these observations are taken is $42^\circ\ 22'\ 48''.60$. This is the result of a long series of observations.

Application of Bessel's Method.

210. As an example of Bessel's method of reduction, let us apply formulæ (XXb) to the foregoing observations of α *Lyræ*.

	Observed Time.	I	$\frac{\sin^2 \frac{1}{2} I}{\sin 1''}$	$-\kappa$	$T - \kappa - \alpha$	m. From Time on Middle Thread.	δ_1	
Dec. 23.	$16^h\ 38^m\ 5^s.8$	$-3^m\ 23^s.1$	$11''.26$	$-.11$	$T = 16^h\ 34^m\ 42^s.74$	$T' = 20^h\ 25^m\ 9^s.0$	$\delta = 38°\ 38'\ 39''.76$	$\tan \delta_1 = 9.9028710$
	$36\ \ 57.2$	$-2\ \ 14.5$	4.94	$-.03$	$\Delta T = +1\ \ 48.56$	$\Delta T' = +1\ \ 48.7$		$\sec(-) = .0572921$
	$35\ \ 49.0$	$-1\ \ 6.3$	1.21	0	$T + \Delta T = 16\ \ 36\ \ 31.30$	$T = 16\ \ 34\ \ 42.0$	$2\delta = 77\ \ 17\ \ 20$	$\cos m = 0$
E	$34\ \ 42.0$	$+\ \ 0.7$	$.00$	0	$\alpha = 18\ \ 31\ \ 40.32$	$\Delta T = +1\ \ 48.6$		$\tan \phi' = 9.9601631$
	$33\ \ 34.5$	$+1\ \ 8.2$	1.28	0	$-\kappa = .00$	$\text{Sum} = 37\ \ 3\ \ 28.3$	$\sin 2\delta = 9.9892$	$\phi' = 42°\ 22'\ 33''.12$
	$32\ \ 28.2$	$+2\ \ 14.5$	4.94	$+.03$	$-1\ \ 55\ \ 9.02$	$18\ \ 31\ \ 44.15$	$\log 4''.94 = .6937$	
	$31\ \ 22.5$	$+3\ \ 20.2$	10.93	$+.11$	$-28°\ 47'\ 15''.3$	$\alpha = 18\ \ 31\ \ 40.32$	$.6829$	
Mean	$16^h\ 34^m\ 42^s.74$		$4''.94$	$.00$		$m = +3.83$		$\tan \delta_1' = 9.9028709$
	$20^h\ 21^m\ 45^s.0$	$+3\ \ 23.0$	$11''.25$	$+.11$	$T = 20^h\ 25^m\ 8^s.04$	$= 57''.5$	Cor. to $\delta = 4''.82$	$\sec(-) = .0574212$
	$22\ \ 54.0$	$+2\ \ 14.0$	4.91	$+.03$	$\Delta T = +1\ \ 48.73$		$\delta_1 = 38°\ 38'\ 44''.58$	$\cos m = 0$
	$24\ \ 1.5$	$+1\ \ 6.5$	1.21	0	$T + \Delta T = 20\ \ 26\ \ 56.77$			$\tan \phi' = 9.9602921$
W.	$25\ \ 9.0$	$-\ \ 1.0$	$.00$	0	$\alpha = 18\ \ 31\ \ 40.32$		$\log 4''.93 = .6928$	$\phi' = 42°\ 23'\ 3''.64$
	$26\ \ 16.2$	$-1\ \ 8.2$	1.28	0	$-\kappa = .00$		log cor. $.6820$	
	$27\ \ 22.5$	$-2\ \ 14.5$	4.94	$-.03$	$+1\ \ 55\ \ 16.45$		Correction $4''.81$	
	$28\ \ 28.1$	$-3\ \ 20.1$	10.92	$-.11$	$+28°\ 49'\ 6''.8$		$\delta_1' = 38°\ 38'\ 44''.57$	
Mean	$20^h\ 25^m\ 8^s.04$		$4''.93$	$.00$				
Dec. 29.	$16^h\ 31^m\ 22^s.3$	$+3^m\ 19^s.7$	$10''.88$	$+.11$	$T = 16^h\ 34^m\ 42^s.01$	$T = 16^h\ 34^m\ 41^s.2$	$\delta = 38°\ 38'\ 38''.08$	$\tan \delta_1 = 9.9028636$
	$32\ \ 28.5$	$+2\ \ 13.5$	4.87	$+.03$	$\Delta T = +1\ \ 48.60$	$\Delta T = +1\ \ 48.6$		$\sec(-) = .0573049$
	$33\ \ 34.3$	$+1\ \ 7.7$	1.26	0	$T + \Delta T = 16\ \ 36\ \ 30.61$	$T' = 20\ \ 25\ \ 10.2$	$2\delta = 77\ \ 17\ \ 16$	$\cos m = 0$
E.	$34\ \ 41.2$	$+\ \ 0.8$	$.00$	0	$\alpha = 18\ \ 31\ \ 40.36$	$\Delta T' = 1\ \ 48.7$		$\tan \phi' = 9.9601685$
	$35\ \ 47.1$	$-1\ \ 5.1$	1.16	0	$-\kappa\ \ .00$	$\text{Sum} = 37\ \ 3\ \ 28.7$	$\sin 2\delta = 9.9892$	$\phi' = 42°\ 22'\ 34''.40$
	$36\ \ 55.2$	$-2\ \ 13.2$	4.85	$-.03$	$-1\ \ 15\ \ 9.75$	$18\ \ 31\ \ 44.4$	$\log 4''.90 = .6902$	
	$38\ \ 5.5$	$-3\ \ 23.5$	11.30	$-.11$	$-28°\ 47'\ 26''.3$	$\alpha = 18\ \ 31\ \ 40.4$	log cor. $= .6794$	
Mean	$16^h\ 34^m\ 42^s.01$		$4''.90$	0		$m = +\ 4.0$	Correction $4''.78$	
	$20^h\ 28^m\ 28^s.3$	$-1\ \ 55.5$	$3''.65$	$-.02$	$T = 20^h\ 26^m\ 32^s.80$	$m = +60''.0$		$\tan \delta_1' = 9.9028541$
	$27\ \ 22.5$	$-\ \ 49.7$	$.67$	0	$\Delta T = 1\ \ 48.72$		$\delta_1 = 38°\ 38'\ 42''.86$	$\sec(-) = .0589040$
W.	$26\ \ 18.4$	$+\ \ 14.4$	$.07$	0	$T + \Delta T = 20\ \ 28\ \ 21.52$			$\cos m = 0$
					$\alpha = 18\ \ 31\ \ 40.36$		$\log 2''.65 = .4233$	$\tan \phi' = 9.9617581$
	$24\ \ 2.0$	$+2\ \ 30.8$	6.21	$+.05$	$-\kappa = .01$		log cor. $= .4125$	$\phi' = 42°\ 28'\ 50''.35$
Mean	$20^h\ 26^m\ 32^s.80$		$2''.65$	$+.007$	$+1\ \ 56\ \ 41.17$		Correction $2''.58$	
					$+29°\ 10'\ 17''.5$		$\delta_1' = 38°\ 38'\ 40''.66$	

In this computation the quantities $\frac{\sin^2 \frac{1}{2}I}{\sin 1''}$ and $-x$ are taken from table A.

From the values of the equatorial intervals already given, we find for the observations over all of the threads $- i_0 = \pm ''.621$. The west transit of December 29th being observed only on threads I, II, III, and V, we have $i_0 = - 318''.787$. c is assumed equal to zero.

The correction $(c + i_0) \frac{\sin \varphi'}{\sin \delta_1}$ is appreciably the same for the two transits of December 23d and for the east transit of December 29th, viz., $\pm ''.67$. For the west transit of December 29th the computation of this term is as follows:

$$\begin{aligned} \log (c + i_0) &= 2.5035006_n \\ \sin \varphi' &= 9.8295232 \\ \operatorname{cosec} \delta_1 &= .2044758 \\ \log \text{correction} &= 2.5374996_n \\ \text{correction} &= -\ 344''.746 \end{aligned}$$

Then we have, December 23d,

$$\begin{array}{rlrl} \text{E. } \varphi' = & 42^\circ\ 22'\ 33''.12 & \text{W. } \varphi' = & 42^\circ\ 23'\ \ 3''.64 \\ b = & \qquad\quad +.41 & & \qquad -.02 \\ (c + i_0) \frac{\sin \varphi'}{\sin \delta_1} = & \qquad\quad -.67 & & \qquad -.67 \\ \varphi = & 42^\circ\ 22'\ 32''.86 & & 42^\circ\ 23'\ \ 2''.95 \end{array}$$

Mean φ, Dec. 23d, *clamp south*, $42^\circ\ 22'\ 47''.90$

$$\begin{array}{rlrl} \text{Dec. 29th, E. } \varphi' = & 42^\circ\ 22'\ 34''.40 & \text{W. } \varphi' = & 42^\circ\ 28'\ 50''.35 \\ b = & \qquad -1\ .25 & & \quad -\qquad 1\ .32 \\ (c + i_0) \frac{\sin \varphi'}{\sin \delta_1} = & \qquad +\ \ .67 & & \quad -5\ 44\ .75 \\ \varphi = & 42^\circ\ 22'\ 33''.82 & & 42^\circ\ 23'\ \ 4''.28 \end{array}$$

Mean , Dec. 29th, *clamp north*, $42^\circ\ 22'\ 49''.05$

The mean of the two values is $\varphi = 42^\circ\ 22'\ 48''.47$

It will be observed that the corrections given in table B are here inappreciable. γ, computed from formula (371) for the west observation of December 29th, is found to be 0.99998433; dividing the quantity $(c + i_0)$ by this factor (365), we find for the correction $344''.752$, instead of $344''.746$ found by neglecting this factor. The difference is inappreciable in this case.

Application of the Method of Least Squares to Prime Vertical Transits.

211. In the preceding discussion we have supposed the stars observed at both the east and west transits, and in both positions of the axis. The method is very simple theoretically, and the results very satisfactory. In the field, time will sometimes be wanting for applying it in the manner there explained. Besides this, many observations would ordinarily be lost by the interference of clouds at the time of one transit or the other. For meeting these difficulties the following modification will be useful:

A number of stars must be observed, some east and some west, the axis being reversed about the middle of the series. Care must be taken to observe about an equal number in both positions of the axis, and about the same number of east and west stars. The declinations of stars observed east should be as nearly as may be the same as those observed west.

We shall suppose the observations reduced to the middle or mean thread by the method of Bessel (Art. 207); then in equation (365) let us write $\tau_1 = T - \alpha - \varkappa$ and $\dfrac{c + i_0}{\gamma} = c'$. Then expanding $\cos(\tau - m)$, the equation becomes

$$c' = -\sin n \sin \delta_1 + \cos n \cos m \cos \delta_1 \cos \tau_1 \\ + \cos n \sin m \cos \delta_1 \sin \tau_1. \quad . \quad (373)$$

Now substituting for $\sin n$, $\cos n \cos m$, and $\cos n \sin m$, their values from (342), this becomes

$$c' = -\cos(\varphi - b) \sin \delta_1 + \sin(\varphi - b) \cos \delta_1 \cos \tau_1 \\ + a \cos \delta_1 \sin \tau_1. \quad . \quad . \quad . \quad (374)$$

Let the auxiliaries φ_1 and z be determined by the equations

$$\left.\begin{aligned} \cos z \sin \varphi_1 &= \sin \delta_1; \\ \cos z \cos \varphi_1 &= \cos \delta_1 \cos \tau_1; \\ \sin z &= \cos \delta_1 \sin \tau_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (375)$$

Then (374) becomes

$$c' = \sin(\varphi - \varphi_1 - b)\cos z + a \sin z.$$

Since $\sin(\varphi - \varphi_1 - b)$ is here of the same order as a and c', we may write this equation

$$\varphi - \varphi_1 - b + a \tan z - c' \sec z = 0. \quad . \quad . \quad (376)$$

Now let $\varphi = \varphi_0 + \varDelta\varphi$, in which φ_0 is an assumed approximate value of φ. Then writing $f = \varphi_0 - \varphi_1 - b$, viz., the algebraic sum of the known terms, we have

$$\varDelta\varphi + a \tan z - c' \sec z + f = 0. \quad . \quad . \quad . \quad (377)$$

Each star observed furnishes one equation of this form for determining the unknown quantities $\varDelta\varphi$, a, and c. A considerable number of stars should be observed, and the resulting equations solved by the method of least squares.

The formulæ for this method are then as follows:

$$\left.\begin{aligned} &\tau_1 = T - \alpha - x; \\ &\cos z \sin \varphi_1 = \sin \delta_1; \\ &\cos z \cos \varphi_1 = \cos \delta_1 \cos \tau_1; \\ &\sin z \qquad\quad\; = \cos \delta_1 \sin \tau_1; \\ &\varDelta\varphi + a \tan z - c' \sec z + f = 0; \\ &f = \varphi_0 - \varphi_1 - b; \\ &\varphi = \varphi_0 + \varDelta\varphi. \end{aligned}\right\} \quad . \quad (\mathrm{XXI})$$

x and δ_1 are determined as explained in Art. 207.

Example.

The following observations were made at Munich by Bessel, 1827, June 28th, with a small transit instrument mounted on a tripod and approximately adjusted in the prime vertical:*

Star.	Circle.	Transit.	Times of Transit over Threads. t_1	t_2	t_3	t_4	t_5	Level.
λ Bootis........	S.	W.	$18^m\ 0^s.0$	$15^m\ 52^s.4$	$10^h\ 13^m\ 42^s.8$	$11^m\ 23^s.2$	$8^m\ 50^s.0$	$-2^d.113$
α Lyræ.........	S.	E.	28 32 .8	29 20 .24	30 8 .8	30 58 .8	31 50 .8	−2 .340
XIII 316........	N.	W.	45 1 .6	46 21 .6	10 47 44 .0	49 4 .8	50 29 .6	−1 .132
i Herculis......	N.	E.	8 38 .0	6 50 .8	11 5 5 .6	3 21 .2	1 32 .0	−1 .798
				Azimuth	disturbed.			
π Lyræ	N.	E.	44 19 .6		11 41 58 .4	40 46 .4	39 31 .2	+ .105
υ Herculis.....	N.	W.	5 53 .2	7 54 .0	12 9 52 .8	11 52 .0	13 52 .4	−1 .122
γ Cygni........	N.	E.	25 58 .0	25 5 .6	24 13 .2	23 23 .6		−2 .123
φ Herculis......	S.	W.	41 7 .2	39 40 .4	12 38 11 .2	36 38 .8	35 0 .0	−1 .353
δ Cygni........	S.	E.	42 44 .0	44 3 .2	45 23 .6	46 46 .4	48 14 .4	−1 .124

The apparent places of the stars for the date of observation, 1827, June 28th, $16^h\ 34^m$, Munich sidereal time, I find to be as follows:

Star.	α	δ
λ Bootis.........	$14^h\ 9^m\ 50^s.20$	$46°\ 53'\ 15''.40$
α Lyræ..........	18 31 8 .14	38 37 49 .01
XIII 316.........	14 1 2 .36	44 40 53 .53
i Herculis........	17 34 38 .04	46 6 20 .56
π Lyræ..........	18 50 7 .75	43 43 28 .14
υ Herculis.......	15 57 27 .45	46 31 23 .50
γ Cygni.........	20 16 4 .61	39 42 34 .46
φ Herculis.......	16 3 21 .83	45 23 40 .34
δ Cygni..........	19 39 38 .03	44 42 52 .86

The values of the equatorial intervals of the threads from the mean thread are as follows:

$i_1 = +598''.08$; $i_2 = +303''.09$; $i_3 = +6''.19$; $i_4 = -294''.91$; $i_5 = -612''.46$.

The correction for inequality of pivots is -0.294† divisions of level for circle north. The value of one division of the level is $4''.49$.

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* See *Astronomische Nachrichten*, vol. ix. p. 415.

† Bessel uses as the correction $-.42$ divisions, which is evidently computed by the erroneous formula $p = \frac{B' - B}{2}\left(\frac{\cos i_1}{\cos i + \cos i_1}\right)$, instead of (297). See *Ast. Nach.*, vi. p. 236.

A mean time chronometer was used, the hourly rate on sidereal time being $+9^s.19$; the correction at 12 hours chronometer time being $5^h\ 4^m\ 44^s.61$.

Bessel gives the approximate values of the latitude and the azimuth of the instrument as follows:

$$\varphi_0 = 48^\circ\ 8'\ 40'';$$
$$a_0 = \ 0^\circ\ 7'\ 48''.$$

If these quantities are not known with accuracy sufficient for forming the equations of condition, a preliminary reduction of a few of the observations will give them.

The values of T, $\varkappa$, and δ_1 are computed precisely as shown in Art. 210. With this series of observations $\varkappa$ in no case exceeds $^s.01$; it has accordingly been neglected.

The computation of τ_1 for each star may now be conveniently arranged as follows:

STAR.		T	ΔT	$T+\Delta T$	α	τ_1	τ_1
λ Bootis...	W.	$10^h 13^m 33^s.68$	$5^h 4^m 28^s.31$	$15^h 18^m\ 1^s.99$	$14^h\ 9^m 50^s.20$	$+\ 1^h\ 8^m 11^s.79$	$+17^\circ\ 2' 56''.85$
α Lyræ....	E.	10 30 10 .29	4 30 .85	15 34 41 .14	18 31 8 .14	− 2 56 27 .00	−44 6 45 .0
XIII 316...	W.	10 47 44 .32	4 33 .55	15 52 17 .87	14 1 2 .36	+ 1 51 15 .51	+27 48 52 .65
ι Herculis.	E.	11 5 5 .52	4 36 .20	16 9 41 .72	17 34 38 .04	− 1 24 56 .32	−21 14 4 .8
π Lyræ. ..	E.	11 41 38 .90	4 41 .80	16 46 20 .70	18 50 7 .75	− 2 3 47 .05	−30 56 45 .75
υ Herculis.	W.	12 9 52 .88	4 46 .13	17 14 39 .01	15 57 27 .45	+ 1 17 11 .56	+19 17 53 .4
γ Cygni...	E.	12 24 40 .10	4 48 .39	17 29 28 .49	20 16 4 .61	− 2 46 36 .12	−41 39 1 .8
φ Herculis.	W.	12 38 7 .52	4 50 .45	17 42 57 .97	16 3 21 .83	+ 1 39 36 .14	+24 54 2 .1
δ Cygni....	E.	12 45 26 .32	5 4 51 .57	17 50 17 .89	19 39 38 .03	− 1 49 20 .14	−27 20 2 .1

As we have an approximate value of the azimuth error, we may write (equation 376)

$$\varphi_0 + \Delta\varphi - \varphi_1 - b + (a_0 + \Delta a) \tan z - (i_0 + c) \sec z = 0.$$

i_0 is zero for all the above stars except π *Lyræ* and γ *Cygni*. In the observation of π *Lyræ* the transit over the second thread was lost. Therefore for this star i_0 is the mean of the equatorial intervals i_1, i_3, i_4 i_5; viz., $-75''.775$.

Similarly for γ *Cygni*, the fifth thread being missed, $i_0 = +153''.1125$.

Writing the sum of the known terms, viz.,

$$\varphi_0 - [\varphi_1 + b - a_0 \tan z + i_0 \sec z] = +f,$$

our equation of condition becomes

$$\Delta\varphi + \Delta a \tan z - c \sec z = -f.$$

The computation of φ_1, tan z, sec z, and f is now arranged as follows:

	λ Bootis.	α Lyræ.	XIII 316.	*i* Herculis.	π Lyræ.
δ_1	46° 53′ 25″.68	38° 37′ 50″.33	44° 40′ 57″.22	46° 6′ 26″.74	43° 43′ 31″.25
tan δ_1	.0286798	9.9026368	9.9951876	.0167925	9.9806703
cos τ_1	9.9804823	9.8561090	9.9466792	9.9694648	9.9333111
tan ϕ_1	.0481975	.0465278	.0485084	.0473277	.0473592
ϕ_1	48° 10′ 22″.10	48° 3′ 47″.93	48° 11′ 35″.47	48° 6′ 56″.78	48° 7′ 4″.22
tan τ	9.48667	9.98655_n	9.72228	9.58947_n	9.77784_n
cos ϕ_1	9.82405	9.82498	9.82388	9.82454	9.82452
og tan z	9.31072	9.81153_n	9.54616	9.41401_n	9.60236_n
log sec z	.00890	.07611	.02538	.01414	.03228
tan z	+ .2045	− .6479	+ .3517	− .2594	− .4003
sec z	1.0207	1.1915	1.0600	1.0331	1.0771
Level-reading......	− 2.113	− 2.340	− 1.132	− 1.798	+ .105
Inequality of pivots	+ .294	+ .294	− .294	− .294	− .294
b	− 8″.17	− 9″.19	− 6″.40	− 9″.39	− 0″.85
i_0 sec z					− 1′ 21″.62
$- a_0$ tan z	− 1′ 35″.71	+ 5′ 3″.24	− 2′ 44″.59	+ 2′ 1″.41	+ 3′ 7″.33
$\begin{bmatrix}\phi_1 + b - a_0 \tan z \\ + i_0 \sec z\end{bmatrix}$	48° 8′ 38″.22	48° 8′ 41″.98	48° 8′ 44″.48	48° 8′ 48″.80	48° 8′ 49″.08
f =	+ 1″.78	− 1″.98	− 4″.48	− 8″.80	− 9″.08

	v Herculis.	γ Cygni.	φ Herculis.	δ Cygni.
δ_1	46° 31′ 31″.34	39° 42′ 35″.37	45° 23′ 44″.94	44° 42′ 56″.60
tan δ_1	.0231351	9.9193426	.0060007	9.9956904
cos τ_1	9.9748853	9.8734442	9.9576263	9.9485819
tan ϕ_1	.0482498	.0458984	.0483744	.0471085
ϕ_1	48° 10′ 34″.43	48° 1′ 19″.32	48° 11′ 3″.84	48° 6′ 5″.04
tan τ	9.54427	9.94911_n	9.66670	9.71340_n
cos ϕ_1	9.82402	9.82532	9.82395	9.82466
log tan z	9.36829	9.77443_n	9.49065	9.53806_n
log sec z	.01153	.06579	.01986	.02445
tan z	+ .2335	− .5949	+ .3095	− .3452
sec z	1.0269	1.1636	1.0468	1.0579
Level-reading............	− 1.122	− 2.123	− 1.353	− 1.124
Inequality of pivots.	− .294	− .294	+ .294	+ .294
b	− 6″.36	− 10″.85	− 4″.75	− 3″.73
i_0 sec z		+ 2′ 58″.16		
$- a_0$ tan z	− 1′ 49″.28	+ 4′ 38″.41	− 2′ 24″.84	+ 2′ 41″.55
$\begin{bmatrix}\phi_1 + b - a_0 \tan z \\ + i_0 \sec z\end{bmatrix}$	48° 8′ 38″.79	48° 8′ 45″.04	48° 8′ 34″.25	48° 8′ 42″.86
f =	+ 1″.21	− 5″.04	+ 5″.75	− 2″.86

Since the azimuth of the instrument was disturbed between the observation of *i* Herculis and π Lyræ, it will be necessary to introduce into the equations a

different value of the azimuth correction for these stars observed after the disturbance took place.

The equations will therefore be *

			$c - o$
λ Bootis,	$\Delta\varphi + .2045\Delta a$	$- 1.0207c = - 1''.78$	$- .50.$
α Lyræ,	$\Delta\varphi - .6479\Delta a$	$- 1.1915c = + 1''.98$	$- .08.$
XIII 316,	$\Delta\varphi + .3517\Delta a$	$+ 1.0600c = + 4''.48$	$- 2.35.$
i Herculis,	$\Delta\varphi - .2594\Delta a$	$+ 1.0331c = + 8''.80$	$- 3.44.$
π Lyræ,	$\Delta\varphi$	$- .4003\Delta a' + 1.0771c = + 9''.08$	$- 1.78.$
υ Herculis,	$\Delta\varphi$	$+ .2335\Delta a' + 1.0269c = - 1''.21$	$+ 3.28.$
γ Cygni,	$\Delta\varphi$	$- .5949\Delta a' + 1.1636c = + 5''.04$	$+ 4.05.$
φ Herculis,	$\Delta\varphi$	$+ .3095\Delta a' - 1.0468c = - 5''.75$	$+ 2.01.$
δ Cygni,	$\Delta\varphi$	$- .3452\Delta a' - 1.0579c = + 2''.86$	$- 1.33.$

From these nine equations of condition the following normal equations are formed:

$$\begin{aligned}
9.0000\Delta\varphi - .3511\Delta a - .7974\Delta a' + 1.0438c &= 23.5000;\\
- .3511\Delta\varphi + .6526\Delta a \qquad\qquad + .6681c &= - 2.3539;\\
- .7974\Delta\varphi \qquad\qquad + .7836\Delta a' - .8424c &= - 9.6825;\\
1.0438\Delta\varphi + .6681\Delta a - .8424\Delta a' + 10.4360c &= 30.6933.
\end{aligned}$$

Solving these equations by the usual methods, we find the following values:

$$\begin{aligned}
\Delta\varphi &= + 1''.38;\\
\Delta a &= - 5''.41;\\
\Delta a' &= - 8''.07;\\
c &= + 2''.50.
\end{aligned}$$

Therefore the latitude as given by this series of transits is

$$\varphi = 48^\circ\ 08'\ 41''.38.$$

Bessel gives as the true value of φ found from other sources $48^\circ\ 8'\ 39''.50$, from which the above value would be only $1''.88$ in error, an agreement which is very satisfactory when it is remembered that the instrument used was a very small one, mounted quite imperfectly, and used in the open air. The residuals given in connection with the equations of condition result from the above values. The weights and probable errors may be computed from these in the usual manner if thought desirable.

* These equations are not the same as those given by Bessel for these observations, the differences being due to the erroneous value of the correction for inequality of pivots, before referred to, and to slightly different values of α and δ for some of the stars.

CHAPTER VII.

DETERMINATION OF LONGITUDE.

212. The difference in longitude of two points on the earth's surface is equal to the angle at the pole formed by the meridian curves passing through the two points. As the earth revolves uniformly on its axis, it will be equal to the difference between the times of transit of the same star over the two meridians, and may be expressed either in degrees, minutes, and seconds of arc, or in hours, minutes, and seconds of time; for astronomical purposes the latter designation is generally preferred.

Any meridian may be assumed as the prime meridian from which to reckon longitudes. At the meridian conference which assembled in Washington, October 1884, Greenwich was chosen as the universal prime meridian. Heretofore most of the leading nations of the world have reckoned longitude from the meridian of their own capital. In conformity with this custom, longitudes within the limits of the United States have been reckoned from the meridian passing through the centre of the dome of the U. S. Naval Observatory at Washington. For local purposes the meridian of Washington will no doubt continue to be employed, but for general scientific purposes longitudes in this country will hereafter be reckoned from Greenwich.

As an astronomical problem, the determination of the difference of longitude between two places consists in an accurate determination of the local time at each place and the

comparison of the times so determined; the difference between the times being the difference of longitude.

The local time will generally be determined with the transit; and when great accuracy is required in the resulting longitude, all of the refinements and precautions to which attention has been called in treating of this subject must be observed. For rough determinations, especially at sea, the time is determined with the sextant or any suitable instrument. Nothing need be added on this point to what has been already said. We shall therefore in this chapter confine our attention to the practical methods of comparing the local time.

There are various methods which may be employed for comparing the local time at two meridians, some of these admitting of a much higher degree of accuracy than others. The most important are the following:

First. By transportation of chronometers;
Second. By the electric telegraph;
Third. Methods depending on the motion of the moon, such as by occultations of stars, eclipses of the sun, lunar culminations, and lunar distances.

Also, some use has been made of terrestrial signals, eclipses of Jupiter's satellites, and eclipses of the moon.

The most accurate of all these methods, when it can be employed, is the telegraphic.

Longitude Determined by Transportation of Chronometers.

213. We shall designate the two stations whose difference of longitude is to be determined by E and W, E being east of W. Let the error and rate of the chronometer be determined at E by any of the methods given for determination of time; then let the chronometer be carried to W and its

error on local time determined at this place. The difference between the time at W given by observation and the time at E which will be given by the chronometer is the difference of longitude. The chronometer may be regulated to either mean or sidereal time. To express the difference of longitude algebraically,

Let $\varDelta T_e$ = chronometer correction at E at chronometer time T_e;

δt = rate per day as shown by chronometer;*

$\varDelta T_w$ = chronometer correction on local time at W at chronometer time T_w;

λ = difference of longitude.

Then $(T_w + \varDelta T_w)$ = true time at W at chronometer time T_w;

$\lambda + T_w + \varDelta T_e + \delta t(T_w - T_e)$ = the corresponding time at E.

Therefore
$$\lambda = \varDelta T_w - [\varDelta T_e + \delta t(T_w - T_e)]. \quad . \quad . \quad (378)$$

Example. At Bethlehem, Pa., 1881, August 7.75, the correction to a mean time chronometer was found to be $+ 6^m 50^s.90$. At Wilkesbarre, Pa., August $10^d 9^h 9^m 17^s.92$, chronometer time, the correction on local time was $+ 4^m 54^s.11$. The daily rate of the chronometer was $+ 1^s.64$; i.e., the chronometer was losing.

Therefore

$\delta t = +1^s.64$		$\varDelta T_e =$	$+ 6^m$	$50^s.90$
$(T_w - T_e) =$	2.63 days	$\delta t(T_w - T_e) =$		4.31
		Sum =	6	55.21
		$\varDelta T_w =$	4	54.11
		$\lambda =$	2	1.1

That is, Wilkesbarre is $2^m 1^s.1$ west of Bethlehem.

* Unless the rate is uncommonly large it will make no difference whether we take chronometer days or true days in applying the correction for rate.

214. The rate is determined at the first station by comparing the results of observations separated by an interval of several days, but it is found that the rate of the chronometer during transportation (called the travelling rate) is seldom the same as its rate when at rest. The travelling rate may be determined, or its effect may be eliminated by transporting the chronometer in both directions.

Let T_e, T_w, T_w', T_e' = the time of leaving E and arriving at W, leaving W and arriving at E, respectively;

$\Delta_e, \Delta_w, \Delta_w', \Delta_e'$ = the corresponding chronometer corrections found by observation;

m = the daily travelling rate.

Then

$(T_w - T_e) + (T_e' - T_w')$ = time during which the chronometer was in transit;

$(\Delta_e' - \Delta_e) - (\Delta_w' - \Delta_w)$ = the corresponding change in the chronometer correction;

$$m = \frac{(\Delta_e' - \Delta_e) - (\Delta_w' - \Delta_w)}{(T_w - T_e) + (T_e' - T_w')}. \quad . \quad . \quad . \quad (379)$$

Previous to the application of the telegraph to the determination of longitude, the construction of chronometers had been brought to such a degree of perfection that the chronometric method was the most accurate one available. Where great accuracy was required, large numbers of chronometers were transported many times in both directions. A most elaborate expedition of this kind was carried out in 1843, by Struve, for determining the difference of longitude between Pulkova and Altona. Sixty-eight chronometers were carried nine times from Pulkova to Altona and eight times from Altona to Pulkova. A similar expedition,* or

* See Report U. S. Coast Survey, 1853, p. 88; 1854, p. 139; 1856, p. 182.

series of expeditions, was conducted by the U. S. Coast Survey during the years 1849, '50, '51, and '55, in which fifty chronometers were transported many times between Boston and Liverpool. The results of the expeditions in the years '49, '50, and '51 showed the necessity of introducing a correction for change of temperature. The expedition of '55 was therefore planned and carried out under the direction of Mr. W. C. Bond, with special reference to this correction. In this year fifty-two chronometers were transported three times in each direction, giving as the difference of longitude between the Cambridge observatory and the observatory at Liverpool—

Voyages from Liverpool to Cambridge, $4^h\ 32^m\ 31^s.92$;
Voyages from Cambridge to Liverpool, $4\ \ 32\ \ 31\ .75$.

Such expeditions are enormously expensive, and the results are not comparable for accuracy with those obtained by the telegraph. As almost every point of much importance on the habitable part of the earth is now or will soon be supplied with telegraphic facilities, chronometric expeditions on the scale of those mentioned may be reckoned as things of the past. Nevertheless the chronometric method is very useful where extreme precision is not required, or where the telegraph cannot be used, as at sea.

The method of conducting a chronometric expedition is briefly as follows: The chronometers at the first station, which we may suppose to be E, are first carefully compared with the standard clock; then they are placed in the vessel, near the middle where the motion will be the least possible, and in a position where they will be accessible for winding and comparing during the voyage. They should be compared daily as a check on the regularity of their rates. A record of the temperature must be kept.

On arriving at W the chronometers are immediately compared with the standard clock as before at E.

215. The errors to which the chronometers are liable are of two kinds: *first*, accidental irregularities which follow no law and are therefore equally liable to affect the result with the plus or minus sign—the larger the number of chronometers the more effectually these will be eliminated; and *secondly*, errors resulting from acceleration or retardation of rate. When the chronometer has been transported a number of times in both directions the effect of a constant acceleration or retardation may be eliminated by reckoning the longitude alternately from each station E and W.

Experiments show the acceleration or retardation of rate to be due to two causes, viz., changes of temperature and the gradual thickening of the lubricating oil. This latter diminishes the amplitude of the vibration and therefore causes an acceleration of rate. Its effect is sensibly proportional to the time.

Although great care is given by the makers to compensating the balance for temperature, it is seldom possible to accomplish this perfectly. It has been found that the effect of changes of temperature may be represented by a term of the form $k(\vartheta - \vartheta_0)^2$, in which ϑ_0 is the temperature of most perfect compensation and ϑ that of actual exposure, and k is a constant which with rare exceptions is positive; that is, exposure to a temperature above or below that of most perfect compensation causes the chronometer to run slower.

The rate of any chronometer may therefore be expressed by the formula

$$u = u_0 + k(\vartheta_0 - \vartheta)^2 - k't; \quad . \quad . \quad . \quad . \quad (380)$$

k' being a constant depending on the thickening of the oil, or any other causes which may be assumed to vary directly with the time.

The constants k, k', and ϑ_0 peculiar to each chronometer can only be determined experimentally.

216. The term depending on the temperature, $k(\vartheta_0 - \vartheta)^2$, having always the same sign, will never vanish; therefore in order to find the total effect of such changes during any interval a strict theory requires the total sum of all these terms for all changes of temperature.

We may proceed as follows:

Let $\tau =$ the interval during which the effect of rate is required;

Let u_0 of formula (380) be taken at the middle of this interval;

Let τ be supposed divided into n equal parts, so small that the temperature during the interval $\frac{\tau}{n}$ may be considered constant;

Let $\vartheta_1, \vartheta_2, \ldots \vartheta_n$ be the values of ϑ for each interval in succession.

Then the accumulated rate for each interval will be as follows:

$$\left.\begin{array}{l}
\left[u_0 + k(\vartheta_1 - \vartheta_0)^2 + k'\dfrac{n-1}{n}\tau\right]\dfrac{\tau}{n}; \\
\left[u_0 + k(\vartheta_2 - \vartheta_0)^2 + k'\dfrac{n-2}{n}\tau\right]\dfrac{\tau}{n}; \\
\quad\cdot \qquad \cdot \qquad\qquad \cdot \qquad\qquad \cdot \\
\left[u_0 + k(\vartheta_{n-1} - \vartheta_0)^2 - k'\dfrac{n-2}{n}\tau\right]\dfrac{\tau}{n}; \\
\left[u_0 + k(\vartheta_n - \vartheta_0)^2 - k'\dfrac{n-1}{n}\tau\right]\dfrac{\tau}{n}.
\end{array}\right\} \quad (381)$$

The sum of all these quantities is the total effect of rate; and as the sum of the coefficients of k' is zero, the value is

$$u_0\tau + k\Sigma_0^n(\vartheta - \vartheta_0)^2\frac{\tau}{n}. \quad . \quad . \quad . \quad . \quad (382)$$

For rigorous accuracy the intervals should be infinitesimal; $\frac{\tau}{n}$ would then be $d\tau$, and the above expression would be

$$u_0\tau + k\int_0^\tau(\vartheta - \vartheta_0)d\tau.$$

As, however, $(\vartheta - \vartheta_0)$ cannot be expressed as a function of τ, the integration is not possible.

For determining $\Sigma_0^n(\vartheta - \vartheta_0)^2$ we write the mean of the observed temperatures (supposed to be the quantities represented above by ϑ_1, ϑ_2, etc.) equal to θ.

Then

$$\begin{aligned}\Sigma_0^n(\vartheta - \vartheta^0)^2 &= \Sigma_0^n[(\theta-\vartheta_0)+ (\vartheta - \theta)]^2 \\ &= \Sigma_0^n(\theta-\vartheta_0)^2+ \Sigma_0^n\, 2(\theta - \vartheta_0)(\vartheta - \theta) + \Sigma_0^n(\vartheta-\theta)^2.\end{aligned}$$

Since θ and ϑ_0 are both constant, we have

$$\Sigma_0^n(\theta - \vartheta_0)^2 = n(\theta - \vartheta_0)^2; \quad . \quad . \quad . \quad . \quad (383)$$

$$\Sigma_0^n\, 2(\theta - \vartheta_0)(\vartheta - \theta) = 2(\theta - \vartheta_0)\Sigma_0^n(\vartheta - \theta) = 0, \quad (384)$$

since θ is the mean of the individual values of ϑ.

Therefore (382) becomes

$$u_0\tau + k(\theta - \vartheta_0)^2\tau + k\Sigma_0^n(\vartheta - \theta)^2\frac{\tau}{n}. \quad . \quad . \quad (385)$$

The value of the quantity $\dfrac{\Sigma_0^n(\vartheta - \theta)^2}{n}$ is computed directly, since ϑ is any observed temperature, and θ the mean of all

the values observed. This will approach more nearly the theoretically exact value the more frequently the temperatures are observed.

Writing

$$\frac{\Sigma_0^n(\vartheta - \theta)^2}{n} = \varepsilon^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (386)$$

we have for the accumulated rate during the interval τ

$$[u_0 + k(\theta - \vartheta_0)^2 + k\varepsilon^2]\tau. \quad . \quad . \quad . \quad . \qquad (387)$$

The quantity in the brackets is the mean rate during the interval τ.

217. In the Coast Survey expedition of 1855 the mean temperature was indicated by a chronometer constructed expressly for this purpose. It was in all respects like one of the ordinary chronometers, except that the arms and rim of the balance were of brass and uncompensated. Its indications of the mean temperature of exposure were found to be much more reliable than could be obtained by the use of ordinary thermometers; its sensitiveness was such that a change of 1° in the temperature produced a change of $6^s.5$ in the daily rate. Experiments made for determining the time required for a chronometer to adapt itself to the temperature of the surrounding air when exposed to a sudden change showed that this was not fully accomplished until five or six hours had elapsed, so that in case of sudden changes the temperature shown by the thermometer might differ widely from the actual temperature of the chronometer balance.

218. In applying (387) to any subsequent interval, τ', u_0 must be replaced by $u_0 - k't$, in which t is the time from the middle of the interval τ to the middle of τ'.

Now suppose the chronometer used for determining the

difference of longitude of two stations E and W. Suppose the corrections Δ_1 and Δ_2 determined at E before starting, at the times T_1 and T_2, and Δ_3 and Δ_4 after reaching W, at times T_3 and T_4, all being reckoned from the same meridian, suppose E.

Let $T_2 - T_1 = \tau_1$; $\quad T_3 - T_2 = \tau_2$; $\quad T_4 - T_3 = \tau_3$.

τ_1 and τ_3 are shore intervals, and τ_2 a sea interval.

Let u_0 = the rate at the middle of the sea interval;
λ = the difference of longitude.

Then from what precedes we have

$$\left.\begin{aligned} \Delta_2 - \Delta_1 &= \left[u_0 + k'\frac{\tau_1 + \tau_2}{2} + k(\theta_1 - \vartheta_0)^2 + k\varepsilon_1^2\right]\tau_1; \\ \lambda + \Delta_3 - \Delta_2 &= \left[u_0 \qquad\qquad\quad + k(\theta_2 - \vartheta_0)^2 + k\varepsilon_2^2\right]\tau_2; \\ \Delta_4 - \Delta_3 &= \left[u_0 - k'\frac{\tau_3 + \tau_2}{2} + k(\theta_3 - \vartheta_0)^2 + k\varepsilon_3^2\right]\tau_3. \end{aligned}\right\} \quad (388)$$

θ_1, θ_2, and θ_3 are the mean temperatures for the intervals, ε_1, ε_2, and ε_3 having the values given by (386). Then from the three equations (388) u_0, k', and λ may be determined.

Let us write
$$\left.\begin{aligned} f &= \frac{\Delta_2 - \Delta_1}{\tau_1} - k(\theta_1 - \vartheta_0)^2 - k\varepsilon_1^2; \\ f'' &= \frac{\Delta_4 - \Delta_3}{\tau_3} - k(\theta_3 - \vartheta_0)^2 - k\varepsilon_3^2. \end{aligned}\right\} \quad (389)$$

We then find, from the first and third of (388),

$$k' = \frac{f - f''}{\frac{1}{2}(\tau_1 + \tau_3) + \tau_2}; \qquad u_0 = \frac{f + f''}{2} + \tfrac{1}{4}k'(\tau_3 - \tau_1). \quad (390)$$

These values substituted in the second of (388) give the value of the longitude λ.

The chronometric method finds its most important application at sea, where a high degree of precision is not important. When the time from port is not very great, this will answer all practical requirements. When the voyage is very long, the result may be rendered much more accurate by applying the corrections for acceleration of rate, the constants k, k', and ϑ_0 having been carefully determined previously.

Determination of Longitude by the Electric Telegraph.

219. The local time at one meridian may be compared with that at another most conveniently and accurately by telegraphic signals.

The most simple method of making this comparison is as follows: The cperator at one station taps the signal key in coincidence with the beat of the chronometer; the instant when the signal is received at the other station is noted by the chronometer at that place. A number of arbitrary signals are sent in this way, when the process is reversed, the operator at the second station sending the signals to the first. The errors of the chronometers will generally be determined by observing transits both before and after exchanging the signals.

Let T_e and $\varDelta T_e$ = the chronometer time and correction at station E at the instant of sending a signal;

T_w and $\varDelta T_w$ = the chronometer time and correction at station W at the instant of receiving this signal;

T_w' and $\Delta T_w'$ = the chronometer time and correction at station W at the instant of sending a return signal;

T_e' and $\Delta T_e'$ = the chronometer time and correction at station E at the instant of receiving this signal;

λ = the difference of longitude;

μ = the transmission time of the electric effect, or the small interval of time which elapses between the instant of pressing the key at one station and the click of the magnet at the other.

Then

$$\lambda - \mu = (T_e + \Delta T_e) - (T_w + \Delta T_w) = \lambda_e;$$
$$\lambda + \mu = (T_e' + \Delta T_e') - (T_w' + \Delta T_w') = \lambda_w.$$

Therefore

$$\left.\begin{aligned} \lambda &= \tfrac{1}{2}(\lambda_w + \lambda_e); \\ \mu &= \tfrac{1}{2}(\lambda_w - \lambda_e). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (391)$$

Thus by eliminating the time required for transmission of signals we have the longitude, or by eliminating the longitude we have the transmission time.

For many purposes the above process will give a sufficient degree of accuracy. For first-class longitudes, however, there are a number of small errors involved which will demand attention. They are as follows:

I. The relative personal equation of the observers in determining the chronometer corrections at the two stations.

II. The personal equations involved in sending and receiving the signals.

III. The time required at the sending station to complete the circuit after the finger touches the key.

IV. The time required at the receiving station for the armature to move through the space in which it plays and give the click—called the *armature time*.

If the two latter could be assumed to be the same at both stations, the above errors would be reduced simply to personal errors. We shall describe some of the methods of dealing with these quantities in first-class longitudes. They may be modified when a less degree of accuracy is demanded.

220. I. *Personal equation.* This may be determined by any of the methods given in Art. 188, and the necessary correction applied. If the relative personal equation is used, it should be determined both before and after the longitude work in order to guard against the effect of its gradual change. The plan followed by the Coast Survey is to exchange signals on five nights, then let the observers exchange stations, when signals are exchanged on five more nights. The personal equation is thus eliminated, provided it has remained constant during the time employed. As this changes with the physical condition of the observer, its variation is probably the chief cause of discrepancy in first-class longitudes.

221. Errors II and III are avoided by using the chronograph. For field-work break-circuit chronometers will generally be used, as they are much more convenient to carry than clocks. Such a chronometer being placed in the circuit may be made to record its beats on the chronographs at both stations. Each chronograph will then contain a record of the beats of both chronometers, the mean of which will be free from the transmission time, but will be affected by any constant difference in the *armature time*, viz., IV above.

222. Another method of sending the signals is the following: The circuit is so arranged that a tap made on the signal key at either station is recorded on the chronographs at both stations. The observer at E then gives a number of taps at intervals of two or three seconds, which are recorded at both places in connection with the beats of the respective chronometers, when the operation is repeated by the observer at

W. For identifying the hour and minute of difference of longitude, the observer at each station informs the one at the other by a telegraphic message what was the hour, minute, and second by his chronometer when the first signal was sent. The hour and minute of one signal being identified, only the seconds and fractional parts of the same need be read for the remaining signals.

223. IV. *The armature time* will be practically the same at both stations, and consequently the effect will be eliminated if the resistance of the line is kept at the same value at both points. For this purpose a rheostat and galvanometer are provided at both stations, by means of which the resistance may be maintained at any required value.

The chronometer is placed in a local circuit acting on a relay, the intensity of the current in the main line being too great for the delicate mechanism of these instruments.

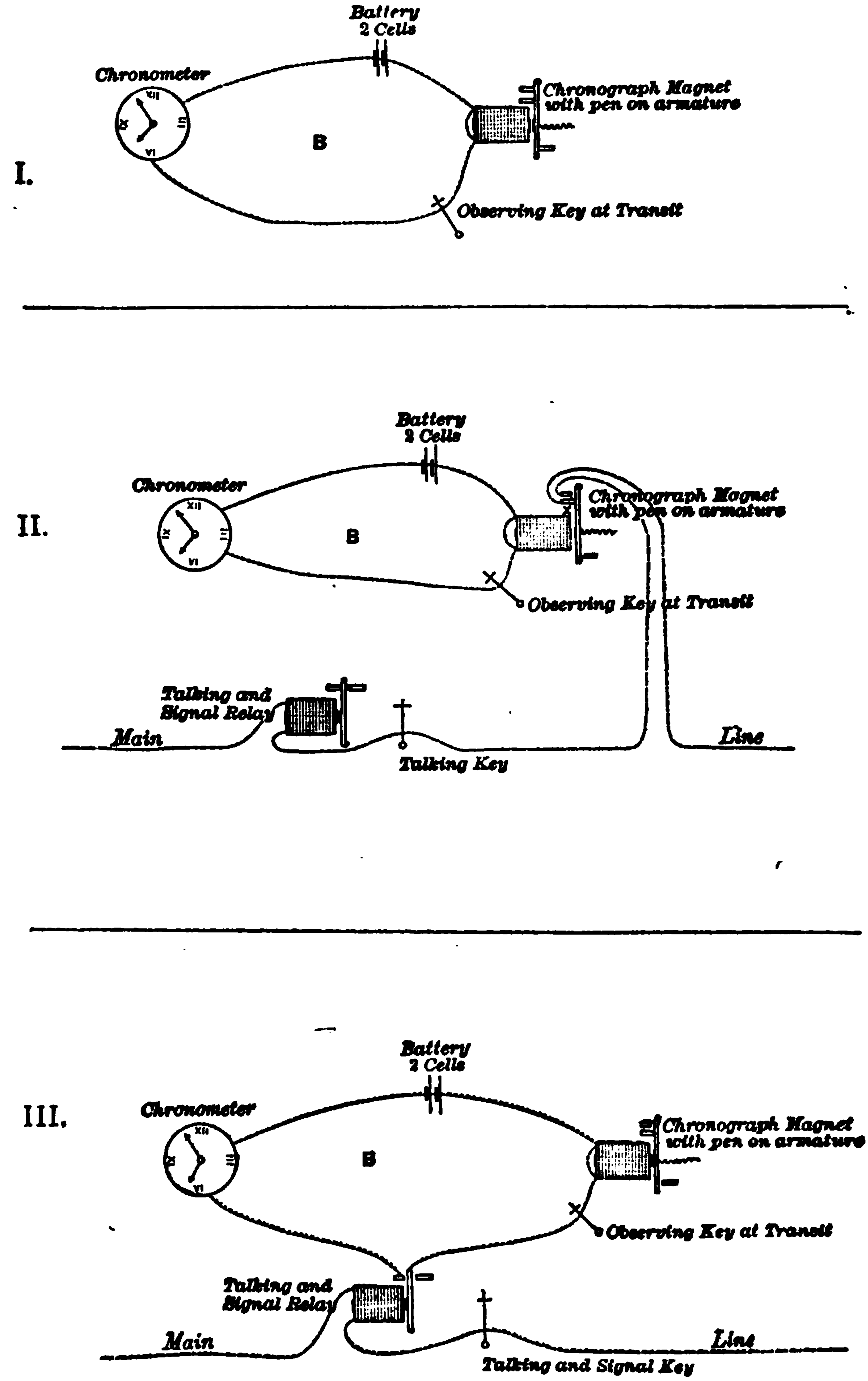
The details will be understood by reference to the following diagrams, taken from a paper by Mr. C. A. Schott.*

I shows a simple circuit for observing transits. The chronometer breaks the circuit B, causing the pen on the armature of the chronograph magnet to record. The observer breaks the circuit with the observing key, also making a record on the chronograph.

II and III show the arrangement of the circuit for chronometer signals: II being at the sending station, III at the receiving station. When the chronometer at the sending station breaks the circuit B, the armature of the chronograph magnet breaks the main circuit at X (II), and the armature of the signal relay at the receiving station breaks the circuit B (III), causing a record to be made on the chronograph.

For sending arbitrary signals the arrangement is the same at both stations, viz., that shown in III. At the sending

* Appendix No. 14, U. S. Coast Survey Report 1880.



I.
Battery
2 Cells
Chronometer
XII
III
VI
IX
B
Chronograph Magnet
with pen on armature
Observing Key at Transit
II.
Battery
2 Cells
Chronometer
B
Chronograph Magnet
with pen on armature
Observing Key at Transit
Talking and
Signal Relay
Main
Talking Key
Line
III.
Battery
2 Cells
Chronometer
B
Chronograph Magnet
with pen on armature
Observing Key at Transit
Talking and
Signal Relay
Main
Line
Talking and Signal Key

Fig. 42

station the main circuit is broken by the signal key, when the armature of the signal relay breaks the circuit B at both stations, causing a record to be made on the chronograph.

In these cases the chronometer is placed directly in the circuit passing to the chronograph, and no provision is made for equalizing the resistance at the two stations. A small difference in the armature time is therefore likely to exist.

IV.

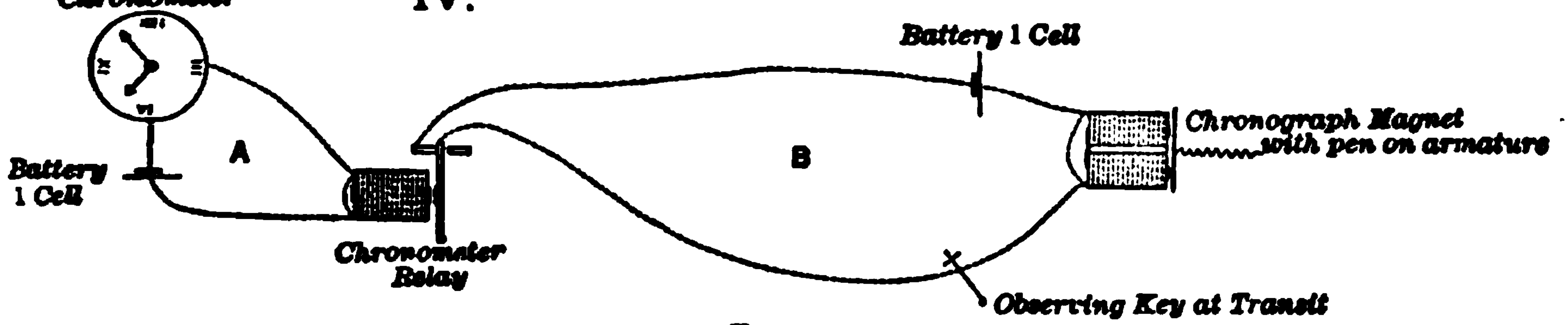


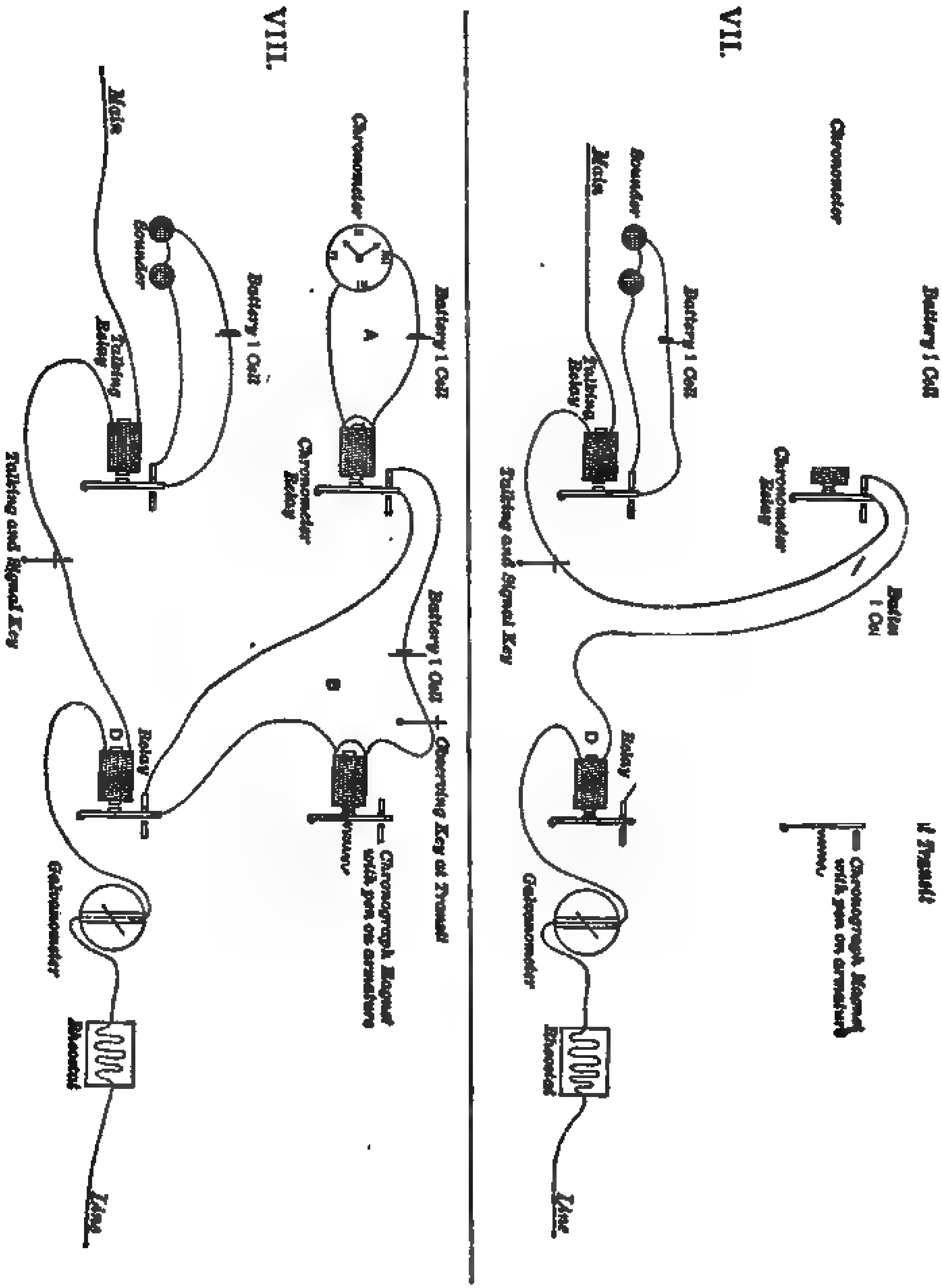
FIG. 44.

224. IV, VII, and VIII show a more complete arrangement of circuits. The chronometer is placed in a local circuit A with a weak battery, in order to avoid the injurious effect of a stronger current on the mechanism. When observing transits the arrangement is as shown in IV. The chronometer breaks the circuit A, the chronometer relay breaks the circuit B, making a record on the chronograph.

The observer breaks circuit B with the observing key, also producing record on chronograph.

VII shows the arrangement for exchanging chronometer signals, being alike at both stations. The chronometer breaks circuit A, when the armature of the chronometer relay breaks the main circuit, the armature of relay D breaking circuit B at both stations.

VIII is arranged for arbitrary signals, both stations being the same. The chronometer breaks circuit A, the armature of chronometer relay breaks circuit B, making record on the chronograph. At the sending station the main circuit is broken by the signal key, when relay D breaks circuit B at both stations.



Main Circuit ______
Local Circuit ______
Arrangement during 1881

FIG. 45.

By means of the rheostat and galvanometer the electric resistance is kept practically the same at both stations, and therefore a constant difference of armature time avoided. In order to eliminate any small outstanding difference in the action of the two sets of electric apparatus, each set may be used at both stations alternately, the instruments being exchanged with the observers at the middle of the series.

225. *Method of Star Signals.* This method of exchanging longitude signals was formerly employed by the Coast Survey. A very full description of the method is given by Chauvenet (Spherical and Practical Astronomy). It is briefly as follows:

The difference of longitude between two points, being simply the time required for a star to pass from the meridian of the east to that of the west station, may be measured by a single clock placed in the electric circuit so as to produce a record on the chronographs at both points. This clock may be at either point, or in fact anywhere in the circuit.

When a star enters the field of the transit instrument at E, the observer records the transit by tapping his signal key in the usual manner, producing a record on both chronographs. When this star reaches the meridian of W, the observer in like manner taps its passage over the threads of his transit instrument, also producing a record at both points.

This method is theoretically very perfect; but as it requires a monopoly of the telegraph lines for several hours every night when signals are exchanged, it has proved somewhat impracticable.

Example.

For the purpose of illustrating this subject I give below the record of a series of longitude signals between Washington, D. C., and Wilkes Barre, Penn., 1881, October 6th.

At Washington the instruments employed were the transit circle, sidereal clock, and chronograph of the U.S. Naval Observatory.

At Wilkes Barre the instruments were a portable transit and mean time chronometer.

At the latter place the following programme was followed: Transits of 16 stars were observed, the instrument being twice reversed; the chronometer was then taken to the telegraph office, 200 feet distant, and the longitude signals exchanged, after which 13 stars were observed with the transit instrument, the axis being reversed once. The 29 equations furnished by the observed transits gave the values of the chronometer correction and rate, also the azimuth and collimation constants of the transit instrument.

The following is the method adopted in exchanging signals:

At Washington the telegraph key was tapped at intervals of about 15 seconds, making a record on the Washington chronograph, and through the telegraph line a click of the sounder at Wilkes Barre. The observer at the latter place, having his eye on the chronometer, noted the instant of this click and recorded the same. After 10 or 15 such signals had been sent from Washington to Wilkes Barre, a similar series was sent in the opposite direction, the operator at Wilkes Barre tapping the key, producing a click of the sounder at that place and a record on the Washington chronograph.

This constitutes a complete series. Two such were exchanged each night when observations were made.

It is obvious that with a chronograph at Wilkes Barre nothing need be changed in the above programme. The record would then be made on the chronograph instead of by the observer, and if thought desirable the intervals between the signals could be much shortened.

The chronometer at Wilkes Barre being regulated to mean

solar time, its correction and rate on sidereal time are somewhat large. The values obtained from the observed transits are as follows:

At $9^h\ 39^m$ chronometer time, $\Delta T = +\ 13^h\ 9^m\ 38^s.903 \pm .024$
Hourly rate, $+\ 9.952$
Rate per minute, $+\ .1659$

Similarly for the Washington clock,

At $22^h\ 30^m$ sidereal time, $\Delta T = -\ 21^s.891 \pm .019$
Hourly rate, $+\ .0360$

The record of the signals with the individual values of the longitude immediately follows:

Washington to Wilkes Barre.

No.	Wilkes B. chronometer.	ΔT.	Washington clock.	ΔT.	Wilkes B. sidereal time.	Washington sidereal time.	Difference of longitude.	v.
1	$9^h\ 39^m\ 13^s.9$	$13^h\ 9^m\ 38^s.94$	$22^h 44^m\ 34^s.44$	$-\ 21^s.88$	$22^h 48^m\ 52^s.84$	$22^h 44^m\ 12^s.56$	$4^m\ 40^s.28$	.03
2	39 28 .9	38 .98	44 49 .50		49 7 .88	44 27 .62	40 .26	.01
3	39 43 .8	39 .02	45 4 .40		49 22 .82	44 42 .52	40 .30	.05
4	39 58 .8	39 .06	45 19 .38		49 37 .86	44 57 .50	40 .36	.16
5	40 13 .6	39 .10	45 34 .30		49 52 .70	45 12 .42	40 .28	.03
6	40 28 .5	39 .14	45 49 .28		50 7 .64	45 27 .40	40 .24	.01
7	40 43 .5	39 .18	46 4 .32		50 22 .68	45 42 .44	40 .24	.01
8	40 58 .8	39 .23	46 19 .56		50 38 .03	45 57 .68	40 .35	.10
9	41 13 .6	39 .27	46 34 .66		50 52 .87	46 12 .78	40 .09	.16
10	9 41 28 .6	13 9 39 .31	22 46 49 .66	$-\ 21\ .88$	22 51 7 .91	22 46 27 .78	4 40 .13	.12

Mean = 4 40 .253 = λ_w

Wilkes Barre to Washington.

1	$9^h\ 45^m\ 11^s.1$	$13^h\ 9^m\ 39^s.93$	$22^h 50^m\ 32^s.78$	$-\ 21^s.88$	$22^h 54^m\ 51^s\ 03$	$22^h 50^m\ 10^s.90$	$4^m\ 40^s.13$	.09
2	45 26 .1	39 .97	50 47 .72		55 6 .07	50 25 .84	40 .23	.01
3	45 36 .0	39 .99	50 57 .74		55 15 .99	50 35 .86	40 .13	.09
4	45 51 .1	40 .04	51 12 .70		55 31 .14	50 50 .82	40 .32	.10
5	46 6 .4	40 .08	51 28 .10		55 46 .48	51 6 .22	40 .26	.04
6	46 20 .7	40 .12	51 42 .52		56 0 .82	51 20 .64	40 .18	.04
7	46 35 .9	40 .16	51 57 .62		56 16 .06	51 35 .74	40 .32	.10
8	46 50 .8	40 .20	52 12 .66		56 31 .00	51 50 .78	40 .22	.00
9	47 6 .1	40 .25	52 28 .10		56 46 .35	52 6 .22	40 .13	.09
10	9 47 21 .1	13 9 40 .29	22 52 43 .00	$-\ 21\ .88$	22 57 1 .39	22 52 21 .12	4 40 .27	.05

Mean = 4 40 .219 = λ_e

Then referring to formulæ (391), we have

$$\lambda = \tfrac{1}{2}(\lambda_w + \lambda_e) = 4^m\ 40^s.236 \text{ Wilkes B. east of Wn.}$$
$$\mu = \tfrac{1}{2}(\lambda_w - \lambda_e) = \quad 0\ .017.$$

In the above the reduction of each signal has been carried out separately, in order to show the precision of the individual values. Practically the labor of reduction may be economized by reducing the means of the recorded times.

Thus from the above we have—

	Wn.—Wilkes B.	Wilkes B.—Wn.
Wilkes Barre chronometer,	$9^h\ 40^m\ 21^s.20$	$9^h\ 46^m\ 14^s.53$
ΔT,	13 9 39 .13	13 9 40 .10
Wilkes B. sidereal time,	$22^h\ 50^m\ 0^s.33$	$22^h\ 55^m\ 54^s.63$
Washington clock,	22 45 41 .95	22 51 36 .29
ΔT,	− 21 .88	− 21 .88
Wn. sidereal time,	$22^h\ 45^m\ 20^s.07$	$22^h\ 51^m\ 14^s.41$

	Wn.—Wilkes B.	Wilkes B.—Wn.
Difference of longitude =	$4^m\ 40^s.26$	$4^m\ 40^s.22$

$\lambda = 4\ \ 40\ .24$ Wilkes B. east of Wn.

$\mu = \quad .02$

This value of λ is affected by the relative personal equation of the observers at Washington and Wilkes Barre, by the personal equation of the observer at Wilkes Barre in recording the signals, and by the difference in armature time at the two stations. (See Articles 220–223.)

Longitude Determined by the Moon.

226. The preceding methods, in circumstances where they are available, leave little to be desired in facility of application or in accuracy of results. Before the invention of the electric

telegraph the most valuable methods for determining longitude were those depending on the moon's motion, chronometric expeditions being generally impracticable. Though the necessity for resorting to these methods is constantly diminishing as the telegraph lines become more widely extended, it will probably never entirely disappear.

There are various methods of utilizing the moon's motion for this purpose, the most important of which are the following:

By eclipses of the sun and occultations of stars.

By moon culminations.

By lunar distances.

By measurements of the moon's altitude or azimuth.

Some use has also been made of lunar eclipses.

All of these methods depend upon the same general principle, viz.: The moon has a comparatively rapid motion of its own, in consequence of which it makes a revolution about the earth in 27⅓ days. The elements of its orbit, together with the effects of the various perturbing forces, being known, it is possible to determine the position of the moon at any given instant of time; thus in the American Ephemeris and Nautical Almanac will be found the right ascension and declination of the moon computed several years in advance for every hour of Greenwich time. Suppose now at a point whose longitude is required the position of the moon to be determined in any convenient manner by observation; the local time being carefully noted, the ephemeris above mentioned gives, either directly or through the medium of a more or less extended computation, the Greenwich time corresponding to this position. A comparison of this Greenwich time with the observed local time gives the difference of longitude required.

227. Some of the applications of this principle are capable of giving very good results; but there is one difficulty inher-

ent in the principle itself which precludes the attainment of an accuracy commensurate with that obtained with the telegraph. The angular velocity of the earth on its axis, which is the measure of time, is twenty-seven times greater than the angular velocity of the moon in its orbit; it follows, therefore, that errors of observation in determining the moon's position, or of the ephemeris, will produce errors in the resulting longitude twenty-seven times as great. So if the errors to be anticipated in determining the place of the moon are of the same order as those of determining and comparing the errors of the clocks by the electric telegraph, we might expect to attain to an ultimate degree of precision by the latter method twenty-seven times greater than by the former.

Longitude by Lunar Distances.

228. This method is chiefly useful on long sea-voyages, where, in consequence of accumulating errors, the indications of the chronometers become unreliable.

The observation consists in measuring with a sextant, or other suitable instrument, the distance of the moon's limb from that of the sun, or from a neighboring star, the time being noted by the chronometer. After this measured distance has received the necessary corrections (to be considered hereafter), the Greenwich time corresponding is taken from the tables of lunar distances of the ephemeris by the methods of Art. 55. The difference between this time and the recorded chronometer time is the error of the chronometer on Greenwich time. An altitude of the sun or a star gives the error on local time; the difference between the two errors is the difference of longitude.

The ephemeris gives the distance, as seen from the centre of the earth, of the moon's centre from the centre of the sun,

from the four larger planets, and from certain fixed stars situated approximately in the path of the moon. They are given at intervals of three hours Greenwich mean time.

By a series of carefully observed lunar distances on both sides of the moon the chronometer error may generally be ascertained within twenty or thirty seconds. A longitude determined in this way should be considered as liable to an error of five miles, a degree of accuracy which answers the requirements of navigation.

229. We shall consider first the distance of the sun and moon.

This distance having been measured and corrected for instrumental errors, such as index error and eccentricity, the result is the apparent distance between the limbs of the sun and moon as seen from the point of observation. In order to have this comparable with the distances of the ephemeris it must be corrected for the semidiameters, parallaxes, and refraction of the two bodies.

In order to apply the necessary corrections a knowledge of the altitudes at the time of observation is necessary. When there are instruments and observers enough, which will frequently be the case at sea, all of the quantities may be observed simultaneously: the altitude of the sun so observed, if that body is sufficiently far from the meridian, may be further utilized for determining the local time.

When it is not expedient to make all these measurements at once the observer may measure the altitudes of the sun and moon immediately after measuring the distance between these bodies, the altitudes at the time of that observation being computed by assuming the change in altitude to be proportional to the change of time, an assumption which will not be much in error if the time is short.

Finally, the altitudes may be computed by formulæ (II), Art. 65, the right ascensions and declinations being taken

from the Nautical Almanac. The apparent altitudes will be derived from these computed values by applying the correction for refraction, table II, and parallax formulæ (VI) and $(VI)_1$, Art. 81. This supposes the longitude to be approximately known; otherwise we lack the means of determining the hour-angle t, required in formulæ (II): but we shall always be in possession of a value sufficiently accurate for this purpose. If in an extreme case this be not true, we may repeat the computation, using the value of the longitude obtained from the first computation as the assumed approximate value.

The corrections necessary to apply to the measured distance may be computed as follows.

Correction for Semidiameter of Sun and Moon.

230. The following quantities are taken from the ephemeris:

$s =$ the geocentric semidiameter of the moon;
$S =$ the geocentric semidiameter of the sun;
$\pi =$ the equatorial horizontal parallax of the moon;
$\Pi =$ the equatorial horizontal parallax of the sun.

The moon being comparatively near the earth, the semidiameter will vary appreciably with the altitude; there will be no appreciable variation in the case of the sun. The moon's semidiameter varies inversely as the distance.

In Fig. 46, $MOB = s.$
Call $MAC = s' =$ apparent semidiameter.

Then
$$\frac{s'}{s} = \frac{\Delta}{\Delta'} = \frac{\sin CAZ}{\sin MOZ} = \frac{\sin (Z + p)}{\sin Z},$$

Z being the geocentric zenith distance of the moon, and p the parallax in zenith distance.

$$\sin(Z+p) = \sin Z \cos p + \cos Z \sin p = \sin Z + \sin p \cos Z, \text{ nearly};$$

from (128), $\sin p = \sin \pi \sin Z$, approximately.

Therefore $$s' = s(1 + \sin \pi \cos Z). \quad . \quad . \quad . \quad . \quad (392)$$

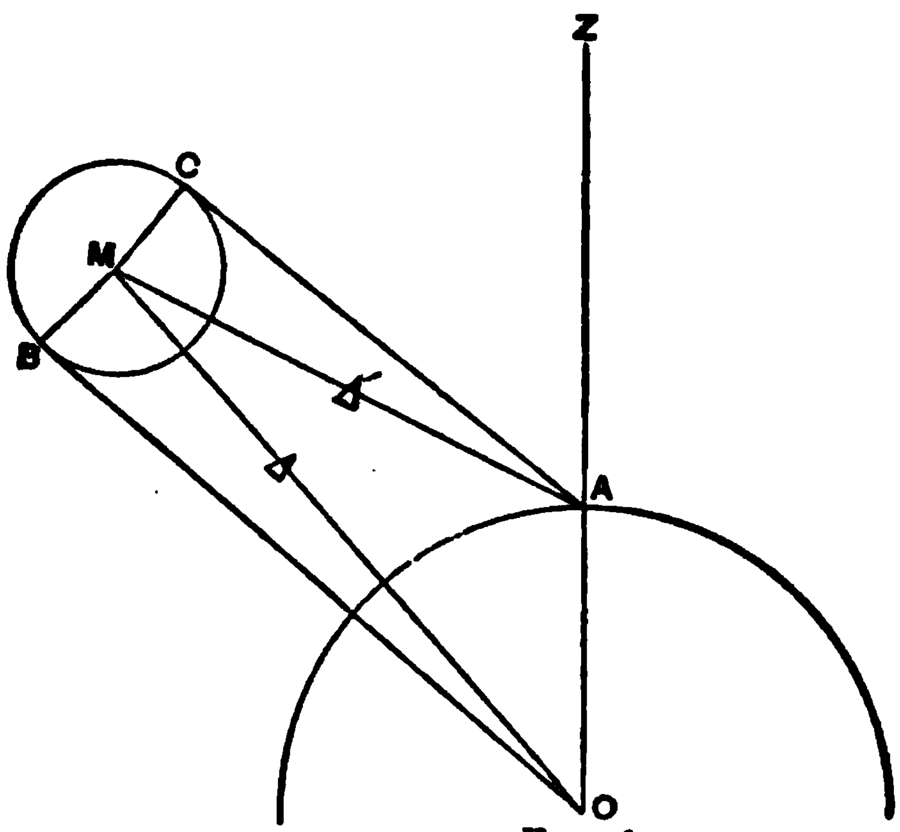


FIG. 46.

The eccentricity of the meridian has been neglected, but the error is inappreciable for this purpose.

The correction for semidiameter will be still further modified by refraction. Owing to this cause the apparent disks of the sun and moon are approximately ellipses, the refraction being less for the upper limb than for the centre, which in turn is less than for the lower limb. We therefore require the radius of the ellipse drawn to the point where the curve is intersected by the great circle joining the centres of the sun and moon.

Regarding the figure of the disk as an ellipse, the conjugate axis will coincide with the vertical circle passing through the centre, the semi-transverse axis will be equal to s' in case of the moon; b, the semi-conjugate axis, is found directly from the refraction table by taking out the refraction for the altitude of the upper and lower limbs respectively and subtracting one half the difference from s'. The angle q formed by the radius s_q' with the conjugate axis is the angle formed with the vertical circle by the great circle joining the centres of the sun and moon; s_q' being the required semidiameter.

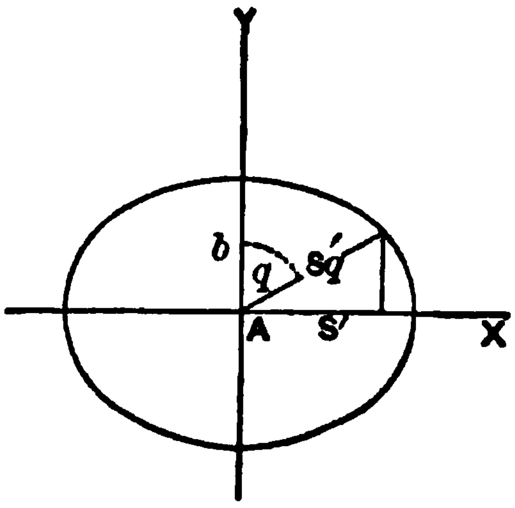


FIG. 47.

To find the angle q.

In the triangle, Fig. 48, Z is the zenith; M and S, the moon and sun.

Then

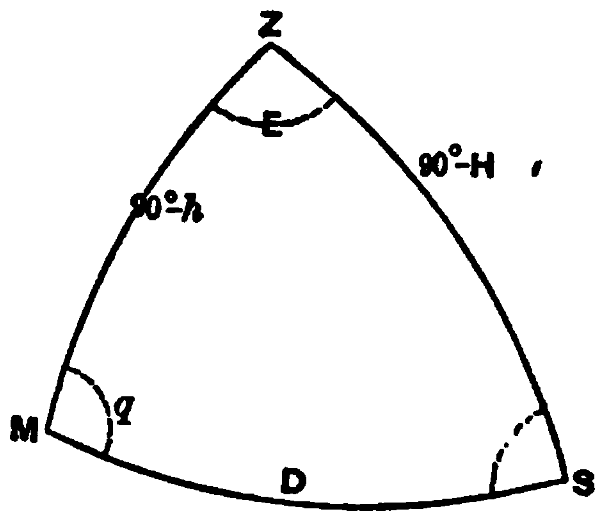


FIG. 48.

$$\sin H = \sin h \cos D + \cos h \sin D \cos q;$$

$$\cos q = \frac{\sin H - \sin h \cos D}{\cos h \sin D};$$

$$\tan \tfrac{1}{2}q = \sqrt{\frac{\sin \frac{1}{2}(D + h - H) \cos \frac{1}{2}(D + h + H)}{\sin \frac{1}{2}(D - h + H) \cos \frac{1}{2}(D - h - H)}}. \quad (393)$$

For computing the angle at the sun, h and H will be interchanged.

Then in the ellipse (Fig. 47) we have

$$x = s_q' \sin q;$$
$$y = s_q' \cos q;$$
$$s'^2 y^2 + b^2 x^2 = s'^2 b^2.$$

Therefore $$s_q' = \frac{s'b}{\sqrt{s'^2 \cos^2 q + b^2 \sin^2 q}}. \quad . \quad . \quad . \quad (394)$$

231. The values of s_q' computed by (394) for both sun and moon are then to be applied to the measured distance of the limbs of those bodies. We thus have the measured distance of the centres as seen from the place of observation. To obtain the required geocentric distance this must now be corrected for refraction and parallax.

Let D', H', and h' = the apparent distance and altitudes of the sun and moon;
D, H, and h = the true geocentric distance and altitudes.

H and h are obtained by applying to H' and h' the corrections for refraction, table II or III, and for parallax formulæ (VI) and (VI)$_1$, Art. 81.

Referring to Fig. 48,

$$\left.\begin{aligned}\cos D' &= \sin H' \sin h' + \cos H' \cos h' \cos E = \cos(H' - h') - \cos H' \cos h' 2\sin^2 \tfrac{1}{2}E; \\ \cos D &= \sin H \sin h + \cos H \cos h \cos E = \cos(H - h) - \cos H \cos h \, 2\sin^2 \tfrac{1}{2}E.\end{aligned}\right\} (395)$$

Multiplying the first of the preceding equations by $\cos H \cos h$, and the second by $\cos H' \cos h'$, then subtracting to eliminate $\sin^2 \frac{1}{2}E$, we find

$$\cos D = \cos(H - h) + \frac{\cos H \cos h}{\cos H' \cos h'}[\cos D' - \cos(H' - h')]. \quad (396)$$

D is therefore expressed in terms of known quantities. The equation is not, however, in convenient form for numerical

computation; therefore we make the following transformation:

$$\left.\begin{array}{lll} \text{Let} \quad \dfrac{\cos H \cos h}{\cos H' \cos h'} = \dfrac{1}{C}; & \dfrac{\cos D'}{C} = \cos D''; \\ \quad H' - h' = d'; & \\ \quad H - h = d; & \dfrac{\cos d'}{C} = \cos d''. \end{array}\right\} \quad (397)$$

It may readily be shown that C will never be so small as to give impossible values to D'' and d''.

(396) then reduces to

$$\cos D - \cos D'' = \cos d - \cos d'';$$

from which

$$\sin \tfrac{1}{2}(D - D'') = \frac{\sin \frac{1}{2}(d + d'')}{\sin \frac{1}{2}(D + D'')} \sin \tfrac{1}{2}(d - d''); \quad . \quad (398)$$

and with accuracy sufficient for practical purposes,

$$D - D'' = \frac{\sin \frac{1}{2}(d + d'')}{\sin \frac{1}{2}(D + D'')} (d - d''). \quad . \quad . \quad (399)$$

As the unknown quantity D is involved in the second member, this equation must be solved by approximation. Writing in the denominator $D' + D''$ for $D + D''$, we obtain a value of D which will generally be sufficiently near the true one. In case the value found in this way differs very widely from D', the computation may be repeated, using this value just found in the denominator of (399).

232. In the above we have assumed the angle E (the difference between the azimuth of the sun and moon) to be the

same for the point of observation as for the centre of the earth. We have seen, however, that the moon has an appreciable parallax in azimuth the value of which is given by formulæ (VI), Art. 81, or (VII), Art. 82.

In order to determine the correction to D due to this quantity, we differentiate the second of (395) with respect to D and E, viz.,

$$dD = \frac{\cos H \cos h \sin E}{\sin D} da, \quad . \quad . \quad . \quad . \quad (400)$$

remembering that $dE = da$.

da is the parallax in azimuth computed by the formulæ above referred to.

Formulæ (392), (393), (394), (397), (399), (400) now give the true geocentric distance D, corresponding to the measured distance D'. Then by the method explained in Art. 55 we take from the ephemeris the Greenwich time corresponding to this distance; the difference between this time and the observed time will then be the chronometer correction on Greenwich time.

If a planet has been used instead of the sun, the same formulæ will be used; but if, as is generally the case, the disk of the planet is bisected by the limb of the moon in making the observation, there will be no correction for semidiameter of planet. The effect of parallax in case of the outer planets will be very small.

If the distance of the moon from a star is measured, there will be no correction for semidiameter or parallax of the star.

233. ***Formulæ for Reducing an Observed Lunar Distance to the Geocentric Distance.***

$$\left.\begin{array}{l} \left.\begin{array}{l} s' = s(1 + \sin\pi\cos z)\,; \\ \tan\tfrac{1}{2}q = \sqrt{\dfrac{\sin\tfrac{1}{2}(D+h-H)\cos\tfrac{1}{2}(D+h+H)}{\sin\tfrac{1}{2}(D-h+H)\cos\tfrac{1}{2}(D-h-H)}}\,; \\ s_q' = \dfrac{s'b}{\sqrt{s'^2\cos^2 q + b^2\sin^2 q}}. \end{array}\right\}\text{Semidiameter.} \\ \text{For parallax of moon, (VI), Art. 81, or (VII), Art. 82.} \\ \text{For parallax of sun, (VIII)}_1\text{, Art. 82.} \\ \left.\begin{array}{ll} \dfrac{\cos H\cos h}{\cos H'\cos h'} = \dfrac{1}{C}; & \dfrac{\cos D'}{C} = \cos D''\,; \\ H' - h' = d'; & \dfrac{\cos d'}{C} = \cos d''. \\ H - h = d; & \end{array}\right\}(397) \\ D - D' = \dfrac{\sin\tfrac{1}{2}(d + d'')}{\sin\tfrac{1}{2}(D + D'')}(d - d''). \quad . \quad (399) \\ dD = \dfrac{\cos H\cos h\sin E}{\sin D}\,da. \quad \text{Correction for parallax in azimuth.} \end{array}\right\}(\text{XXII})$$

These formulæ have been written down rigorously, but in practice many abridgments may generally be made in the application.

Example. 1856, March 9th, $5^h\ 14^m\ 6^s$ local mean time, the following distance of the nearest limbs and altitudes of the lower limbs of the sun and moon were measured:

$$D' = 44°\ 36'\ 58''.6; \qquad H' = 8°\ 56'\ 23''; \qquad h' = 52°\ 34'\ 0''.$$

These values are corrected for instrumental errors.

Barometer 29.5 inches; Attached thermometer 60°; Detached ther. 58°; Latitude $\varphi = 35°$; Assumed longitude $L = 150° = 10^h$ west of Greenwich.

From the Nautical Almanac we take the following quantities:

		Sun.		Moon.
Right ascension,	$\alpha =$	$23^h\ 22^m\ 27^s$		$2^h\ 11^m\ 47^s$
Declination,	$\delta =$	$-\ 4°\ 3'\ 6''$		$14°\ 18'\ 41''$
Semidiameter,	$S =$	$16\ \ 8\ .0$	$s =$	$16\ \ 23.1$
Horizontal parallax,	$\Pi =$	$8\ .6$	$\pi =$	$60\ \ 1\ .9$
Sidereal time, mean noon,		$23^h\ 11^m\ 5^s$		

From the refraction table we find, for the altitudes above given,

	Sun.	Moon.
Refraction, lower limb,	$5'\ 42''.9$	$43''.1$
Approx. altitude of centre,	$9°\ 6'\ 48''$	$52°\ 49'\ 40''$

We now compute the apparent or augmented semidiameter of the moon by the first of (XXII), and then the oblique semidiameter of both sun and moon by the second and third of these formulæ.

$$z = 37°\ 10' \qquad \cos z = 9.9014$$
$$\pi = 1°\ 0'\ 1''.9 \qquad \sin \pi = 8.2419$$
$$\text{Sum} = 8.1433$$
$$\log (1 + \sin \pi \cos z) = .0060$$
$$s = 983.1 \qquad \log = 2.9926$$
$$s' = 996.8 \qquad \log = 2.9986$$

$$\begin{aligned}
\text{Measured } D' &= 44°\ 36'\ 58''.6 \\
s' &= \quad 16\ \ 36\ .8 \\
S &= \quad 16\ \ \ 8\ .0 \\
\text{Approximate } D' &= 45\ \ \ 9\ 43\ .4
\end{aligned}$$

Then for computing q:

Sun.			Moon.		
$D=$	$45°\ 10'$		$D=$	$45°\ 10'$	
$H=$	$52\ \ 51$		$H=$	$9\ \ 12$	
$h=$	$9\ \ 12$		$h=$	$52\ \ 51$	
$\frac{1}{2}(D+h-H)=$	$0\ \ 45.5$	$\sin=8.1217$	$\frac{1}{2}(D+h-H)=$	$44\ \ 25$	$\sin=9.8450$
$\frac{1}{2}(D+h+H)=$	$53\ \ 36$	$\cos=9.7734$	$\frac{1}{2}(D+h+H)=$	$53\ \ 36$	$\cos=9.7734$
$\frac{1}{2}(D-h+H)=$	$44\ \ 25$	$\text{cosec}=.1550$	$\frac{1}{2}(D-h+H)=$	$0\ \ 45.5$	$\text{cosec}=1.8783$
$\frac{1}{2}(D-h-H)=$	$-8\ \ 26$	$\sec=\ \ 47$	$\frac{1}{2}(D-h-H)=$	$-8\ \ 26$	$\sec=\ \ 47$
		$\text{Sum}=8.0548$			$\text{Sum}=1.5014$
$\frac{1}{2}q=$	$6°\ \ 5'$	$\tan\frac{1}{2}q=9.0274$	$\frac{1}{2}q=$	$79°\ 56'$	$\tan\frac{1}{2}q=.7507$
$q=$	$12\ \ 10$		$q=$	$159\ \ 52$	

Then from the refraction table we find—

Refraction—upper limb	$= 5'\ 24''.8$	lower limb	$= 43''.1$
centre	$= 5\ 33\ .6$	centre	$= 42\ .7$
Therefore	$b = 15'\ 59\ .2$		$b = 16'\ 36''.4$

$\log b = 2.9819$	$\log b^2 = 5.9638$	$\log b = 2.9984$	$\log b^2 = 5.9968$
	$\sin^2 q = 8.6476$		$\sin^2 q = 9.0736$
	4.6114		5.0704
	$A^* = 1.3407$		$A^* = .8720$
$\log S' = 2.9859$	$\log S'^2 = 5.9718$	$\log s' = 2.9986$	$\log s'^2 = 5.9972$
	$\cos^2 q = 9.9803$		$\cos^2 q = 9.9452$
	5.9521		5.9424
	$B^* = 1.3601$		$B^* = .9267$
	5.9715		5.9971
ac log den. $= 7.0143$	log d. $= 2.9857$	ac log den. $= 7.0014$	log d. $= 2.9986$
$\log S_q = 2.9821$		$\log s_q' = 2.9984$	
$S_q = 15'\ 59''.6$		$s_q' = 16'\ 36''.4$	

Obs. $D' = 44°\ 36'\ 58.''6$
True $D' = 45\quad 9\quad 34.\ 6$

An approximate value of the azimuth of the moon is required for computing the parallax; also of the sun for computing the small correction dD given by the last of (XXII). The formulæ for this computation are †

$$\tan M = \frac{\tan \delta}{\cos t}; \qquad \tan a = \frac{\cos M}{\sin(\varphi - M)} \tan t.$$

Converting the mean time of observation into sidereal time (Art. 94), we find

$\theta = 4^h\ 26^m\ 3^s$	
Sun $\alpha = 23\quad 22\quad 27$	Moon $\alpha = 2^h\ 11^m\ 47^s$
$t = (\theta - \alpha) = 5\quad 3\quad 36$	$t = 2\quad 14\quad 16$
$t = 75°\ 54'$	$t = 33°\ 34'$

Sun.		Moon.	
$\delta = -\ 4°\ 3'$	$\tan = 8.8501_n$	$\delta = 14°\ 19'$	$\tan = 9.4067$
$t = 75\quad 54$	$\cos = 9.3867$	$t = 33\quad 34$	$\cos = 9.9208$
$M = -16\quad 12$	$\tan = 9.4634_n$	$M = 17\quad 1$	$\tan = 9.4859$
$\varphi = 35\quad 0$		$\varphi = 35\quad 0$	
$\varphi - M = 51\quad 12$	$\operatorname{cosec}(\varphi - M) = .1083$	$\varphi - M = 17°\ 59'$	$\operatorname{cosec}(\varphi - M) = .5104$
	$\cos M = 9.9824$		$\cos M = 9.9806$
	$\tan t = .6000$		$\tan t = 9.8219$
$a = 78°\ 29'$	$\tan a = .6907$	$a = 64°\ 3'$	$\tan a = .3129$

* Addition logarithms.

† (II), Art. 65.

For parallax, (VIII)$_1$, Art. 82, (VII), Art. 82,

$$z' - z = \Pi \sin z'; \qquad \gamma = (\varphi - \varphi') \cos a.$$

$$\sin (z' - z) = \frac{\rho \sin \pi \cos (\varphi - \varphi') \sin (z' - \gamma)}{\cos \gamma};$$

$$\sin (a' - a) = \frac{\rho \sin \pi \sin (\varphi - \varphi') \sin a'}{\sin z}$$

$z' = 80° \; 53' \; 11''.4$	
$\log \pi = 0.9345$	$\log (\varphi - \varphi') = 2.81158$
$\sin z' = 9.9945$	$\cos a = 9.64106$
$\log (z' - z) = 0.9290$	$\log \gamma = 2.45264$
$z' - z = \qquad 8''.5$	$\gamma = \qquad 4' \; 44''$
Therefore $H = 9° \; 6' \; 57''.1$	$z' = 37° \; 10' \; 6''.3$
	$z' - \gamma = 37 \;\; 5 \;\; 22$

$\log \rho = 9.99952$	$\log \rho = 9.99952$
$\sin \pi = 8.24208$	$\sin \pi = 8.24208$
$\cos (\varphi - \varphi') = \qquad 0$	$\sin (\varphi - \varphi) = 7.49715$
$\sin (z' - \gamma) = 9.78036$	$\sin a' = 9.95384$
$\sec \gamma = \qquad 0$	$\operatorname{cosec} z = .22494$
$\sin (z' - z) = 8.02196$	$\sin (a' - a) = 5.91753$
$z' - z = \quad 36' \; 9''.6$	$a' - a = \qquad 17''.1$
$h = 53° \; 26' \; 3''.3$	

We now compute (397) ana (399):

$H = \quad 9° \; 6' \; 57''.1$	$\cos = 9.9944799$
$h = \quad 53 \;\; 26 \;\; 3 \;.3$	$\cos = 9.7750603$
$H' = \quad 9 \;\; 12 \;\; 22 \;.2$	$\sec = .0056304$
$h' = \quad 52 \;\; 50 \;\; 36 \;.4$	$\sec = .2189667$
$d = -44 \;\; 19 \;\; 6 \;.2$	$\log \frac{1}{C} = 9.9941373$
$d' = -43 \;\; 38 \;\; 14 \;.2$	

$\cos D' = 9.8482718$	$D' = \quad 45° \; 9' \; 34''.6$
$\cos D'' = 9.8424091$	$D'' = \quad 45 \;\; 55 \;\; 7 \;.0$
	$d = -44 \;\; 19 \;\; 6 \;.2$
$\cos d' = 9.8595724$	
$\cos d'' = 9.8537097$	$d'' = -44 \;\; 26 \;\; 13 \;.7$
	$\frac{1}{2}(d + d'') = -44 \;\; 22 \;\; 40 \;.0$
	$\frac{1}{2}(D' + D'') = \quad 45 \;\; 32 \;\; 21 \;.8$
	$d - d'' = 427.5$

First Approximation.	Second Approximation.
$\sin \frac{1}{2}(d + d'') = 9.84472_n$	$\sin \frac{1}{2}(d + d'') = 9.84472_n$
$\log (d - d'') = 2.63094$	$\log (d - d'') = 2.63094$
$\operatorname{cosec} \frac{1}{2}(D' + D'') = .14647$	$\operatorname{cosec} \frac{1}{2}(D + D'') = .14409$
$\log (D - D'') = 2.62213_n$	$\log (D - D'') = 2.61975_n$
$D - D'' = -418.9$	$D - D'' = -6' \, 56''.6$
$D = 45° \, 48' \, 8''.$	$D = 45° \, 48' \, 10''.4$
$\frac{1}{2}(D + D'') = 45 \; 51 \; 37 .5$	$dD = 3 .5$
	$D = 45 \; 48 \; 13 .9$

Correction for parallax in azimuth:

$$E = A' - a = 14° \, 26'$$
$$\cos H = 9.9945$$
$$\cos h = 9.7751$$
$$\sin E = 9.3966$$
$$\operatorname{cosec} D = .1445$$
$$\log (a' - a) = 1.2330$$
$$\log dD = 0.5437$$
$$dD = 3''.5$$

We have now to take from the Nautical Almanac the Greenwich time corresponding to this distance by the method explained in Art. 55. For 1856, March 9th, we find the following distances of the sun and moon:

12^h	$D = 43° \, 59' \, 31''$	$PL = .2493$	
			17
15	45 40 54	.2510	
			17
18	47 21 53	.2527	

We have therefore to interpolate between 15^h and 18^h.
Referring to formula (106), we have

$\Delta' =$	$7' \, 19''.9$	$\log = 2.6433$
		$PL\Delta = .2510$
$t =$	$13^m \; 4^s$	$\log t = 2.8943$
Therefore $T =$	$15^h \; 13^m \; 4^s$	
* Correction for 2d difference	-1	
Resulting Greenwich time	15 13 3	
Local time of observation	5 14 6	
Resulting longitude	9 58 57	

The above solution of this problem is only one among many, as it has received much attention from mathematicians on account of its importance to

* Taken from table I at the end of the Nautical Almanac.

navigators. The majority of the solutions are only approximate, the design being to reduce the numerical work to a minimum without at the same time sacrificing too much in the way of accuracy. Such methods may be found in any work on navigation, and will be preferred where only an approximate solution is required.

As may be seen, the solution which we have given may be considerably abridged without a great sacrifice of accuracy. The differences between the oblique and vertical semidiameters of the sun and moon are very small, and the correction for parallax in azimuth is not large. When we remember that the least reading of the sextant is 10″, and that measurements of this kind are quite difficult, it will be seen that often little will be lost by neglecting this part of the computation.

Longitude by Moon Culminations.

234. The right ascension of the moon may be determined by means of a transit instrument, mounted at the place whose longitude is required, and the local time of observation compared with the Greenwich time corresponding to this right ascension, either by taking this time from the ephemeris of the moon, or by means of similar observations made at Greenwich, or some place whose longitude from Greenwich is known.

Comparison by means of the Ephemeris.

235. The transit instrument having been adjusted as accurately as may be, the transit of the moon's bright limb is observed, together with a number of stars suitable for determining the errors of the instrument and the clock correction. The corrections necessary to give the moon's right ascension, from the observed time of transit of the limb, are then applied according to formulæ (XIX), Art. 195. The last term of the formula may be taken from the table of moon culminations where it is given under the heading "Sidereal time of semidiameter passing meridian."

236. To insure greater accuracy, the moon's right ascension may be derived by comparing the observed time of transit with that of about four stars differing but little from the moon in declination, two culminating before the moon and two after. A list of stars suitable for this purpose was formerly given in the ephemeris, under the heading "Moon culminating stars," but it has been discontinued since 1882. It is an easy matter for the observer to select suitable stars from the general list of the ephemeris.

Let A_0 = the right ascension of the moon's bright limb at the instant of culmination;
A = the right ascension of the moon's centre;
Θ = clock time of observed transit of limb, corrected for all known instrumental errors and for rate;
$\alpha\,.\,\theta$ = right ascension and time of transit respectively of a star, the time being corrected for instrumental errors and rate of clock;
S_1 = sidereal time of semidiameter passing the meridian, taken from ephemeris.

Then
$$\left.\begin{aligned} A_0 - \alpha &= \Theta - \theta; \\ A_0 &= \alpha + (\Theta - \theta); \\ A &= A_0 \pm S_1. \end{aligned}\right\} \quad (401)$$

This quantity A is then the local sidereal time of transit of the moon's centre.

237. We have now to take from the ephemeris of the moon the Greenwich mean time T corresponding to this value A of the moon's right ascension; the mean time T must then be converted into the corresponding Greenwich sidereal time Θ_0. Then λ being the difference of longitude, we have

$$\lambda = \Theta_0 - A. \quad (402)$$

The time T may be interpolated to second differences from the ephemeris, as follows:

Let A_1 = the ephemeris value nearest to A;
T_1 = the corresponding time.

Then $T_1 + t$ = the required time corresponding to A.

$$A_1 = f(T_1);$$
$$A = f(T_1 + t) = A_1 + \frac{dA_1}{dT}t + \frac{d^2A_1}{dT^2}\frac{t^2}{2}.$$

Let ΔA = the difference of right ascension for 1 minute, taken from the ephemeris;
δA = difference between two consecutive values of ΔA.

δA then equals the change in ΔA in one hour. Then if t is supposed expressed in seconds, we shall have to second differences inclusive

$$\frac{dA}{dT} = \frac{\Delta A}{60}; \qquad \frac{d^2A}{dT^2} = \frac{\delta A}{3600};$$
$$A = A_1 + \frac{t}{60}\left[\Delta A + \delta A \cdot \frac{1}{2} \cdot \frac{t}{3600}\right].$$

From which
$$t = \frac{60[A - A_1]}{\Delta A + \delta A \dfrac{t}{7200}};$$

and with sufficient accuracy,

$$t = \frac{60[A - A_1]}{\Delta A}\left[1 - \frac{t}{7200}\frac{\delta A}{\Delta A}\right]. \quad . \quad . \quad . \qquad (403)$$

Writing $x = \dfrac{60\,[A - A_1]}{\Delta A}$, $\qquad x'' = -\dfrac{x^2}{7200}\dfrac{\delta A}{\Delta A}$,

then (403) becomes $\qquad t = x + x''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (404)$

Example. Among the observations of the moon made at Washington I find the following:

1877, May 23d. Observed right ascension of the moon's centre,		$A =$	$13^h\ 28^m\ 5^s.02$
From ephemeris of the moon, $T_1 = 14^h$,		$A_1 =$	$13\ \ 27\ \ 3\ .91$
$\Delta A = \ \ 2.0996$		$A - A_1 =$	$1\ \ 1\ .11$
$\delta A = +.0029$	$60(A - A_1) = 3666.6$	$\log =$	3.56426
		$\log \Delta A =$	$.32213$
	$x = 29^m\ 6^s.4$	$\log x =$	3.24213
	$x'' = \ \ -.6$	$\log x^2 =$	6.48426
	$t = 29\ \ 5\ .8$	$\log(-\delta A) =$	7.46240_n
		$ac \log \Delta A =$	9.67787
		$ac \log 7200 =$	6.14267
$T_1 + t = 14^h\ 29^m\ 5^s.8$		$\log x'' =$	9.76720_n

This is now the Greenwich mean time corresponding to the Washington sidereal time A. In order to compare the two, $T_1 + t$ must be converted into sidereal time.

	$T_1 + t =$	14^h	29^m	$5^s.8$
Table III, Appendix N. A.,			2	22 .77
Sidereal time Greenwich M. N. =		4	4	48 .56
Greenwich sidereal time	$\Theta_0 =$	18	36	17 .1

$$\lambda = \Theta_0 - A = 5^h\ 8^m\ 12^s.1,$$

the required difference of longitude.

238. If the ephemeris were perfect, very little could be done further in the way of perfecting this method. The errors of the ephemeris, however, are not inconsiderable, and in consequence it cannot be used directly as above, except when an approximate value of the longitude is sufficient. For the year 1877 the average correction to the right ascen-

sions of the ephemeris, as derived from 66 observations at Washington, was $-^s.31$, which would have produced an error of 8^s in the longitude if the observations had been used for that purpose.

Either of two different methods may be used for eliminating from the result these errors of the ephemeris.

First. Correction of the ephemeris. This method is due to Prof. Peirce.* The ephemeris is compared with all available observations of the moon made at Greenwich, Washington, and other fixed observatories during the lunation, and in this way a series of corrections to the ephemeris obtained which, as they depend on all available data, are much more reliable than simply the place of the moon observed at any one observatory.

Peirce found that for each semilunation the corrections to the right ascension of the ephemeris could be represented by the formula

$$X = A + Bt + Ct^2; \quad . \quad . \quad . \quad . \quad . \quad (405)$$

X being the correction required, t the time reckoned from any assumed epoch (which should be chosen near the middle of the period under consideration for greater convenience), and A, B, and C being constants determined from the observations made at Washington, Greenwich, etc. The ephemeris when so corrected is used as already explained.

239. *Second. Corresponding observations.* The difference in the longitude of any two points may be found by comparing the values of the right ascension of the moon observed on the same night at both places.

The times of transit of the moon's bright limb and of the comparison stars are observed at both places and the corrections applied as already explained to find the right ascen-

* Report of U. S. Coast Survey 1854, p. 115 of Appendix.

sion of the centre at the instant of transit. It will be a little better if the same comparison stars are used at both stations.

Let L_1 and L_2 = the assumed longitudes of the two stations;*

λ = the true difference of longitude;

A_1 and A_2 = right ascensions of moon's centre from observations at L_1 and L_2;

H = variation of right ascension for one hour of longitude, while passing from meridian of L_1 to that of L_2.

Then
$$A_2 - A_1 = \lambda H;$$
$$\lambda = \frac{A_2 - A_1}{H}. \quad . \quad . \quad . \quad . \quad (406)$$

H is taken from the table of moon culminations, where it is given for the instant of transit of the moon's centre over the meridian of Washington. When used as in (406) its value must be interpolated for a longitude midway between L_1 and L_2.

Example. As an example of the determination of longitude by corresponding observations, let us take the transit of the moon, the observations and reduction of which are given in Art. 196.

We have there found for 1883, October 15:

Right ascension of moon's first limb, $1^h\ 15^m\ 50^s.08$.
Second † limb, $1\ \ 18\ \ 11\ .76$.

At Washington the right ascensions of the limbs were observed as follows:

First limb, $1^h\ 16^m\ \ 7^s.38$.
Second limb, $1\ \ 18\ \ 28\ .69$.

* Reckoned from Washington or Greenwich according as we use the ephemeris computed for Washington or Greenwich. One of the longitudes, L_1 or L_2, must be known with some accuracy.

† This is corrected for defective illumination.

Taking the mean in each case as the observed right ascension of the centre, we have

$$A_1 = 1^h\ 17^m\ \ 0^s.92\,;$$
$$A_2 = 1\ \ 17\ \ 18\ .035.$$

$$A_2 - A_1 = \ 17^s.115.$$

From ephemeris, $H =$ $153^s.88$;

$$\lambda = \frac{A_2 - A_1}{H} = 0^h.1112 = 6^m\ 40^s.3.$$

The difference of longitude between Washington and Bethlehem determined telegraphically is $6^m\ 40^s.2$. This close agreement is of course accidental; a deviation of four or five seconds from the true value would not have been surprising.

If we reduce the observations of the two limbs separately, we find:

$$\text{First limb,}\quad \lambda = 6^m\ 44^s.7.$$
$$\text{Second limb,}\quad \lambda = 6\ \ 36\ .0.$$

The mean being the same as above. This is an illustration of the necessity of employing transits of both limbs. Frequently the difference of longitude determined separately from transits of each limb will show much wider deviations than this, even when all possible care is taken to avoid error.

To illustrate the method of Art. 236 for deriving the moon's right ascension by means of comparison stars, take the following transits of the moon: *f Piscium* and *ν Piscium* observed at the Sayre observatory, 1883, October 15.

Object.	Clock Time.
f Piscium	$1^h\ 11^m\ 55^s.67$
Moon I	1 15 55 .55
Moon II	1 18 17 .23
ν Piscium	1 35 30 .41

These times are corrected for instrumental errors, and that of the second limb of the moon for defective illumination. The clock-rate is inappreciable.

	f Piscium.	*ν* Piscium.		*f* Piscium.	*ν* Piscium.
θ	$1^h\ 11^m\ 55^s.67$	$1^h\ 35^m\ 30^s.41$	α	$1^h\ 11^m\ 50^s.06$	$1^h\ 35^m\ 24^s.87$
☾	1 15 55 .55	1 15 55 .55	$A_{☾}$	1 15 49 .94	1 15 50 .01
☾′	1 18 17 .23	1 18 17 .23	$A_{☾}'$	1 18 11 .62	1 18 11 .69
☾ − θ	+ 3 59 .88	− 19 34 .86	Mean of $A_{☾}$	1 15 49 .98	
☾′ − θ	+ 6 21 .56	− 17 13 .18	$A_{☾}'$	1 18 11 .66	

This method of deriving the moon's right ascension is employed with most advantage when the same comparison stars are used at both places whose difference of longitude is required, as then uncertainties in the places of the stars will produce no appreciable effect on the result.

In our example we have preferred to use the value of the moon's right ascension derived in Art. 196, since the value of ΔT there used was obtained from transits of a number of stars, and thus a result obtained more likely to be reliable than the one above, which depends only on two stars.

240. If the difference in longitude between the two places is more than two hours, the above method requires some modification, as then the third differences in the hourly motion H will be appreciable.

The right ascensions A_1 and A_2 are obtained from observation precisely as before; then the right ascensions are taken from the ephemeris for the time of culmination at the two meridians, using for this purpose the assumed values of the longitude.

Let α_1 and α_2 = values of the right ascension taken from the ephemeris for the assumed longitudes L_1 and L_2;

$\Delta\alpha$ = correction to the ephemeris.

Then $\alpha_1 + \Delta\alpha$ and $\alpha_2 + \Delta\alpha$ = true values of the right ascension.

If then L_2 and L_1 are the true values of the longitude, $(\alpha_2 + \Delta\alpha) - (\alpha_1 + \Delta\alpha) = \alpha_2 - \alpha_1$ will be equal to $A_2 - A_1$.

Let $L_2 - L_1 + \Delta L$ = true difference of longitude. Then ΔL is the correction to the assumed difference of longitude.

Let $$x = (A_2 - A_1) - (\alpha_2 - \alpha_1).$$

Then $$\Delta L = \frac{x}{H}. \qquad (407)$$

H being, as above, the hourly change in the moon's right ascension, ΔL will here be expressed in hours. To reduce to seconds we multiply by 3600, viz.,

$$\Delta L = x\frac{3600}{H}. \qquad (408)$$

This process is sufficiently simple in theory, but if the table of moon culminations is employed the moon's right ascension must be interpolated to fourth or fifth differences, which will involve considerable labor. By using the hourly ephemeris of the moon the interpolation need only be carried to second differences. In any case we must assume the moon's motion in right ascension given in the ephemeris to be correct.

The hourly motion, H, is taken from the ephemeris for the time of observation at the meridian whose longitude is to be determined.

Example. 1883, October 16, the moon's right ascension was determined by meridian observation at Greenwich and Bethlehem as given below. The transit of the second limb was observed, the Bethlehem observations being made and reduced precisely as in the example of Art. 196.

$$\text{At Greenwich, } A_1 = 2^h\ 6^m\ 17^s.46.$$
$$\text{At Bethlehem, } A_2 = 2\ \ 19\ \ 32\ .18.$$

From the hourly ephemeris of the moon we now take the right ascension of the moon's centre. Since the argument is the Greenwich mean time, we must convert the above values of the right ascension, which are equal to the sidereal times of observation, into the corresponding Greenwich mean solar time, using for the longitude of Bethlehem the best approximation to the true value which we possess. Thus:

Local sidereal time................		$A_2 = 2^h\ 19^m\ 32^s.18$
Assumed longitude from Greenwich.		5 1 31 .9
Greenwich sidereal time...........	$2^h\ 6^m\ 17^s.46$	7 21 4 .08
Sidereal time, mean noon..........	13 38 38 .61	13 38 38 .61
Sidereal interval past noon........	12 27 38 .85	17 42 25 .47
Table II, Appendix of Ephemeris..	2 2 .48	2 54 .05
Greenwich mean time.............	12 25 36 .37	17 39 31 .42
For these times we find..........	$\alpha_1 = 2^h\ 6^m\ 17^s.61$	$\alpha_2 = 2^h\ 19^m\ 32^s.38$

From the table of moon culminations—page 379 of Ephemeris—we find for the hourly motion in right ascension at the time of the Bethlehem observation,

$$H = 158^s.58.$$

Then by formula (408), $\Delta L = - {}^s.05 \times \dfrac{3600}{158.58} = - 1^s.1.$

We have assumed $L = 5^h\ 1^m\ 31^s.9.$

Final value of longitude, $5\ \ 1\ \ 30\ .8.$

241. The determination of the moon's right ascension by the difference between the time of transit of the moon and a neighboring star does not do away with the necessity for correcting the observed times for all known errors of the transit instrument as explained in Articles 195 and 196. What we require is the right ascension of the moon's centre at the instant of transit over the meridian of the place of observation. Since this right ascension is constantly changing, if there is an uncorrected error of τ seconds in the reduced time, it

is precisely the same as though the moon were observed with an instrument perfectly mounted in a meridian differing from this one by r seconds. Thus an uncorrected instrumental error affects the resulting longitude by its full amount.

In order to obtain the best result from the method of moon culminations the observations should be arranged so as to include about an equal number of each limb; that is, the moon should be observed about the same number of times before and after full moon. In this way uncertainties in the value of the semi-diameter will be eliminated, and to some extent the personal equation of the observer in estimating the instant of transit of the limb. As the difference between the values of the longitude, determined from the first and second limbs respectively, from observations embracing an entire year, frequently amounts to 10^s, the importance of this will be obvious.

In a discussion of the limit of accuracy attainable in the determination of longitude by moon culminations, Prof. Peirce gives* $^s.101$ as the probable error of a single determination of the right ascension of the moon. The probable error of the difference between two observed right ascensions would then be $^s.142$; the probable error of the resulting longitude is twenty-seven times this, or $3^s.83$. By using an ephemeris corrected as before explained, this probable error of a single determination is somewhat reduced.

If now the law of distribution of error, which forms the basis of the method of least squares, were the only thing to be considered in making and combining observations, we could by a sufficient accumulation of individual determinations reduce this probable error to an unlimited extent. In this case, however, as in all cases where quantities are determined by observation, the errors of a purely accidental character are so combined with others of a constant character that accumulation of observation beyond a certain limit adds but little to the accuracy of the final result.

Prof. Peirce estimates the ultimate limit of accuracy which we can hope to reach in determining a quantity by observation at about four times the accuracy of the most carefully executed single determination. If then we assume that it is possible to determine the difference in the moon's right ascension within $^s.1$ by a single observed transit at each place, this would give a value of the longitude accurate to within $2^s.7$. The ultimate degree of accuracy which could be attained would then be within $^s.67$ of the truth. Owing, however, to the unexplained discrepancies in the results from the two limbs of the moon, this ultimate error is probably too small. Prof. Peirce places the limit at $1^s.00$, a limit which might be reached by observing all available culminations for two or three years, but which would not be much reduced by a further accumulation of observations.

* Report of U. S. Coast Survey 1854, p. 112 of Appendix.

Determination of Longitude by Occultations of Stars.

242. The observation of occultations of stars by the moon and of eclipses of the sun furnishes, next to the telegraphic method, the most accurate means of determining the difference of longitude between two places.* Prof. Peirce estimates the ultimate accuracy attainable by this method as within one tenth of a second of time.

The mathematical theory of eclipses and occultations of stars and of planets by the moon, and of fixed stars by planets, may all be embraced in one general discussion. It is not proposed to enter here into the general problem of eclipse prediction, as it would lead us beyond what is designed to be the scope of this work. We shall therefore confine ourselves to so much of the problem as relates to the occultation of fixed stars by the moon.

General Theory.

243. The distance of a fixed star is so great in comparison with the distance of the moon that the rays of light from the star enveloping the moon may be regarded as forming a cylindrical surface, the radius of the cylinder being equal to the radius of the moon. If this cylinder intersects the earth, the star will be hidden from all parts of the earth's surface within the cylinder. Let a line be supposed drawn from the star through the centre of the moon: this line will form the axis of the cylinder, and the point where it pierces the celestial sphere coincides with the place of the star.

* When the places are favorably situated for a chronometric determination that method may be preferable, but a high degree of precision is not possible when the chronometers are transported by land.

Now let a plane be passed through the centre of the earth perpendicular to this line: this plane is called the fundamental plane, and is taken as the plane of XY in considering the rectangular co-ordinates of the points entering into the problem. The axis of X is the line in which the fundamental plane intersects the plane of the equator, the positive axis of Y is directed towards the north, and the axis of Z is parallel to the axis of the cylinder; the origin of co-ordinates being the centre of the earth.

244. *To find the distance of any point on the earth's surface from the axis of the cylinder.*

Let α, δ = the right ascension and declination of the star;
A, D, r = the right ascension, declination, and distance from the centre of the earth, of the moon's centre;
x, y, z = the rectangular co-ordinates of the moon's centre.

Let the axis of X be positive in the direction of the end whose right ascension is equal to $90° + \alpha$.

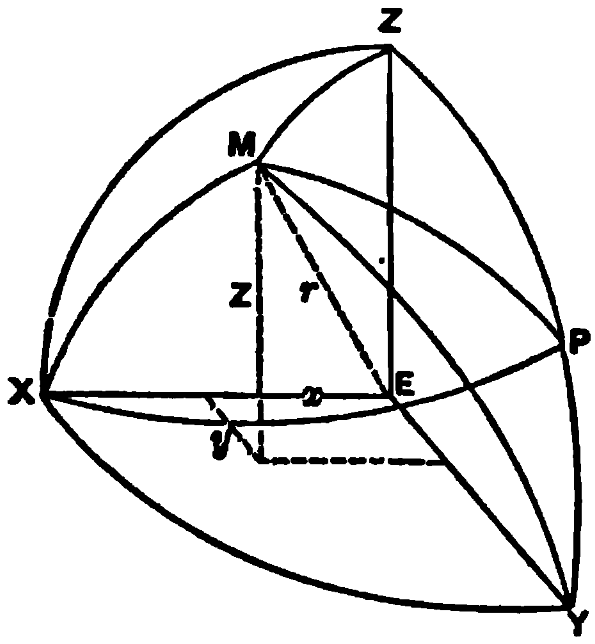


FIG. 49.

Then E, Fig. 49, being the centre of the earth, M the moon, and P the pole, we have

$$x = r \cos MX;$$
$$y = r \cos MY;$$
$$z = r \cos MZ.$$

From the triangle MPX,

$$MP = 90° - D; \qquad PX = 90°; \qquad MPX = 90° - (A - \alpha).$$

Therefore $\qquad \cos MX = \cos D \sin (A - \alpha).$

Similarly from triangles MPZ and MPY we find the values

of cos MZ and cos MY, from which result the following equations:

$$\left.\begin{aligned} x &= r \cos D \sin (A - \alpha); \\ y &= r[\sin D \cos \delta - \cos D \sin \delta \cos (A - \alpha)]; \\ z &= r[\sin D \sin \delta + \cos D \cos \delta \cos (A - \alpha)]. \end{aligned}\right\} \quad (409)$$

As the axis of the cylinder is parallel to the axis of Z and passes through the centre of the moon, x and y will be the co-ordinates of the point where this axis pierces the fundamental plane.

For our purposes z will not be required. For computing x and y with extreme accuracy it is convenient to transform (409) as follows:

Let π = the equatorial horizontal parallax of the moon.

Then $r = \dfrac{1}{\sin \pi}$, expressed in terms of the equatorial radius of the earth,

$$\text{and } x = \frac{\cos D \sin (A - \alpha)}{\sin \pi};$$

$$y = \frac{\sin (D-\delta)\cos^2 \frac{1}{2}(A-\alpha) + \sin(D+\delta)\sin^2 \frac{1}{2}(A-\alpha)}{\sin \pi}. \quad (410)$$

245. *To find the distance of any point on the earth's surface from the axis of the cylinder.*

Let ξ, η, and ζ = the rectangular co-ordinates of the point;
ρ, φ, and φ' = respectively the radius of the earth, the geographical and geocentric latitude of this point;
μ = the local sidereal time.

Then in Fig. 49, if we suppose M to be a point on the surface of the earth whose co-ordinates are ξ, η, and ζ, we have

$$\xi = \rho \cos MX; \qquad \eta = \rho \cos MY; \qquad \zeta = \rho \cos MZ.$$

In the triangle MPX

$$MP = 90^\circ - \varphi'; \qquad MPX = 90^\circ - (\mu - \alpha); \qquad PX = 90^\circ.$$

Therefore $\qquad \cos MX = \cos \varphi' \sin (\mu - \alpha).$

In the triangle MPY

$$PY = \delta; \qquad MPY = 180^\circ - (\mu - \alpha).$$

Therefore

$$\cos MY = \sin \varphi' \cos \delta - \cos \varphi' \sin \delta \cos (\mu - \alpha),$$

and similarly for $\cos MZ$, so that finally

$$\left.\begin{aligned} \xi &= \rho \cos \varphi' \sin (\mu - \alpha); \\ \eta &= \rho[\sin \varphi' \cos \delta - \cos \varphi' \sin \delta \cos (\mu - \alpha)]; \\ \zeta &= \rho[\sin \varphi' \sin \delta + \cos \varphi' \cos \delta \cos (\mu - \alpha)]. \end{aligned}\right\} \quad (411)$$

These formulæ may be computed in this form, or they may be adapted to logarithmic computation, as follows:

$$\left.\begin{aligned} \rho \sin \varphi' &= b \sin B; \\ \rho \cos \varphi' \cos (\mu - \alpha) &= b \cos B; \\ \\ \xi &= \rho \cos \varphi' \sin (\mu - \alpha); \\ \eta &= b \sin (B - \delta); \\ \zeta &= b \cos (B - \delta). \end{aligned}\right\} \quad . \quad . \quad . \quad (412)$$

$(\mu - \alpha)$ is the hour-angle of the star as seen from the given point on the earth's surface at the instant for which ξ, η, and ζ are computed.

Let H = the Washington hour-angle of the star;
h_0 = the local hour-angle = $\mu - \alpha$;
λ = the west longitude of the point ξ, η, ζ.

Then $$h_0 = H - \lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (413)$$

Let Δ = the distance of the point from the axis of the shadow.

Then $$\Delta = \sqrt{(x - \xi)^2 + (y - \eta)^2}. \quad . \quad . \quad . \quad . \qquad (414)$$

At the instant of the beginning or ending of an occultation, it is evident that the point ξ, η, ζ will be in the surface of the cylinder, and the distance from the centre Δ is equal to the radius of the cylinder, which in turn is equal to the radius of the moon, or .2723, expressed in terms of the earth's equatorial radius. Therefore

The condition for the beginning or ending of an occultation at any place is

$$k = .2723 = \sqrt{(x - \xi)^2 + (y - \eta)^2}. \quad . \quad . \quad . \qquad (415)$$

Prediction of the Principal Phases of an Occultation.

246. The instant of beginning and ending of an occultation are called respectively the time of *immersion* and *emersion*. We shall at first suppose it to be known that an occultation of the star under consideration will be visible from the given place on a given day, and shall develop the formulæ for determining these two phases—viz., of *immersion* and *emersion*.

For this purpose we require the solution of equation (415) for the local time T, of which x, y, ξ, and η are functions. The equation is transcendental and of such a form that a direct solution is not possible. In fact it will readily appear that an infinite number of values of T must satisfy this equa-

tion, since the same star may suffer occultation an indefinite number of times.

Equation (415) must therefore be solved by approximation, the most convenient method being as follows: x and y are computed for a time T as near as may be to that of the required phase. For the first approximation the time chosen is commonly that of the geocentric conjunction of the moon and star in right ascension. This time is readily found from the hourly ephemeris of the moon by finding the Greenwich time when the moon's right ascension is equal to the right ascension of the star. If, as will commonly be the case in the United States, the meridian from which the longitude is reckoned is that of Washington, the above time will be converted into Washington time by subtracting the difference of longitude between Washington and Greenwich, viz., $5^h\ 8^m\ 12^s.09$.

The object of this computation will generally be to determine the time of immersion and emersion, to assist in observing the occultation. For this purpose great accuracy will not be necessary; in fact an error of a whole minute in the computed time would not, ordinarily, be a serious matter. The general formulæ may therefore be much abridged. In any case it would be superfluous to use the rigorous formulæ in the first approximation.

247. We first compute x, y, ξ, and η for the instant of geocentric conjunction of the moon and star in right ascension, viz., when $A = \alpha$. For this instant (410) may be written

$$x = 0; \qquad y = \frac{D - \delta}{\sin \pi}. \quad . \quad . \quad . \quad . \quad . \quad (416)$$

For the short interval between conjunction and immersion or emersion we may then assume the change in x and y to be proportional to the time.

Let x' and y' = the changes in x and y in one hour, mean solar time.

Differentiating the expression for x in (410), and for y in (416), we have for the instant of conjunction

$$dx = \frac{dA}{\sin \pi} \cos D; \qquad dy = \frac{dD}{\sin \pi}.$$

Let $\varDelta A$ and $\varDelta D$ = the hourly changes in the moon's right ascension and declination taken from the ephemeris.

Then $$x' = \frac{\varDelta A}{\sin \pi} \cos D; \qquad y' = \frac{\varDelta D}{\sin \pi}. \quad . \quad . \quad . \quad (417)$$

x, y, x' and y', being independent of the place of observation, may be computed for any future time, and will be available for all parts of the earth from which the occultation is visible. Their values are given in the American Ephemeris for all the principal stars occulted throughout the year. When required for this purpose they may therefore be taken directly from that publication.

248. We must next compute ξ, η, ξ' and η'—the latter being the change in ξ and η for one hour mean solar time. ξ and η are given by formulæ (411) or (412).

For computing ξ' and η' we differentiate the first and second of (411) with respect to $(\mu - \alpha)$, viz.:

$$d\xi = \rho \cos \varphi' \cos (\mu - \alpha)\, d(\mu - \alpha);$$
$$d\eta = \rho \cos \varphi' \sin \delta \sin (\mu - \alpha)\, d(\mu - \alpha).$$

$(\mu - \alpha)$ is the hour-angle of the star. Let us now substitute for $d(\mu - \alpha)$ the change which takes place in the value of this hour-angle in one mean solar hour.

$1^h 0^m 0^s$ mean solar time = $1^h 0^m 9^s.856$ sidereal time = $54148''$.

Therefore $$d(\mu - \alpha) = 54148'' \sin 1'' . \quad . \quad . \quad . \quad . \quad (418)$$

$$\left.\begin{aligned} \xi' &= [9.419157]\, \rho \cos \varphi' \cos (\mu - \alpha); \\ \eta' &= [9.419157]\, \rho \cos \varphi' \sin (\mu - \alpha) \sin \delta. \end{aligned}\right\} . \quad (419)$$

Let k = the moon's radius expressed in terms of the earth's radius $= .2723$;

T = approximate time of immersion or emersion;*

$T + \tau$ = true time of phase.

τ will then be an unknown correction to T to be determined.

x, y, ξ, and η having been computed for the time T, their true values will be

$$x + x'\tau; \qquad y + y'\tau; \qquad \xi + \xi'\tau; \qquad \eta + \eta'\tau. \quad (420)$$

Let the auxiliary quantities Q, m, M, n, N be determined as follows:

$$\left.\begin{aligned} \dagger k \sin Q &= (x - \xi) + (x' - \xi')\tau; \\ k \cos Q &= (y - \eta) + (y' - \eta')\tau; \end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad (421)$$

$$\left.\begin{aligned} m \sin M &= (x - \xi); \qquad & n \sin N &= (x' - \xi'); \\ m \cos M &= (y - \eta); \qquad & n \cos N &= (y' - \eta'). \end{aligned}\right\} . \quad (422)$$

Then (421) become

$$\begin{aligned} k \sin Q &= m \sin M + \tau n \sin N; \\ k \cos Q &= m \cos M + \tau n \cos N. \end{aligned}$$

From these we derive

$$\begin{aligned} k \sin (Q - N) &= m \sin (M - N); \\ k \cos (Q - N) &= m \cos (M - N) + n\tau. \end{aligned}$$

* For the first approximation the time of conjunction in right ascension may be used as before explained.

† It will be observed that these two equations are identical with (415).

Let us write $Q - N = \psi$.

Then
$$\left.\begin{aligned}\sin\psi &= \frac{m \sin (M - N)}{k};\\ \tau &= \frac{k \cos\psi}{n} - \frac{m \cos (M - N)}{n}.\end{aligned}\right\} \quad (423)$$

Thus we have our equation solved for τ and consequently for $T + \tau$. Since the algebraic sign of $\cos\psi$ is not determined, the last equation gives two values of τ, that value corresponding to the minus sign of $\cos\psi$ giving the time of immersion, that given by the plus sign being the time of emersion.

The resulting times will only be approximations to the true values, since in deriving them we have neglected the second and higher orders of differences in the variation of x, y, ξ, and η.

If we require the time more accurately, we may now assume these approximate values of T and recompute formulæ (411), (419), (422), and (423), thus obtaining a second approximation to the values of T for immersion and emersion.

Position Angle of the Star.

249. The accurate observation of the star's emersion will be greatly facilitated if we know in advance the exact point on the moon's limb where its appearance may be expected. This point is determined by its position angle, which is the angle measured from the north point of the moon's limb around towards the east to the point in question. We may perhaps define this angle more clearly as follows:

Suppose two great circles drawn from the moon's centre respectively through the pole and the star: the position angle will then be the angle between these circles, measured from that drawn through the pole around towards the east.

In equations (421) x, y, ξ, and η being the rectangular co-ordinates of the moon's centre, and of the place of observation on the earth's surface, let us suppose a system of rectangular axes drawn through the latter point and parallel to the old axes. $(x - \xi)$ and $(y - \eta)$ will be the rectangular co-ordinates of the moon's centre in reference to this new system.

Since k is the moon's radius, equations (421) require Q to be the position angle of the moon's centre, measured from the axis of Y. Now it is evident that when the star is in contact with the moon's limb, which is the condition expressed by equations (421), the position angle of the star measured from the north point of the moon's limb will differ from the position angle of the moon's centre measured from the axis of Y by 180°.

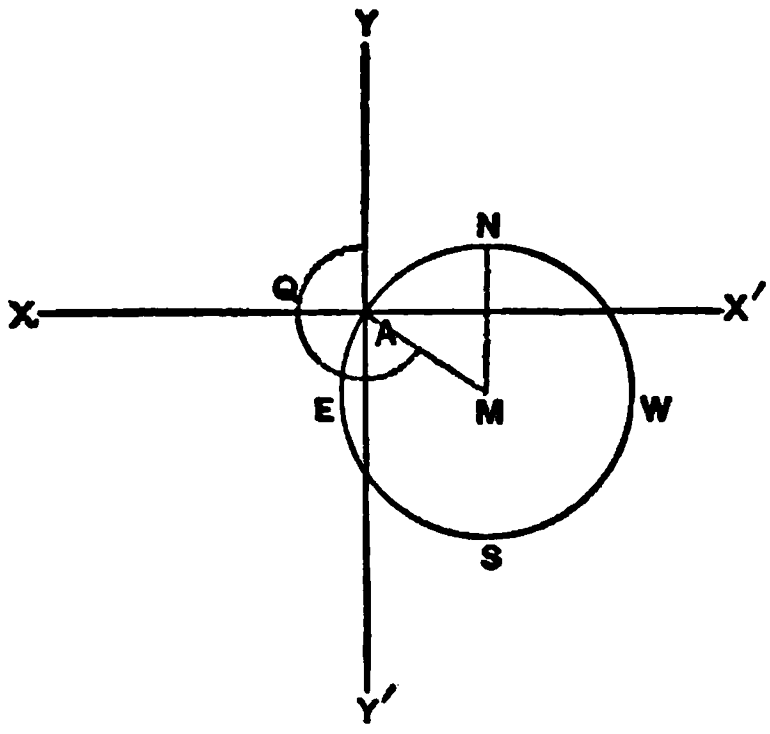


FIG. 50.

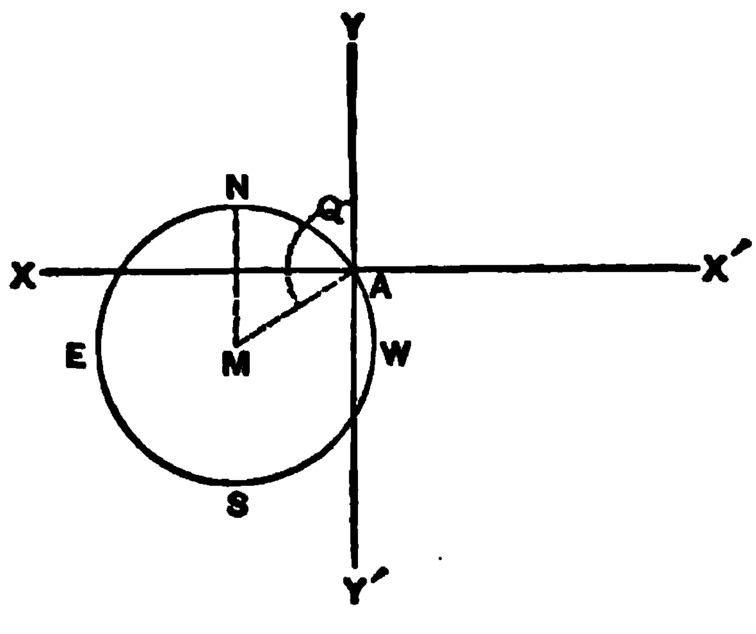


FIG. 51.

Thus, in Fig. 50, the star at immersion being at A, NMA is the position angle required. Calling this angle P, we have

$$P = Q - 180°.$$

At emersion, as shown in Fig. 51, the position angle P will be the angle $NESWA$. Therefore

$$P = Q + 180°.$$

Then since, equations (423), $Q = N + \psi$, we have

$$\left.\begin{array}{ll}\text{For immersion} & P = N + \psi - 180°; \\ \text{For emersion} & P = N + \psi + 180°.\end{array}\right\} \quad . \quad . \quad (424)$$

If the telescope used is mounted equatorially and provided with a position micrometer,* this point may be kept in view very readily by placing the micrometer-thread tangent to the moon's limb at the point.

If the telescope is not provided with a micrometer, a single thread may be placed in the focus of a common eye-piece, and a rough graduation marked around the rim. This thread may then be set in the direction of the tangent to the moon's limb as before.

250. If the telescope has only an altitude and azimuth motion, it will be convenient to measure the angle from the vertex, or highest point of the moon's limb, instead of the north point.

Consider the triangle formed by the zenith, the pole, and the moon's centre.

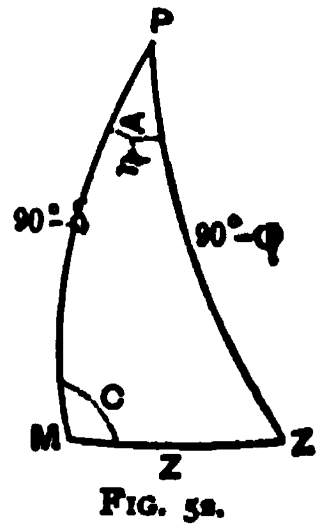


FIG. 52.

Let V = the position angle measured from the moon's vertex.

Then, referring to Fig. 52,

$$V = P - C. \quad . \quad . \quad . \quad . \quad . \quad . \quad (425)$$

* In a position micrometer the reticule revolves in a plane perpendicular to the line of collimation of the telescope, and the threads may be placed at any angle with the meridian by means of a graduated circle. On the other hand, by the same circle the angle formed with the hour-circle of a star by the line joining it with any other star in the field of the telescope may be measured.

To determine C, apply to the triangle the formulæ of spherical trigonometry, viz.:

$$\left.\begin{aligned}\sin Z \sin C &= \cos\varphi \sin(\mu - A);\\ \sin Z \cos C &= \sin\varphi \cos\delta - \cos\varphi \sin\delta \cos(\mu - A).\end{aligned}\right\} \quad (426)$$

Since C will not be required with extreme precision, and at the time for which C is required the right ascension of the star differs but little from that of the moon, we may write, bearing in mind the values given by equations (411),

$$\left.\begin{aligned}\sin Z \sin C &= \xi;\\ \sin Z \cos C &= \eta;\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (427)$$

and since at the instant of contact the values of ξ and η are, by equations (420), $\xi + \xi'\tau$ and $\eta + \eta'\tau$,

$$\tan C = \frac{\xi + \xi'\tau}{\eta + \eta'\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (428)$$

251. In connection with the elements for predicting the occultation of a given star, found in the American Ephemeris, there are given the limiting parallels of latitude within which the star will be occulted. It does not necessarily follow, however, that because a place is within the limits there given the star will be occulted at that place. The limiting curves do not coincide with parallels of latitude, as we might show by investigating the theory farther, or as may be seen by referring to the charts of solar eclipses to be found in any number of the ephemeris.

In case the point falls outside the limit of occultation, it will be shown in computing τ from equations (423), when we should find $m \sin(M - N) > k$, thus making $\sin\psi > 1$, an impossible value.

As the observation of occultations near this limit is not of

great value for the determination of longitude, it will not be worth while to make a very close computation to ascertain whether the occultation actually does occur when it is found to be near the limit.

252. The successive steps in preparing to observe the occultation of a Nautical Almanac star at a given place, assuming it to be visible at that place,* are therefore as follows:

I. We take from the "Elements for the Prediction of Occultations" of the American Ephemeris the Washington mean time of geocentric conjunction T_0, the Washington hour-angle H, also Y, x', y', and the star's apparent declination δ.

II. T_0 and H are reduced to the local time and hour-angle by applying the correction for longitude, λ.

$\rho \sin \varphi'$ and $\rho \cos \varphi'$ are to be found by the use of table A.

TABLE A.

ϕ	log F	log G
0°	.00000	.00291
5	.00001	.00290
10	.00004	.00286
15	.00010	.00281
20	.00017	.00274
25	.00026	.00265
30	.00036	.00255
35	.00048	.00243
40	.00060	.00231
45	.00073	.00218
50	.00085	.00206
55	.00098	.00193
60	.00109	.00182
65	.00119	.00171
70	.00128	.00163
75	.00136	.00155
80	.00141	.00150
85	.00143	.00147
90	.00145	.00145

This table is for computing $\rho \cos \varphi'$ and $\rho \sin \varphi'$, which will be given by the formulæ

$$\rho \cos \varphi' = F \cos \varphi;$$
$$\rho \sin \varphi' = \frac{\sin \varphi}{G}.$$

* We shall subsequently show how to select from the list of stars of the American Ephemeris those whose occultation is likely to be visible from a given place on a given day.

III. We then compute ξ, η, ξ', and η' for the local mean solar time, $(T_0 - \lambda)$, by the formulæ

$$\left.\begin{aligned} \xi &= \rho \cos \varphi' \sin h_0; \\ \eta &= \rho \sin \varphi' \cos \delta - \rho \cos \varphi' \sin \delta \cos h_0. \end{aligned}\right\} . \quad (411)$$

$$\left.\begin{aligned} \xi' &= [9.4192]\ \rho \cos \varphi' \cos h_0; \\ \eta' &= [9.4192]\ \rho \cos \varphi' \sin h_0 \sin \delta. \end{aligned}\right\} . \quad . \quad . \quad . \quad (419)$$

In which $$h_0 = H - \lambda = \mu - \alpha.$$

IV. m, M, n, and N are computed by

$$\left.\begin{aligned} m \sin M &= x - \xi; & n \sin N &= x' - \xi'; \\ m \cos M &= y - \eta; & n \cos N &= y' - \eta'; \end{aligned}\right\} . \quad (422)$$

then ψ and τ by
$$\left.\begin{aligned} \sin \psi &= \frac{m \sin (M - N)}{k}; \\ \tau &= \pm \frac{k \cos \psi}{n} - \frac{m \cos (M-N)}{n}. \end{aligned}\right\} (423)$$

Calling the value corresponding to the plus sign τ_1, and that corresponding to the minus sign τ_2, we have

$$\begin{aligned} &\text{Time of immersion} = T_0 + \tau_1 = T_1; \\ &\text{Time of emersion} \ \ = T_0 + \tau_2 = T_2. \end{aligned}$$

V. With these values T_1 and T_2 we now repeat the computation for a second approximation to the true values of the time of immersion and emersion. h_0 in (411) and (419) will become $(h_0 + \tau_1)$ for immersion, and $(h_0 + \tau_2)$ for emersion. T_1 will give us two values of τ; one a small value giving a more accurate time for immersion, the other a large value giving an inaccurate time of emersion. In the same way T_2

gives a small and accurate value of τ for emersion and a large inaccurate value for immersion.

The values of x and y to be used in this second approximation will be given by the formulæ

$$x = x'\tau_1, \qquad y = Y + y'\tau_1, \qquad \text{for immersion,}$$
and
$$x = x'\tau_2, \qquad y = Y + y'\tau_2, \qquad \text{for emersion.}$$

The values of τ given above will be expressed in hours. If it is considered desirable to express them in minutes we may use, instead of n, a quantity n', viz.,

$$n' = \frac{n}{60} = [8.2218]n.$$

As a check upon the values of the times finally obtained, we compute for these times the values of x, y, ξ, and η. If the times are correct these quantities will satisfy the equation

$$(x - \xi)^2 + (y - \eta)^2 = 0.07413.$$

253. Instead of carrying through the computation of numbers III and IV with the hour-angle h_0 of geocentric conjunction, we may obtain a rough approximation to the time of immersion and emersion, as follows:

We first require the interval of time between geocentric and apparent conjunction in right ascension. At the instant of apparent conjunction $x = \xi$; or writing for x and ξ their values,

$$\tau_0 x' = \rho \cos \varphi' \sin (h_0 + \tau_0).$$

In which τ_0 is the interval required and h_0 is, as before, the hour-angle at the station at the time of geocentric conjunction.

We have

$$\sin(h_0 + \tau_0) = \sin h_0 \cos \tau_0 + \cos h_0 \sin \tau_0$$
$$= \sin h_0(1 - 2\sin^2 \tfrac{1}{2}\tau_0) + \cos h_0\, 2 \sin \tfrac{1}{2}\tau_0 \cos \tfrac{1}{2}\tau_0;$$

and finally,

$$\sin(h_0 + \tau_0) = \sin h_0 + 2 \sin \tfrac{1}{2}\tau_0 \cos(h_0 + \tfrac{1}{2}\tau_0).$$

τ will never be very large, so we may write

$$2 \sin \tfrac{1}{2}\tau_0 = \tau_0\, 54148''.\, \sin 1'' = [9.4192]\tau_0,$$

since the unit in which τ is expressed is the mean solar hour. Therefore

$$\tau_0 x' = \rho \cos \varphi' \sin h_0 + [9.4192]\, \rho \cos \varphi' \cos(h_0 + \tfrac{1}{2}\tau_0)\,.\,\tau_0.$$

Write
$$\left.\begin{aligned} \rho \cos \varphi' \sin h_0 &= \xi_0; \\ [9.4192]\, \rho \cos \varphi' \cos(h_0 + \tfrac{1}{2}\tau_0) &= \xi'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (429)$$

Then
$$\tau_0 = \frac{\xi_0}{x' - \xi'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (430)$$

In the first approximation the τ_0 in the value of ξ' may be neglected; or we may assume it equal to $\frac{1}{2}h_0$, which will generally be a little more accurate.

As the average duration of an occultation is about one hour, we may therefore, in ordinary cases, assume as the hour-angle in equations (411) and (419)—

$$\left.\begin{aligned} &\text{For immersion, } h_0 + \tau_0 - 30^m; \\ &\text{For emersion, } \quad h_0 + \tau_0 + 30^m. \end{aligned}\right\} \quad . \quad . \quad . \quad (431)$$

The value of τ_0 may be taken from Downes's table, given in connection with the subject of occultations in the American Ephemeris.

Example.

Required the time of immersion and emersion of the star α^2 *Libræ* at Bethlehem, 1883, September 6th. $\varphi = 40° \ 36' \ 24''$; $\lambda = -0^h \ 6^m \ 40^s.2$.

From p. 424 of the American Ephemeris we find

$$\begin{array}{llll}
 & \text{Washington mean time } T_0 = & 6^h \ 18^m.4 & Y = +.6374 \\
 & H = & +2 \ \ 36 \ .9 & x' = +.5332 \\
\text{From table A, §252.} & \delta = & -15° \ 33'.4 & y' = -.1173 \\
\log \rho \sin \varphi' = 9.8112 & T_0 - \lambda = & 6^h \ 25^m.1 & h_0 = 2^h \ 43^m.6 \\
\log \rho \cos \varphi' = 9.8810 & & & h_0 = 40° \ 54'
\end{array}$$

Instead of computing at once the values of ξ, η, ξ', and η' with this value of h_0, let us first determine the times of immersion and emersion roughly by (429)-(431).

$$\begin{array}{rrr}
\sin h_0 = 9.8160 & \cos h_0 = 9.8785 & x' = .5332 \\
\rho \cos \varphi' = 9.8810 & \rho \cos \varphi' = 9.8810 & \\
\overline{} & \text{constant log} = 9.4192 & \\
\log \xi_0 = 9.6970 & \overline{} & \\
\log (x' - \xi') = 9.5824 & \log \xi' = 9.1787 & \xi' = .1509 \\
\overline{} & & \overline{} \\
{}^*\tau_0 = 1^h.302 \qquad \log \tau_0 = \ .1146 & & x' - \xi' = .3823
\end{array}$$

The computation is now as follows:

$$\begin{array}{rr}
\text{Immersion.} & \text{Emersion.} \\
h_0 = 2^h \ 43^m.6 & h_0 = 2^h \ 43^m.6 \\
\dagger \tau_0 = 1 \ \ 18 \ .1 & \tau_0 = 1 \ \ 18 \ .1 \\
-30 & +30 \\
\hline
h_0' = 3^h \ 31^m.7 & h_0' = 4^h \ 31^m.7 \\
= 52° \ 55' & = 67° \ 55'
\end{array}$$

We now compute ξ, η, ξ', and η', as follows:

---

* We might have used Downes's table above referred to, where we find $\tau_0 = 74^m$.

† Strictly τ_0 should here be reduced to sidereal interval, but the approximation is so rough that it is not important.

Immersion.

$$\sin\delta = 9.4284_n$$
$$\cos\delta = 9.9838$$
$$\sin h_0' = 9.9018$$
$$\rho\cos\varphi'\sin\delta = 9.3094_n$$
$$\cos h_0' = 9.7803$$
$$\log\xi = 9.7828$$

Check.*

$$\boxed{\begin{aligned}(x-\xi)^2 &= .0320\\ (y-\eta)^2 &= .0414\\ \text{Sum} &= .0734\end{aligned}}$$

$$\xi = +\ 0.6064$$

$$\rho\cos\varphi'\sin\delta\cos h_0' = 9.0897_n \qquad \text{Nat. No.} = -\ .1229$$
$$\rho\sin\varphi'\cos\delta = 9.7950 \qquad \text{Nat. No.} = +\ .6238$$
$$\eta = +\ .7467$$

$$\log\rho\cos\varphi'\cos h_0' = 9.6613 \qquad x = x'\tau = +\ .4276$$
$$\rho\cos\varphi'\sin\delta\sin h_0' = 9.2112_n \qquad y = Y + y'\tau = +\ .5433$$

$$\log\xi' = 9.0805 \qquad (x-\xi) = -\ .1788$$
$$\log\eta' = 8.6304_n \qquad (y-\eta) = -\ .2034$$

$$\sin M = 9.8197_n$$
$$m\sin M = 9.2524_n$$
$$m\cos M = 9.3083_n$$
$$\tan M = 9.9441 \qquad M = 221^\circ\ 19'.2$$
$$\log m = 9.4327$$

$$\xi' = .1204$$
$$\eta' = -\ .0427$$
$$x' = .5332$$
$$y' = -\ .1173$$

$$\sin N = 9.9930 \qquad x' - \xi' = .4128$$
$$n\sin N = 9.6158 \qquad y' - \eta' = -\ .0746$$
$$n\cos N = 8.8727_n$$
$$\tan N = .7431_n \qquad N = 100^\circ\ 14'.5$$
$$\log n = 9.6228$$
$$\log n' = 7.8446 \qquad M - N = 121^\circ\ 4'.7$$
$$\sin(M-N) = 9.9328 \qquad \cos(M-N) = 9.7128_n$$
$$\log m = 9.4327 \qquad \log m = 9.4327$$
$$\log\frac{1}{k} = .5650 \qquad \log\frac{1}{n'} = 2.1554$$
$$\sin\psi = 9.9305 \qquad 1.3009_n \qquad \text{Nat. No.} = -\ 20^m.00$$
$$\psi = 58^\circ\ 26'.2 \qquad \cos\psi = 9.7189$$
$$\log k = 9.4350$$
$$\log\frac{1}{n'} = 2.1554$$
$$1.3093 \qquad \text{Nat. No.} = \pm\ 20^m.39$$
$$\text{Immersion } \tau_1 = -\ 0\ .39$$
$$\text{Emersion (inaccurate) } \tau_2 = +\ 40\ .39$$

$$T_0 - \lambda = 6^h\ 25^m.1$$
$$\tau_0 - 30^m = +\ 48\ .1$$
$$\tau_1 = -\ 0\ .39$$
$$T = 7^h\ 12^m.81$$

* The comparison with the true value of k^2, viz., .0741, shows the adopted value of h_0' for

Emersion.

$$
\begin{aligned}
\sin\delta &= 9.4284_n \\
\cos\delta &= 9.9838 \\
\sin h_0' &= 9.9669 \\
\rho\cos\varphi'\sin\delta &= 9.3094_n \\
\cos h_0' &= 9.5751 \\
\hline
\log\xi &= 9.8479
\end{aligned}
\qquad
\begin{array}{|rl|}
\hline
\multicolumn{2}{|c|}{\text{Check.*}} \\
(x-\xi)^2 &= .0657 \\
(y-\eta)^2 &= .0753 \\
\text{Sum} &= .1410 \\
\hline
\end{array}
$$

$$
\begin{aligned}
\xi &= +\ 0.7045 \\
\hline
\text{Nat. No.} &= -\ \ .0766 \\
\text{Nat. No.} &= \ \ \ \ .6238 \\
\eta &= +\ \ .7004
\end{aligned}
$$

$$
\begin{aligned}
\rho\cos\varphi'\sin\delta\cos h_0' &= 8.8845_n \\
\rho\sin\varphi'\cos\delta &= 9.7950
\end{aligned}
$$

$$
\begin{aligned}
\rho\cos\varphi'\cos h_0 &= 9.7561 \\
\rho\cos\varphi'\sin\delta\sin h_0' &= 9.2763_n
\end{aligned}
\qquad
\begin{aligned}
x = x'\tau &= +\ .9608 \\
y = Y + y'\tau &= +\ .4260 \\
\hline
(x-\xi) &= \ \ \ .2563 \\
(y-\eta) &= -\ .2744
\end{aligned}
$$

$$
\begin{aligned}
\log\xi' &= 8.8753 \\
\log\eta' &= 8.6955_n
\end{aligned}
$$

$$
\begin{aligned}
\sin M &= 9.8341 \\
m\sin M &= 9.4087 \\
m\cos M &= 9.4384_n \\
\hline
\tan M &= 9.9703 \\
\log m &= 9.5746
\end{aligned}
\qquad M = 136^\circ\ 57'.6
$$

$$
\begin{aligned}
\xi' &= \ \ \ .0750 \\
\eta' &= -\ .0496 \\
x' &= \ \ \ .5332 \\
y' &= -\ .1173 \\
\hline
x' - \xi' &= \ \ \ .4582 \\
y' - \eta' &= -\ .0677
\end{aligned}
$$

$$
\begin{aligned}
\sin N &= 9.9953 \\
n\sin N &= 9.6611 \\
n\cos N &= 8.8306_n \\
\hline
\tan N &\doteq .8305_n \\
\log n &= 9.6658 \\
\log n' &= 7.8876 \\
\sin(M-N) &= 9.7946 \\
\log m &= 9.5746 \\
\log\frac{1}{k} &= .5650 \\
\hline
\sin\psi &= 9.9342 \\
\psi &= 59^\circ\ 15'0
\end{aligned}
\qquad
\begin{aligned}
N &= 98^\circ\ 24'.3 \\
M - N &= 38^\circ\ 33'.3 \\
\cos(M-N) &= 9.8932 \\
\log m &= 9.5746 \\
\log\frac{1}{n'} &= 2.1124 \\
\hline
& \ \ \ 1.5802 \qquad \text{Nat. No.} \quad +\ 38^m.0 \\
\cos\psi &= 9.7087 \\
\log k &= 9.4350 \\
\log\frac{1}{n'} &= 2.1124 \\
\hline
& \ \ \ 1.2561 \qquad \text{Nat. No.} \quad \pm\ 18^m.0
\end{aligned}
$$

$$
\begin{aligned}
\text{Emersion } \tau_2 &= -\ 20\ .0 \\
\text{Immersion (inaccurate) } \tau_1 &= -\ 56\ .0
\end{aligned}
$$

$$
\begin{aligned}
T_0 - \lambda &= 6^h\ 25^m.1 \\
\tau_0 + 30^m &= 1\ \ 48\ \ .1 \\
\tau_2 &= -20\ \ .01 \\
\hline
T &= 7^h\ 53^m.19
\end{aligned}
$$

---

immersion to be nearly correct. That for emersion, however, is considerably in error.

As a check on the accuracy of these values we now recompute x, y, ξ, and η, when we find

$$(x-\xi)^2+(y-\eta)^2=.07426; \qquad (x-\xi)^2+(y-\eta)^2=.07447.$$

We have therefore a very close approximation to the true time of immersion, the time for emersion being a little less accurate. A partial recomputation of the latter gives a correction of $-0^m.16$, making the final value of $T = 7^h\ 53^m.03$. This latter computation is altogether unnecessary for practical purposes.

For computing the position angle P at emersion,* formula (424), we obtain a value which will generally be sufficiently exact by using the last values of N and ψ obtained in computing τ. In this case we have

$$\begin{aligned} N &= \ \ 98^\circ\ 24'; \\ \psi &= \ \ 59\ \ 15\ ; \\ P = N+\psi+180^\circ &= 337\ \ 39\ . \end{aligned}$$

If the angle at the vertex V is required, we have, (428) and (425),

$$\tan C=\frac{\xi+\xi'\tau}{\eta+\eta'\tau}; \qquad V=P-C.$$

Using the values just derived, viz.,

$$\xi=.7045, \quad \xi'=.0750, \quad \eta=+.7004, \quad \eta'=-.0496, \quad \tau_2=-0^h.3335,$$

we find $\qquad C = 43^\circ\ 28'. \qquad$ Therefore $\qquad V = 294^\circ\ 11'.$

254. In predicting the occultations which will be visible at a given place within a given time, the first operation will be to go over the list of occultations of the ephemeris and select those which may be visible. The conditions of possible visibility are:

1. The limiting parallels of the last column must include the latitude of the place.

2. The hour-angle $H-\lambda$, taken without regard to sign, must be less than the semidiurnal arc of the star; in other words the star must be above the horizon.

3. The sun must be below the horizon, or at least not much above it, at the local mean time $(T-\lambda)$, unless the star is bright enough to be seen in the day-time.

Remark 1. If the place is near one of the limiting parallels of latitude an occultation may or may not occur. If it is desirable to observe such stars as are

* This angle is not required for immersion.

occulted near the north or south limbs of the moon, such doubtful ones may be included in our list, and the occurrence or non-occurrence of an occultation will be shown in the computation of the time of immersion and emersion. As before shown, if the occultation is not visible at the place under consideration it will be indicated by $\sin \psi$ becoming > 1 in the formula $\sin \psi = \dfrac{m \sin (M - N)}{k}$.

Remark 2. In most cases we may see by inspection whether condition 2 is fulfilled. For those stars near the limit it may be necessary to compute roughly the hour-angle of the star when in the horizon, for which we have

$$\cos t = - \tan \delta \tan \varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

If then $(H - \lambda)$ is numerically less than t this condition is fulfilled.

A small table computed for the latitude of the place, giving t with the argument δ, is convenient in examining this condition and the next.

Remark 3. For determining whether the sun is above or below the horizon, we may compute roughly the times of sunrise and sunset by the method given above for the star, or, since it is not required with great accuracy, we may take it from a common almanac.

In going over the list of the ephemeris, the computer will write the value of λ on the lower edge of a piece of paper, and pausing over each star for which condition 1 is fulfilled, he will see whether 2 and 3 are also fulfilled. If either fails the computer passes on. In those cases where he is unable to decide by inspection whether either of the two fail, the star will be marked for further examination after the list has been gone over.

Where many predictions are to be made for a given place the work may be much reduced by computing tables for the given latitude by means of which the computation of ξ, η, ξ', η', and τ is facilitated. The necessary directions for forming and using such tables are given in the American Ephemeris, to which the reader is referred.

III

Graphic Process.

255. If the observer possesses a celestial chart containing the stars whose occultation is to be predicted, the necessary computation may be made by a very simple graphic process. The scale of the chart must be large, and the method will be principally useful in case of clusters like the Pleiades, where a considerable number of stars undergo occultation within a short time.

The right ascension and declination of the moon are taken from the ephemeris for intervals of half an hour throughout the time covered by the occultations; the correction for parallax must then be applied. The resulting apparent places of the moon are then laid down on the chart, and a curve being drawn through

the points we have the apparent path of the moon's centre; this line being then properly subdivided between the half-hour points furnishes a graphic time-table of the moon's centre. Each star whose distance from this line is less than the augmented semidiameter* of the moon will suffer occultation. From such a star as a centre, with the moon's augmented semidiameter as a radius, let a circle be drawn; this circle cuts the path of the moon's centre in two points the position of which on the curve will give the time of immersion and emersion of the star, and the direction of the star from the point of intersection gives the position angle on the moon's limb.

Computation of Longitude.

256. It has now been shown how we may predict the time of beginning and ending of an occultation, as seen from a point on the earth's surface whose longitude is known. The fundamental equation which expresses the condition necessary for such an occurrence is

$$k^2 = (x - \xi)^2 + (y - \eta)^2. \quad . \quad . \quad . \quad . \qquad (415)$$

If now all of the data of the problem were perfectly known, and if no error entered into the observed time of the occultation, this equation would be completely satisfied. Since, however, such perfection is not attainable, we may employ the observed time of an occultation for determining the corrections to the values of the constants used.

The correction which it is the immediate object of this discussion to consider is that of the longitude assumed. In order, however, that this may be obtained with all possible precision, we must endeavor to obtain or eliminate as far as possible the corrections to the other quantities which enter into the equation if the values employed are at all uncertain.

257. Before making the transformation which (415) requires in order to adapt it to our purpose, let us examine the quantities entering into each term separately, in order to see

* Formula (392).

what may be regarded as definitively known and what quantities may require corrections.

k. The moon's semidiameter may be determined from occultations more accurately than in any other way. A correction Δk to the value employed may therefore be introduced as one of the unknown quantities of our equation.

ξ, η. Referring to the expressions for the value of these quantities, equations (411), we see that they depend upon α and δ, the right ascension and declination of the star; μ, the local sidereal time; ρ, the earth's radius; and φ', the geocentric latitude. α and δ should be so well determined that they may be regarded as absolute, that is, no stars should be used for this purpose whose places are not so well determined as to require no further consideration. μ, the local time, must be accurately determined by the transit instrument (see Chap. VI). The time determined by observation will generally be sidereal. The ephemeris of the moon given in the Nautical Almanac is arranged for mean solar intervals, so that when this is employed it may be necessary to convert the sidereal time into mean solar time, or the reverse in some cases. It will be remembered that this conversion supposes the longitude known. We shall therefore require an approximate value of the longitude, which we shall suppose to be accurate enough so that no appreciable error will result from employing it for the above reduction. If a case should ever occur, which is not likely, where this preliminary value was so erroneous that appreciable errors in the subsequent computation resulted from its employment, then it would be necessary to repeat that part of the computation which was affected by it, using the value of the longitude obtained from the first reduction. In this way we should obtain a second approximation to the true value.

φ. The latitude must be well determined by the zenith telescope or other suitable instrument.

ρ depends upon the eccentricity of the earth's meridian passing through the place of observation. A satisfactory determination of this quantity from occultations is not possible, but Bessel introduces a term into the equation depending on the correction to the assumed eccentricity, in order to show its effect on the final result. This term will be retained for the sake of completeness, though in the practical application of the formulæ it will generally be disregarded.

x and y. Equations (409). Besides quantities already considered these contain A, D, and r, the right ascension, declination, and distance of the moon. Corrections to the assumed values of all these quantities will be introduced into the equations. Those to the right ascension and declination can be well determined from an occultation observed at any place whose position is known. In order, however, to determine r, or the moon's parallax on which r depends, observations must be combined which are made at widely different points on the earth's surface, whose difference of longitude has been previously well determined. The correction to the parallax will be retained for completeness.

258. Let us now suppose a series of occultations observed at two points, the longitude of one of which is well determined. The immediate object is to determine the longitude of the second point. If one star only is observed at the second point, we must assume all the quantities entering into the equation to be known with one exception. If we assume the longitude to be the unknown quantity, we obtain from our data a value of that quantity which is affected by all of the errors of the data. If the star is also observed at the first point, this observation may be employed to correct the tabular right ascension and declination of the moon, and the longitude of the second point determined by the aid of these corrected values. If more stars are observed sufficiently near together so that the errors may be regarded as constant

during the time elapsed, then the correction to the semi-diameter can be included as an unknown quantity. As we have remarked before, the errors of the parallax cannot be well separated from the longitude. If then the number of occultations observed is greater than that of the unknown quantities which can be well determined, a solution of the resulting equations by the method of least squares will give the most probable values of the quantities, expressed in terms of the constants, and of those quantities which cannot be separated from the constants.

259. We now proceed to develop the equation in the form required. The method is that of Bessel. The meridian from which the longitude is reckoned will be called the first meridian.

Let t = the local time of an observed occultation—mean or sidereal;

w = the west longitude of the place of observation.

Then $t + w$ = the time at the first meridian.

Let τ = an arbitrary time at the first meridian sufficiently near $(t + w)$ so that the change in x and y during the interval $(t + w - \tau)$ may be assumed to be proportional to the time.

x_0 and y_0 are the values of x and y at the time τ.

Let Δx, Δy, Δk, be the corrections required to reduce the values of x, y, and k employed to the true values. These corrections depend on the various outstanding errors above considered.

The true values of these quantities, corresponding to the instant of observation, will then be

$$k + \Delta k; \qquad x_0 + x'(t + w - \tau) + \Delta x; \qquad y_0 + y'(t + w - \tau) + \Delta y.$$

x' and y' are as before the changes in x and y in one hour,—mean or sidereal according as one or the other is employed.

Let Δee = the correction to the assumed value of e^2; e being the eccentricity of the meridian.

Then ξ and η will require the corrections $\frac{d\xi}{dee}\Delta ee$ and $\frac{d\eta}{dee}\Delta ee$.

As these quantities, ξ and η, do not depend upon the longitude, they will be correctly given by equations (411), and require no other corrections.

Using the corrected values of x, y, ξ, η, and k, equation (415) becomes

$$(k + \Delta k)^2 = \left[x_0 - \xi + x'(t + w - \tau) + \Delta x - \frac{d\xi}{dee}\Delta ee\right]^2 + \left[y_0 - \eta + y'(t + w - \tau) + \Delta y - \frac{d\eta}{dee}\Delta ee\right]^2. \quad (431)$$

w is supposed known with precision enough so that the values of x' and y', which change with the time, will be known with sufficient accuracy.

Let
$$\left.\begin{aligned} m\sin M &= (x_0 - \xi); & n\sin N &= x'; \\ m\cos M &= (y_0 - \eta); & n\cos N &= y'. \end{aligned}\right\} \quad . \quad . \quad (432)$$

Equation (431) may then be written

$$[k+\Delta k]^2 = \left[m\sin M + n\sin N(t+w-\tau) + \Delta x - \frac{d\xi}{dee}\Delta ee\right]^2 + \left[m\cos M + n\cos N(t+w-\tau) + \Delta y - \frac{d\eta}{dee}\Delta ee\right]^2, \quad (433)$$

which may be placed in the form

$$[k+\Delta k]^2=\left[n(t+w-\tau)+m\cos(M-N)+\Delta x\sin N+\Delta y\cos N-\frac{d(\xi\sin N+\eta\cos N)}{dee}\Delta ee\right]^2$$
$$+\left[m\sin(M-N)+\Delta x\cos N-\Delta y\sin N-\frac{d(\xi\cos N-\eta\sin N)}{dee}\Delta ee\right]^2. \quad (434)$$

Let us write

$$\left.\begin{aligned}\lambda &= \Delta x\sin N+\Delta y\cos N-\frac{d(\xi\sin N+\eta\cos N)}{dee}\Delta ee;\\ -\lambda' &= \Delta x\cos N-\Delta y\sin N-\frac{d(\xi\cos N-\eta\sin N)}{dee}\Delta ee.\end{aligned}\right\} \quad (435)$$

Then $[k+\Delta k]^2=[n(t+w-\tau)+m\cos(M-N)+\lambda]^2$
$$+[m\sin(M-N)-\lambda']^2. \quad (436)$$

Let $$m\sin(M-N)=k\sin\psi. \quad . \quad . \quad . \quad . \quad . \quad (437)$$

Then neglecting terms of the second and higher orders in λ' and Δk, (436) may be written as follows:

$$t+w-\tau=\frac{k}{n}\cos\psi-\frac{m}{n}\cos(M-N)+\frac{\Delta k}{n}\sec\psi$$
$$+\frac{\lambda'}{n}\tan\psi-\frac{\lambda}{n}. \quad (438)$$

We have $\dfrac{k}{n}\cos\psi-\dfrac{m}{n}\cos(M-N)=\dfrac{m}{n}\dfrac{\sin(M-N+\psi)}{\sin\psi}$,

a form which is a little more convenient when $\sin\psi$ is not very small.

Equation (438) then gives

$$w = \frac{m}{n}\frac{\sin(M-N+\psi)}{\sin\psi} - (t-\tau) + \frac{\Delta k}{n}\sec\psi + \frac{\lambda'}{n}\tan\psi - \frac{\lambda}{n}, \quad (439)$$

and the equation is solved for w.

As will be seen, this value of w is ambiguous, ψ being determined from (437) in terms of the sine, with nothing to fix the algebraic sign of cos ψ. As before, however, equation (423), the sign of cos ψ will be $-$ in case of immersion and $+$ for emersion. This will always be the case except when the occultation takes place very near the north or south limb of the moon, when there will sometimes be exceptions to the rule. Such occultations, however, are worth very little for longitude purposes, and therefore will not require further consideration here.

260. x' and y' vary so slowly that the above equation will give a very close approximation to the true result, even when $(t + w - \tau)$ is some hours in duration. It will, however, be best to arrange the computation so that $(t + w - \tau)$ is a small quantity, as the labor is less in dealing with small quantities than with large ones, and there is less liability to error.

The unit of time in the small terms of (438) and (439) is one hour. If then w and $(t - \tau)$ are expressed in the usual way in hours, minutes, and seconds, it will be convenient to express these small terms in seconds. If then the time of the ephemeris and of observation are both sidereal or both mean solar, these terms should be multiplied by 3600. If, however, the ephemeris time is mean solar, and that of observation sidereal, we must multiply by 3609.856.

261. Let us now consider more fully the quantities λ and λ'. These depend upon the corrections to the moon's co-ordinates, viz., Δx and Δy, and upon the correction to the eccentricity, Δee. These will be considered separately.

The co-ordinates x and y are variable quantities, and the corrections which they require on account of the inaccuracy of the data, viz., Δx and Δy, will also be variables. It will be more convenient for present purposes to express these in terms of quantities which remain constant throughout the entire occultation.

We have
$$\left.\begin{aligned} x &= x_0 + n \sin N(t + w - \tau); \\ y &= y_0 + n \cos N(t + w - \tau); \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (440)$$

from which we have

$$\left.\begin{aligned} x \sin N + y \cos N &= \quad x_0 \sin N + y_0 \cos N + n(t + w - \tau); \\ -x \cos N + y \sin N &= -x_0 \cos N + y_0 \sin N. \end{aligned}\right\} \quad (441)$$

The last of these is practically independent of the time, and therefore may be regarded as constant throughout the entire occultation.

Let $\varkappa = - x_0 \cos N + y_0 \sin N = - x \cos N + y \sin N.$

Then squaring and adding equations (441),

$$x^2 + y^2 = \varkappa^2 + [x_0 \sin N + y_0 \cos N + n(t + w - \tau)]^2. \quad (442)$$

This expression is a minimum when the last term is zero.

Let the value of $(t + w)$ corresponding to this minimum be T. Then

$$x_0 \sin N + y_0 \cos N + n(T - \tau) = 0;$$

$$\left.\begin{aligned} T &= \tau - \frac{1}{n}(x_0 \sin N + y_0 \cos N); \\ \varkappa &= - x_0 \cos N + y_0 \sin N. \end{aligned}\right\} \quad . \quad . \quad . \quad (443)$$

Therefore $\varkappa = \sqrt{x^2 + y^2}$ is the minimum distance of the axis of the cylinder from the centre of the earth, and T is the time at the first meridian corresponding to this minimum.

We now have
$$\left.\begin{aligned} x \sin N + y \cos N &= n(t + w - T); \\ - x \cos N + y \sin N &= \varkappa. \end{aligned}\right\} \quad . . \quad (444)$$

Referring now to the values of λ and λ', equation (435), we have for the part of these quantities depending on x and y—

$$\begin{aligned} &\text{For } \lambda, \quad \Delta x \sin N + \Delta y \cos N; \\ &\text{For } \lambda', \ - \Delta x \cos N + \Delta y \sin N. \end{aligned}$$

Differentiating equations (444), we have for these quantities

$$\begin{aligned} \Delta x \sin N + \Delta y \cos N &= - n\Delta T + (t + w - T)\Delta n; \\ - \Delta x \cos N + \Delta y \sin N &= \Delta \varkappa. \end{aligned}$$

Therefore that part of the terms $(\lambda' \tan \psi - \lambda)$ due to Δx and Δy is

$$n\Delta T + \Delta \varkappa \tan \psi - (t + w - T)\Delta n. \quad . \quad . \quad (445)$$

The corrections Δx and Δy are by this formula expressed in terms of ΔT, $\Delta \varkappa$, and Δn, which will be constant for the same occultation.

262. It remains to consider the effect of an error in the eccentricity, viz., Δee, which is considered here for the sake of completeness, though it might be neglected without seriously impairing the practical value of the theory.

From (134) and (140) we have

$$\rho \cos \varphi' = \frac{\cos \varphi}{\sqrt{1 - ee \sin^2 \varphi}}; \qquad \rho \sin \varphi' = \frac{\sin \varphi (1 - ee)}{\sqrt{1 - ee \sin^2 \varphi}}. \quad (446)$$

$$\frac{d\rho \cos \varphi'}{dee} = \frac{1}{2} \beta\beta \rho \cos \varphi'; \qquad \frac{d\rho \sin \varphi'}{dee} = \frac{1}{2}\beta\beta \rho \sin \varphi' - \beta.$$

In which
$$\beta = \frac{\rho \sin \varphi'}{1 - ee}.$$

Then $$\frac{d\xi}{dee} = \frac{d\xi}{d\rho\sin\varphi'}\,\frac{d\rho\sin\varphi'}{dee} + \frac{d\xi}{d\rho\cos\varphi'}\,\frac{d\rho\cos\varphi'}{dee};$$
$$\frac{d\eta}{dee} = \frac{d\eta}{d\rho\sin\varphi'}\,\frac{d\rho\sin\varphi'}{dee} + \frac{d\eta}{d\rho\cos\varphi'}\,\frac{d\rho\cos\varphi'}{dee}.$$

Referring now to the values of ξ and η, equations (411), we have

$$\frac{d\xi}{d\rho\cos\varphi'} = \sin(\mu - \alpha); \qquad \frac{d\xi}{d\rho\sin\varphi'} = 0;$$
$$\frac{d\eta}{d\rho\cos\varphi'} = -\sin\delta\cos(\mu - \alpha); \qquad \frac{d\eta}{d\rho\sin\varphi'} = \cos\delta.$$

Therefore $$\frac{d\xi}{dee} = \frac{1}{2}\beta\beta\xi; \qquad \frac{d\eta}{dee} = \frac{1}{2}\beta\beta\,\eta - \beta\cos\delta. \tag{447}$$

Referring now to the values of λ and λ', (435), we have for the terms depending on Δee—

$$\left.\begin{aligned} &\text{For } \lambda, -\frac{d(\xi\sin N + \eta\cos N)}{dee}\Delta ee = \left[-\frac{1}{2}\beta\beta(\xi\sin N + \eta\cos N) + \beta\cos\delta\cos N\right]\Delta ee; \\ &\text{For } \lambda', \quad \frac{d(\xi\cos N - \eta\sin N)}{dee}\Delta ee = \left[-\frac{1}{2}\beta\beta(-\xi\cos N + \eta\sin N) + \beta\cos\delta\sin N\right]\Delta ee. \end{aligned}\right\} . \tag{448}$$

Let us write $$\xi = x_0 - (x_0 - \xi) = x_0 - m\sin M;$$
$$\eta = y_0 - (y_0 - \eta) = y_0 - m\cos M.$$

Substituting these values in (448) and reducing by (443), we find—

$$\left.\begin{aligned} &\text{For } \lambda, \left\{-\tfrac{1}{2}\beta\beta[n(\tau - T) - m\cos(M - N)] + \beta\cos\delta\cos N\right\}\Delta ee; \\ &\text{For } \lambda', \left\{-\tfrac{1}{2}\beta\beta[\qquad\quad \varkappa + m\sin(M - N)] + \beta\cos\delta\sin N\right\}\Delta ee. \end{aligned}\right\} \tag{449}$$

We have from (437) and (438), neglecting the small terms of the latter,

$$-m\cos(M - N) = (t + w - \tau)n - k\cos\psi;$$
$$m\sin(M - N) = \qquad\qquad\qquad k\sin\psi;$$

which substitution will give us for (449)

$$\left.\begin{array}{l}\{-\frac{1}{2}\beta\beta[n(t+w-T)-k\cos\psi]+\beta\cos\delta\cos N\}\Delta ee;\\ \{-\frac{1}{2}\beta\beta[\qquad\qquad\quad \varkappa+k\sin\psi]+\beta\cos\delta\sin N\}\Delta ee.\end{array}\right\} \quad (450)$$

Therefore that part of $(\lambda'\tan\psi-\lambda)$ which depends upon Δee is

$$\left[\frac{1}{2}\beta\beta[n(t+w-T)-\varkappa\tan\psi-k\sec\psi]-\frac{\beta\cos\delta\cos(N+\psi)}{\cos\psi}\right]\Delta ee.\,. \quad (451)$$

Therefore by (445) and (451) the last three terms of equation (438) or (439) will be as follows:

$$\begin{aligned}\frac{\Delta k}{n}\sec\psi+\frac{\lambda'}{n}\tan\psi-\frac{\lambda}{n} &= \Delta T+\frac{k}{n}\tan\psi\,\Delta\varkappa\\ &+\frac{k}{n}\sec\psi\,\Delta k-\frac{\Delta n}{n}(t+w-T)\\ &+\frac{k}{n}\Delta ee\left[\frac{1}{2}\beta\beta[n(t+w-T)-\varkappa\tan\psi-k\sec\psi]-\frac{\beta\cos\delta\cos(N+\psi)}{\cos\psi}\right]. \quad (452)\end{aligned}$$

Each term is expressed in seconds of time, and h is the number of seconds in one hour of the kind of time employed in the ephemeris of the moon. If the times employed in the ephemeris and in observation are both sidereal or both mean solar, $h = 3600$. If the ephemeris time is mean solar and the time of observation sidereal, $h = 3609.86$.

263. We have now obtained an expression for the small terms of our equation, in which the quantities depending on the corrections to the moon's place are expressed in terms of quantities which are constant during the time of the occultation. It will be advantageous, however, to express them directly in terms of the corrections to the quantities given in the ephemeris, viz., to the moon's right ascension, declination, and horizontal parallax.

Let $\Delta(A - \alpha) =$ the correction to the assumed difference of right ascension of the moon and star;

$\Delta(D - \delta) =$ the correction to the assumed difference of declination;

$\Delta\pi =$ the correction to the assumed parallax.

We have, equation (409),

$$x = \frac{\cos D \sin (A - \alpha)}{\sin \pi}; \qquad y = \frac{\sin D \cos \delta - \cos D \sin \delta \cos (A - \alpha)}{\sin \pi}. \quad (453)$$

Writing for brevity $x = \dfrac{X}{\sin \pi}$, $\quad y = \dfrac{Y}{\sin \pi}$,

and differentiating, we have

$$\Delta x = \frac{\Delta X}{\sin \pi} - x\frac{\Delta \pi}{\tan \pi}; \qquad \Delta y = \frac{\Delta Y}{\sin \pi} - y\frac{\Delta \pi}{\tan \pi}.$$

These equations in connection with (444) give the following:

$$\frac{\Delta X \sin N + \Delta Y \cos N}{\sin \pi} - n(t + w - T)\frac{\Delta \pi}{\tan \pi} = -n\Delta T + \Delta n(t + w - T);$$
$$\frac{-\Delta X \cos N + \Delta Y \sin N}{\sin \pi} - \varkappa\frac{\Delta \pi}{\tan \pi} = \Delta \varkappa.$$

It will presently be shown that $\dfrac{\Delta \pi}{\tan \pi} = -\dfrac{\Delta n}{n}$,

and therefore

$$\left.\begin{aligned} -n\Delta T &= \frac{\Delta X \sin N + \Delta Y \cos N}{\sin \pi}; \\ \Delta \varkappa &= \frac{-\Delta X \cos N + \Delta Y \sin N}{\sin \pi} - \varkappa\frac{\Delta \pi}{\tan \pi}. \end{aligned}\right\} \quad (454)$$

264. The value of Δn will now be more fully considered.

We have, equations (432), $n \sin N = x'$;
$$n \cos N = y'.$$

From these, $$n^2 = x'^2 + y'^2.$$

Differentiating, $$n\Delta n = x'\Delta x' + y'\Delta y'. \quad . \quad . \quad . \quad . \quad . \qquad (455)$$

x' and y', it will be remembered, are the changes in x and y respectively in one hour. Regarding them as the differential coefficients of x and y with respect to the time, we have

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{X}{\sin \pi}\right) = \frac{dX}{dt} \frac{1}{\sin \pi} = x';$$
$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{Y}{\sin \pi}\right) = \frac{dY}{dt} \frac{1}{\sin \pi} = y'.$$

$\frac{dX}{dt}$ and $\frac{dY}{dt}$ depend upon the hourly change of the moon in right ascension and declination, which changes are given with accuracy by the ephemeris. Any correction to the values of x' and y' will therefore depend upon π.

We may therefore write

$$\Delta x' = \Delta \frac{a}{\sin \pi} = -x' \frac{\Delta \pi}{\tan \pi};$$
$$\Delta y' = \Delta \frac{b}{\sin \pi} = -y' \frac{\Delta \pi}{\tan \pi}.$$

Substituting in equation (455), it becomes

$$n \,.\, \Delta n = -(x'^2 + y'^2)\frac{\Delta \pi}{\tan \pi}.$$

Therefore $$\frac{\Delta n}{n} = -\frac{\Delta \pi}{\tan \pi},$$ the value assumed above.

265. Returning now to equations (454), we see that

$$\frac{\Delta\pi}{\tan\pi}, \quad \frac{\Delta X}{\sin\pi}, \quad \text{and} \quad \frac{\Delta Y}{\sin\pi}$$

may be regarded as constant throughout the duration of the occultation, since they are expressed in terms of ΔT and $\Delta\varkappa$, which are constant, and Δn and N, which are practically so. The values of $\dfrac{\Delta X}{\sin\pi}$ and $\dfrac{\Delta Y}{\sin\pi}$ will then result from the differentiation of equations (453), viz.:

$$\begin{aligned}
X &= \cos D \sin(A-\alpha);\\
Y &= \sin D \cos\delta - \cos D \sin\delta \cos(A-\alpha);\\
\Delta X &= \cos D \cos(A-\alpha)\Delta(A-\alpha) - \sin D \sin(A-\alpha)\Delta D;\\
\Delta Y &= [\cos D \cos\delta + \sin D \sin\delta \cos(A-\alpha)]\Delta D\\
&\quad + \cos D \sin\delta \sin(A-\alpha)\Delta(A-\alpha)\\
&\quad - [\sin D \sin\delta + \cos D \cos\delta \cos(A-\alpha)]\Delta\delta.
\end{aligned}$$

At the time of conjunction of the sun and moon A becomes equal to α. Therefore

$$\frac{\Delta X}{\sin\pi} = \frac{\cos D}{\sin\pi}\Delta(A-\alpha); \qquad \frac{\Delta Y}{\sin\pi} = \frac{\cos(D-\delta)}{\sin\pi}\Delta(D-\delta). \tag{456}$$

Therefore taking D and π for the instant of conjunction of the moon and star in right ascension, and regarding $\Delta(A-\alpha)$ and $\Delta(D-\delta)$ as the corrections to the assumed differences of right ascension and declination at this instant, also writing unity for $\cos(D-\delta)$, π for $\sin\pi$ and $\tan\pi$, we have, from (454),

$$\left.\begin{aligned}
-\Delta T &= \frac{\cos D\Delta(A-\alpha)}{n\pi}\sin N + \frac{\Delta(D-\delta)}{n\pi}\cos N;\\
\Delta\varkappa &= -\frac{\cos D\Delta(A-\alpha)}{\pi}\cos N + \frac{\Delta(D-\delta)}{\pi}\sin N - \varkappa\frac{\Delta\pi}{\pi};\\
\frac{\Delta n}{n} &= -\frac{\Delta\pi}{\pi}.
\end{aligned}\right\} \tag{457}$$

Substituting these values in (452), and writing for brevity

$$\nu = \frac{h}{n\pi'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (458)$$

we have

$$\begin{aligned}\frac{\Delta k}{n}\sec\psi + \frac{\lambda'}{n}\tan\psi - \frac{\lambda}{n} = &- \nu[\sin N\cos D\Delta(A-\alpha) + \cos N\Delta(D-\delta)] \\ &+ \nu[-\cos N\cos D\Delta(A-\alpha) + \sin N\Delta(D-\delta)] \\ &+ \nu\sec\psi\,\pi\Delta k + \nu[n(t+w-T) - \varkappa\tan\psi]\Delta\pi \\ &+ \nu\left[\frac{1}{2}\beta\beta[n(t+w-T) - \varkappa\tan\psi - k\sec\ \ ] - \frac{\beta\cos\delta\cos(N+\psi)}{\cos\psi}\right]\pi\Delta ee. \qquad (459)\end{aligned}$$

This equation gives the expression for the last three terms of (438) or (439), in which $\Delta\pi$ and Δee are completely separated from the other corrections.

266. Let us now write

$$\left.\begin{aligned}\Omega &= h\left[\frac{k}{n}\cos\psi - \frac{m}{n}\cos(M-N)\right] - (t-\tau); \\ \gamma &= \sin N\cos D\Delta(A-\alpha) + \cos N\Delta(D-\delta); \\ \vartheta &= -\cos N\cos D\Delta(A-\alpha) + \sin N\Delta(D-\delta); \\ E &= n(t+w-T) - \varkappa\tan\psi; \\ F &= \left[\frac{1}{2}\beta\beta[n(t+w-T) - \varkappa\tan\psi - k\sec\psi] - \frac{\beta\cos\delta\cos(N+\psi)}{\cos\psi}\right]\pi.\end{aligned}\right\} \qquad (460)$$

Then equation (438) becomes

$$w = \Omega - \nu\gamma + \nu\tan\psi\vartheta + \nu\sec\psi\pi\Delta k + \nu E\Delta\pi + \nu F\Delta ee. \qquad (461)$$

This equation is now in a form which is well adapted to the purpose in view.

w, γ, ϑ, $\pi\Delta k$, $\Delta\pi$, and Δee may in certain cases be treated as unknown quantities, but they can never all be determined at the same time from the same series of equations.

$\nu\gamma$ is a constant, and its value is independent of the longitude of the place of observation. In order to make its de-

termination possible, therefore, the occultation should be observed at one place at least whose longitude is known. In case such an observation is not available, y may be determined from meridian observations of the moon, if such are available, made on the same night or sufficiently near the same time that ΔA and ΔD may be well determined from them. Of course if the ephemeris of the moon were perfect this would be unnecessary, as then ΔA and ΔD would be zero.

267. In case simply the immersion or emersion of a star has been observed at two places, the longitude of one of which is well determined, the power of the data will be exhausted with the determination of w and y. If both the immersions and emersions have been observed, we may also determine $\pi\Delta k$ and ϑ as unknown quantities, but in no case can $\Delta\pi$ be determined from occultations unless w has been previously well determined. Still less can a satisfactory determination of Δee be obtained in this manner. The two last terms may, however, be retained in the solution of the equations in order to show the effect on the resulting longitude of an error in π or in ee. At the same time it will make it possible to apply the necessary correction to the longitude, if from any source values of these quantities become known more accurate than those assumed in the computation.

For the determination of Δk from single occultations both immersion and emersion must be observed, but contacts at the bright limb can be observed much less satisfactorily than at the dark limb.

The best results are obtained from the occultations of groups of stars like the Pleiades, in which the relative positions of the stars are well determined. The passage of the moon through such a group furnishes a number of equations of condition of the form (461), equal to that of the observed disappearances or reappearances of the stars occulted. As

before remarked, observations at the dark limb can be made with much greater accuracy than at the bright limb (except perhaps in case of a few of the brighter stars). If it is thought desirable, therefore, only observations made at the dark limb need be used in the equations, especially so if stars are observed both north and south of the moon's equator.

On account of the advantages offered by the Pleiades for this purpose, Prof. Peirce developed the equations in a form especially adapted to this group, for use in the longitude work of the U. S. Coast Survey. The reader who is sufficiently interested in the subject may refer to the reports of the U. S. Coast Survey, 1855–56–57–61, in the latter of which is given a numerical example of the application of the method.

Correction for Refraction and for Elevation above Mean Sea Level.

268. The fundamental equation which has been used as the basis of our analysis expresses the condition that the point from which the immersion or emersion is observed is situated in the surface of a right cylinder enveloping the moon and star. At the same time it has been supposed to be in the spheroidal surface of the earth.

The refraction which the ray suffers in passing through the atmosphere causes the elements of this cylinder to be curved lines instead of right lines; or, more correctly, the surface is not that of a cylinder. Further, it follows from the irregularities of the earth's surface that the point from which the observation is made will not in general be in the surface of the mean ellipsoid. Neither of our surfaces therefore conforms exactly to the mathematical form assumed. The effect upon the observed time of an occultation will

always be small, but in extreme cases must be taken into account in an accurate investigation.

If we consider a ray of light as it comes to the eye at the instant when the star is apparently in contact with the moon's limb, this ray will form a curved line, the asymptote of which will cut the vertical line of the observer at a point where the contact would be seen at the same instant as that observed if no refraction existed. The effect of refraction will then be taken into account if we substitute this point for the point occupied by the observer.

Let h' = the altitude of this fictitious point above the observer's position;

h = the altitude of the observer's position above the mean sea level.

Then $h + h'$ = the altitude of the fictitious point above the mean sea level.

Let us then suppose the observation to be made from a point at this elevation above the surface of the mean ellipsoid.

The necessary transformation will be accomplished by changing $\rho \cos \varphi'$ and $\rho \sin \varphi'$ into $\rho \cos \varphi' + (h + h') \cos \varphi$ and $\rho \sin \varphi' + (h + h') \sin \varphi$; or, by formulæ 446,

$$\rho \cos \varphi' \left[1 + (h + h') \sqrt{1 - ee \sin^2 \varphi}\right]$$

and
$$\rho \sin \varphi' \left[1 + (h + h') \frac{\sqrt{1 - ee \sin^2 \varphi}}{1 - ee}\right].$$

h and h' will always be very small fractions when expressed in parts of the earth's radius; therefore no appreciable error will result from neglecting the products of these

quantities by ee. Also $(1 + h + h')$ will be practically equal to $(1 + h)(1 + h')$, the small term hh' being of no account.

The necessary correction for elevation above the mean sea level will therefore be obtained by adding to $\log \rho$ $\log (1 + h)$, and the correction for refraction by adding $\log (1 + h')$.

Expanding $\log (1 + h)$, we have

$$\log (1 + h) = M(h - \frac{h^2}{2} + \text{etc.})$$

$M = .43429448$ is the modulus of the common system of logarithms.

h is here expressed in terms of the earth's radius. If it is given in feet we shall have, instead of the above, $\frac{h}{20923597}$. Therefore, neglecting squares and higher powers of h,

$$\log (1 + h) = h(.000\,000\,02076) \quad . \quad . \quad . \quad . \quad 462$$

If, for instance, the elevation is 1000 feet, the correction to be applied to $\log \xi$ and $\log \eta$ will be .000 0208.

The factor $(1 + h')$ will now be considered.

In the general theory of refraction the atmosphere is regarded as composed of concentric strata the thickness of which is uniform and may be regarded as infinitesimal. If the distance of any point in a ray of light from the earth's centre be r, i the angle between the tangent and normal at the point to which r is drawn, then it is shown by the theory of refraction that $\mu r \sin i$ is a constant, μ being the index of refraction for the infinitesimal stratum at the point under consideration.

For the point where the ray enters the eye let r_0, μ_0, and z' be the special values of r, μ, and i. Then z' will be the apparent zenith distance of the star, and from the foregoing

$$\mu_0 r_0 \sin z' = \mu r \sin i. \quad . \quad . \quad . \quad . \quad . \quad (463)$$

If the first point is taken so far away as to be beyond the limit of the earth's atmosphere, then the refraction at this point is zero and μ becomes unity.

The above equation then becomes

$$\mu_0 r_0 \sin z' = r \sin i. \quad . \quad (464)$$

In the figure,

$OP = r_0$; $\quad PQ = h'$;

$Or = r$; $\quad OrQ = i$.

$ZQr = z$ is the true zenith distance of the star observed.

Then from the triangle rQO

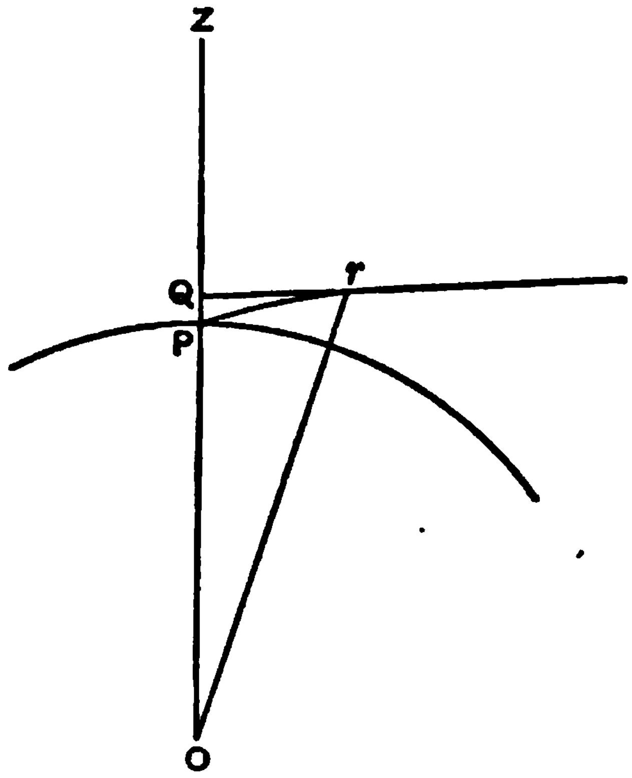


FIG. 53.

$$(r_0 + h') \sin z = r \sin i;$$

and from equation (464)

$$(r_0 + h') \sin z = \mu_0 r_0 \sin z',$$

from which $$1 + \frac{h'}{r_0} = \mu_0 \frac{\sin z'}{\sin z}.$$

r_0 will not differ appreciably for this purpose from the

equatorial radius of the earth; so that if we regard h' as expressed in terms of this quantity we have

$$\log(1 + h') = \log \frac{\sin z'}{\sin z} + \log \mu_0 . . . . \quad (465)$$

The mean value of μ_0 is 1.000 2800.

A table is readily arranged for log $(1 + h')$, with the argument z, the zenith distance of the star. By referring to the value of ζ—equations (411)—we see that ζ is very nearly equal to cos z. For this purpose we may consider it the same.

The following is Bessel's table for log $(1 + h')$. In addition to the argument z we have given cos z, for which we may use log ζ without appreciable error.

TABLE B.

z	log cos z	log $(1 + h')$	z	log cos z	log $(1 + h')$
0°	.0000	0.0000000	82° 0'	9.1436	0.0000069
10	9.9934	0.0000000	83 0	9.0859	0.0000086
20	9.9730	0.0000000	84 0	9.0192	0.0000111
30	9.9375	0.0000001	85 0	8.9403	0.0000147
40	9.8843	0.0000001	85 30	8.8946	0.0000169
50	9.8081	0.0000002	86 0	8.8436	0.0000198
60	9.6990	0.0000005	86 30	8.7857	0.0000234
62	9.6716	0.0000006	87 0	8.7188	0.0000280
64	9.6418	0.0000007	87 30	8.6397	0.0000337
66	9.6093	0.0000008	88 0	8.5428	0.0000412
68	9.5736	0.0000009	88 30	8.4179	0.0000511
70	9.5341	0.0000012	88 50	8.3088	0.0000594
72	9.4900	0.0000015	89 00	8.2419	0.0000643
74	9.4403	0.0000019	89 10	8.1627	0.0000695
76	9.3837	0.0000025	89 20	8.0658	0.0000753
78	9.3179	0.0000033	89 30	7.9408	0.0000817
80	9.2397	0.0000046	89 40	7.7648	0.0000888
81	9.1943	0.0000056	89 50	7.4637	0.0000967
82	9.1436	0.0000069	90 00	....	0.0001054

Example. The following occultations of stars of the Pleiades group were observed at Washington and Greenwich on September 26, 1839:

	At Greenwich.	At Washington.
Star.	Sidereal Time.	Sidereal Time.
g Celæno	$5^h\ 23^m\ 53^s.85$	$22^h\ 51^m\ 19^s.99$
e Taygeta	5 56 50 .63	23 1 0 .68
c Maja	5 58 17 .43	23 17 46 .52

These are all emersions observed at the dark limb of the moon.

The observations at Washington were made at Gilliss's observatory on Capitol Hill, the position of which is assumed to be Latitude $\varphi = 38°\ 53'\ 32''.8$

West longitude $5^h\ 8^m\ 1^s.75$

The latitude of Greenwich $\varphi = 51°\ 28'\ 38''.4$

We now take from Bessel's catalogue of the Pleiades the right ascensions and declinations of the stars for 1839.0 and reduce them to apparent place for 1839, September 26, Greenwich 3^h and 6^h sidereal time, viz.:

	$\alpha\ 3^h$	$\alpha\ 6^h$	$\delta\ 3^h$	$\delta\ 6^h$
g Celæno ...	$53°\ 49'\ 34''.68$	$53°\ 49'\ 34''.72$	$23°\ 46'\ 56''.47$	$23°\ 46'\ 56''.48$
e Taygeta ...	53 55 27 .47	53 55 27 .51	23 57 40 .96	23 57 40 .97
c Maja	54 4 47 .27	54 4 47 .31	23 51 50 .01	23 51 50 .02

The right ascension, declination, and horizontal parallax of the moon for four consecutive hours—viz., 3^h, 4^h, 5^h, and 6^h Greenwich sidereal time—are as follows:

	* Moon's A	D	π
3^h	$52°\ 40'\ 29''.52$	$24°\ 8'\ 55''.07$	$60'\ 10''.19$
4^h	53 18 58 .26	24 18 44 .85	60 8 .88
5^h	53 57 31 .09	24 28 24 .41	60 7 .57
6^h	54 36 8 .03	24 37 53 .73	60 6 .25

We now compute x and y for these dates for each of the stars from formulæ (410), viz.,

$$x = \frac{\cos D \sin (A-\alpha)}{\sin \pi}; \qquad y = \frac{\sin (D-\delta) \cos^2 \frac{1}{2}(A-\alpha) + \sin (D+\delta) \sin^2 \frac{1}{2}(A-\alpha)}{\sin \pi}.$$

* These values are given by Peirce, Coast Survey Report 1861, pp. 204, 205. They were computed directly from Hansen's tables. When the Nautical Almanac is used the intervals will be mean solar hours.

The computation is given in full for *g* Celæno.

	3^h	4^h	5^h	6^h
log π	3.5575301	3.5573724	3.5572148	3.5570558
S	4.6855527	4.6855527	4.6855527	4.6855527
sin π	8.2430828	8.2429251	8.2427675	8.2426085
cosec π	1.7569172	1.7570749	1.7572325	1.7573915
A	52° 40′ 29″.52	53° 18′ 58″.26	53° 57′ 31″.09	54° 36′ 8″.03
α	53 49 34 .68	53 49 34 .69	53 49 34 .70	53 49 34 .72
$A - \alpha$	−1 9 5 .16	−0 30 36 .43	+0 7 56 .39	+0 46 33 .31
sin $(A - \alpha)$ {	4.6855457	4.6855692	4.6855745	4.6855616
	3.6175413$_n$	3.2639744$_n$	2.6779626	3.4461191
cos D	9.9602268	9.9596679	9.9591146	9.9585670
cosec π	1.7569172	1.7570749	1.7572325	1.7573915
log x	.0202310$_n$	9.6662864$_n$	9.0798842	9.8476392
x	−1.047686	−0.463753	+0.120194	+0.704108
D	24° 8′ 55″.07	24° 18′ 44″.85	24° 28′ 24″.41	24° 37′ 53″.73
δ	23 46 56 .47	23 46 56 .47	23 46 56 .48	23 46 56 .48
$D - \delta$	0 21 58 .60	0 31 48 .38	0 41 27 .93	0 50 57 .25
$D + \delta$	47 55 51 .54	48 5 41 .32	48 15 20 .89	48 24 50 .21
$\frac{1}{2}(A - \alpha)$	− 34 32 .58	− 15 18 .22	+ 3 58 .20	+ 23 16 .66
sin $\frac{1}{2}(A - \alpha)$ {	4.6855676	4.6855735	4.6855748	4.6855716
	3.3165113	2.9629467	2.3769418	3.1450906
$\sin^2 \frac{1}{2}(A - \alpha)$	6.0041578	5.2970404	4.1250332	5.6613244
sin $(D + \delta)$	9.8706018	9.8717195	9.8728115	9.8738782
Sum 1	5.8747596	5.1687599	3.9978447	5.5352026
$\cos^2 \frac{1}{2}(A - \alpha)$	9.9999562	9.9999914	9.9999994	9.9999798
sin $(D - \delta)$ {	4.6855719	4.6855687	4.6855644	4.6855590
	3.1201131	3.2806649	3.3958381	3.4853310
Sum 2	7.8056412	7.9662250	8.0814019	8.1708698
$S2 - S1$	1.9308816	2.7974651	4.0835572	2.6356672
Zech*	.0050625	.0006918	.0000358	.0010038
cosec π	1.7569172	1.7570749	1.7572325	1.7573915
log y	9.5676209	9.7239917	9.8386702	9.9292651
$y =$	+ .369506	.529653	.689716	.849699

* This is the quantity taken from Zech's addition and subtraction logarithmic table.

We thus have values of x and y computed for four consecutive hours, from which we can now compute the values of x' and y' to the third order of differences inclusive by means of formulæ (101), $(101)_1$, and $(101)_2$, viz.:

	x	x'	y	y'
3^h	-1.047686	.583910	.369506	.160189
4^h	$-\ .463753$	.583948	.529653	.160105
5^h	$+\ .120194$	.583938	.689716	.160023
6^h	$+\ .704108$	.583882	.849699	.159941

For the other stars observed we find—

Taygeta.

	x	x'	y	y'
3^h	-1.136840	$+.583978$	$+.191768$	.159690
4^h	$-\ .552839$	.584017	.351424	.159623
5^h	$+\ .031182$	.584018	.511015	.159560
6^h	$+\ .615185$	.583981	.670546	.159503

Maja.

3^h	-1.278300	$+.584071$	.290289	.159105
4^h	$-\ .694197$	.584128	.449353	.159024
5^h	$-\ .110057$	.584145	.608340	.158951
6^h	$+\ .474080$	.584122	.767257	.158884

Computation of ξ, η, and ζ.

(ζ is only required for determining the correction due to refraction.)

Formulæ (412) are as follows:

$$\rho \sin \varphi' = b \sin B; \qquad \xi = \rho \cos \varphi' \sin (\mu - \alpha);$$
$$\rho \cos \varphi' \cos (\mu - \alpha) = b \cos B; \qquad \eta = b \sin (B - \delta);$$
$$\zeta = b \cos (B - \delta).$$

With the known values of φ for Greenwich and Washington, we obtain ρ and φ' by the use of formulæ (V), Art. 77.

The computation is then as follows:

	Greenwich.	Washington.
φ'	51° 17′ 24″.8	38° 42′ 18″.3
$\sin \varphi'$	9.8922748	9.7960967
$\log \rho$	9.9991135	9.9994302
$\cos \varphi'$	9.7961411	9.8923033
$\rho \sin \varphi'$	9.8913883	9.7955269
$\rho \cos \varphi'$	9.7952546	9.8917335
μ	$5^h\ 23^m\ 53^s.85$	$22^h\ 51^m\ 19^s.99$
μ	80° 58′ 27″.8	342° 49′ 59″.9
α	53 49 34 .7	53 49 34 .7
$\mu - \alpha$	27 8 53 .1	289 0 25 .2
$\cos(\mu - \alpha)$	9.9493072	9.5127960
$\sin(\mu - \alpha)$	9.6592427	9.9756518_n
$\log \xi$	9.4544973	9.8673853_n
ξ	+.284772	−.736861
$b \cos B$	9.7445618	9.4045295
$\sin B$	9.9107179	9.9668001
$b \sin B$	9.8913883	9.7955269
$\tan B$	.1468265	.3909974
B	54° 30′ 21″.21	67° 52′ 51″.33
δ	23 46 56 .47	23 46 56 .47
$B - \delta$	30 43 24 .74	44 5 54 .86
$\sin(B - \delta)$	9.7083326	9.8425436
$\log b$	9.9806704	9.8287268
$\cos(B - \delta)$	9.9343179	9.8562113
$\log \eta$	9.6890030	9.6712704
η	.488656	.469105
$\log \zeta$	9.9149883	9.6849381
z	34° 41′	61° 3′

z has been computed for the purpose of taking into account the correction for refraction. With this value we find from table B, Art. 268, log $(1 + k')$ = .000 000 1 and .000 000 5 respectively, which values are to be added to log ξ and log η. As they are so small as to be practically inappreciable, they have been neglected.

Also, we have for the above times of observation—

	Taygeta.		Maja.	
	Greenwich.	Washington.	Greenwich.	Washington.
ξ	+.360523	−.725974	+.362353	−.704226
η	+.504728	+.455553	+.506584	+.436040

With the assumed value of the longitude of the observatory at Washington, viz., $5^h\ 8^m\ 1^s.75$, we reduce the Washington times to Greenwich time, and assuming the values of τ sufficiently near these times that x and y may be assumed to vary uniformly during the interval, we compute M, m, N, n, and ψ by the formulæ

$$m \sin M = x_0 - \xi; \qquad n \sin N = x'; \qquad \sin \psi = \frac{m}{k} \sin (M - N).$$
$$m \cos M = y_0 - \eta; \qquad n \cos N = y';$$

The computation for Celæno is then as follows:

	Greenwich.	Washington.
Wash. time		$22^h\ 51^m\ 19^s.99$
Gh. time	$5^h\ 23^m\ 53^s.85$	$3\ \ 59\ \ 21\ .74$
Assumed τ	$5^h.4$	$4^h.0$
x_0	.353765	−.463753
ξ	.284772	−.736861
$x_0 -$	.068993	+.273108
y_0	.753720	.529653
η	.488656	.469105
$y_0 - \eta$	.265064	.060548
$\log m \sin M$	8.8388050	9.4363344
$\sin M$	9.4012192	9.9895810
$\log m \cos M$	9.4233508	8.7820998
$\tan M$	9.4154542	.6542346
M	14° 35′ 22″.8	77° 29′ 59″.0
$\log m$	9.4375858	9.4467534
x'	.583916	.583948
y'	.159990	.160105
$\log n \sin N$	9.7663504	9.7663742
$\sin N$	9.9842810	9.9842609
$\log n \cos N$	9.2040928	9.2044049
$\tan N$	.5622576	.5619693
N	74° 40′ 38″.3	74° 40′ 3″.4
$\log n$	9.7820694	9.7821133
$M - N$	299° 54′ 44″.5	2° 49′ 55″.6
$\sin (M - N)$	9.9379135$_n$	8.6938108
$\log m$	9.4375858	9.4467534
$ac \log k$	.5650000	.5650000
$\sin \psi$	9.9404993$_n$	8.7055642

Since the *emersions* were the phases observed, cos ψ is plus; therefore

	Greenwich.	Washington.
$\psi =$	$299° \ 18' \ 43''.7$	$2° \ 54' \ 35''.5.$

We now compute Ω from the formula

$$\Omega = h\left[\frac{k}{n}\cos\psi - \frac{m}{n}\cos(M - N)\right] - (t - \tau);$$

where $h = 3600; \quad \log h = 3.5563025.$

	Greenwich.	Washington.
$\cos\psi$	9.6898123	9.9994397
* $\log k$	9.4350000	9.4350000
$\log \frac{1}{n}$	.2179306	.2178867
S_1	2.8990454	3.2086289
Nat. No.	$792^s.58$	$1616^s.70$
$\frac{hk}{n}\cos\psi$	$13^m \ 12^s.58$	$26^m \ 56^s.70$
$\cos(M - N)$	9.6978174	9.9994692
$\log m$	9.4375858	9.4467534
$\log \frac{1}{n}$	.2179306	.2178867
S_2	2.9096363	3.2204118
Nat. No.	$812^s.15$	$1661^s.16$
$\frac{hm}{n}\cos(M-N)$	$13^m \ 32^s.15$	$27^m \ 41^s.16$
$t - \tau$	$- \ 6.15$	$-5^h \ 8^m \ 40^s.01$
Ω	$- \ 13^s.42$	$+5^h \ 7^m \ 55^s.55$

In a similar manner we find for the other stars—

For Taygeta,	Ω	$- \ 9^s.30$	$+5^h \ 7^m \ 55^s.67;$
For Maja,	Ω	$- \ 9^s.79$	$+5^h \ 7^m \ 53^s.08.$

We next compute T, $\varkappa$, and ν by formulæ (443) and (458), viz.:

$$T = \tau - \frac{1}{n}(x_0 \sin N + y_0 \cos N);$$

$$\varkappa = - x_0 \cos N + y_0 \sin N;$$

$$\nu = \frac{h}{n\varkappa}.$$

* It is not necessary for this purpose to know the value of k with extreme accuracy, since the correction Δk to the assumed value appears as one of the terms of our equation.

For Celæno we have

τ 5.4

$\sin N$ 9.98428	$\log x_0 \cos N$ 8.97073	$\log \frac{1}{\pi}$ 6.44270
$\log x_0$ 9.54871	Zech .83098	$\log \frac{1}{n}$ .21793
$\cos N$ 9.42202	$\log y_0 \sin N$ 9.86149	$\log k$ 3.55630
$\log y_0$ 9.87721		
	$\log \varkappa$ 9.80171	$\log \nu$ .21693
$\log x_0 \sin N$ 9.53299	$\varkappa$ .6334	ν 1.6479
Zech .43345		
$\log y_0 \cos N$ 9.29923		
$\log(x_0 \sin N + y_0 \cos N)$ 9.73268		
$\log \frac{1}{n}$ .21793		
9.95061		
Nat. No. .8925		
T 4.5075		

We now compute the coefficients for the final equations of the form (461), viz.:

$$\nu \tan \psi, \qquad \nu E = \nu[n(t + w - T) - \varkappa \tan \psi], \qquad \text{and} \qquad \nu \sec \psi.$$

	Greenwich.	Washington.
$t + w$	5.3983	3.9894
$t + w - T$	.8908	−.5181
$\log (t + w - T)$	9.94978	9.71441_n
$\log n$	9.78207	9.78211
Sum	9.73185	9.49652_n
$\log \varkappa$	9.80171	9.80171
$\tan \psi$	$.25069_n$	8.70612
Sum	$.05240_n$	8.50783
Zech	.16969	.04241
$\log E$	.22209	9.53893_n
$\log \nu$	.21693	.21693
$\log \nu E$	.43902	9.75586_n
νE	2.7480	−.5700
$\sec \psi$	.31019	.00056
$\log \nu \sec \psi$	.52712	.21749
$\log \nu \tan \psi$	$.46762_n$	8.92305
$\nu \sec \psi$	3.3661	1.6500
$\nu \tan \psi$	−2.9351	.0838

Computing the coefficients for the other two stars in the same way, we obtain the following six equations:

$$\left.\begin{array}{llrl}
\text{Celæno:} & \text{G.}\ w & = & -0^h\,0^m\,13^s.42 - 1.648\gamma - 2.935\vartheta + 3.366\pi\Delta k + 2.748\Delta\pi;\ [1] \\
 & \text{W.}\ w' & = & 5\ \ 7\ \ 55\,.55 - 1.648\gamma + .084\vartheta + 1.650\pi\Delta k - .570\Delta\pi.\ [4] \\
\text{Taygeta:} & \text{G.}\ w & = & -0\ \ 0\ \ \ 9\,.30 - 1.648\gamma - .598\vartheta + 1.753\pi\Delta k + 1.507\Delta\pi;\ [2] \\
 & \text{W.}\ w' & = & 5\ \ 7\ \ 55\,.67 - 1.648\gamma + 1.048\vartheta + 1.953\pi\Delta k - 1.084\Delta\pi.\ [5] \\
\text{Maja:} & \text{G.}\ w & = & -0\ \ 0\ \ \ 9\,.79 - 1.648\gamma - 2.328\vartheta + 2.852\pi\Delta k + 2.492\Delta\pi;\ [3] \\
 & \text{W.}\ w' & = & 5\ \ 7\ \ 53\,.08 - 1.648\gamma - .062\vartheta + 1.650\pi\Delta k - .442\Delta\pi.\ [6]
\end{array}\right\}\ .\quad (A)$$

If we assume γ, ϑ, $\Delta\pi$, and $\pi\Delta k$ to be the same in all of these equations—an assumption which involves no appreciable error—we shall have six equations between those quantities and w'. w, the longitude of Greenwich, will be zero.

It is evident, however, that for various reasons a direct solution of these equations will not be expedient. In the first place, the large terms involved would render the operation very laborious, and further it will not be possible to separate $\Delta\pi$ from the remaining quantities without assuming both w and w' to be known.

We therefore proceed as follows: Assuming the equations to be of equal weight, we subtract the first from the third, the first from the fifth, and the third from the fifth; then we subtract the second from the fourth, the second from the sixth, and the fourth from the sixth. We then have the following six equations:

$$\left.\begin{array}{rl}
0 = & 4.12 + 2.337\vartheta - 1.613\pi\Delta k - 1.241\Delta\pi;\ [2]-[1] \\
0 = & 3.63 + .607\vartheta - .514\pi\Delta k - .256\Delta\pi;\ [3]-[1] \\
0 = & -\ .49 - 1.730\vartheta + 1.099\pi\Delta k + .985\Delta\pi;\ [3]-[2] \\
0 = & +\ .12 + .964\vartheta + .303\pi\Delta k - .514\Delta\pi;\ [5]-[4] \\
0 = & -\ 2.47 - .146\vartheta - .000\pi\Delta k + .128\Delta\pi;\ [6]-[4] \\
0 = & -\ 2.59 - 1.110\vartheta - .303\pi\Delta k + .642\Delta\pi.\ [6]-[5]
\end{array}\right\}\ .\quad (B)$$

By means of these six equations of condition we now determine the most probable values of ϑ and $\pi\Delta k$. The value of $\Delta\pi$, however, cannot be well determined, as we have before remarked. If it were not known *a priori* that such was the case, it would be shown from the normal equations, which would be practically indeterminate for this quantity. We shall therefore determine ϑ and $\pi\Delta k$ in terms of $\Delta\pi$ in order to show what effect an error in π will have upon the longitude.

By the method of Art. 21 we derive from the above equations the following two normal equations:

$$\left.\begin{array}{rl}
11.0056\vartheta - 5.3545\pi\Delta k = & -\ 16.0306 + 5.9864\Delta\pi; \\
-\ 5.3545\vartheta + 4.2574\pi\Delta k = & 8.2287 - 2.8656\Delta\pi.
\end{array}\right\}\ .\ .\quad (C)$$

From which
$$\left.\begin{aligned}\pi\Delta k &= \quad ''.2588 + .0289\Delta\pi;\\ \vartheta &= -\ 1''.3301 + .5577\Delta\pi.\end{aligned}\right\} \quad \ldots\ldots\ldots \quad \text{(D)}$$

We now substitute these values in the first, third, and fifth of equations (A), writing zero for w, the longitude of Greenwich, when we find the following values for 1.648γ:

$$\left.\begin{aligned}1.648\gamma &= -\ 8.645 + 1.209\Delta\pi;\\ 1.648\gamma &= -\ 8.055 + 1.226\Delta\pi;\\ 1.648\gamma &= -\ 5.955 + 1.276\Delta\pi.\end{aligned}\right\} \quad \ldots\ldots\ldots \quad \text{(E)}$$

$$\text{Mean } 1.648\gamma = -\ 7.552 + 1.237\Delta\pi; \qquad \gamma = -\ 4''.582 + .751\Delta\pi.$$

We now substitute these values of $\pi\Delta k$, ϑ, and γ in the second, fourth, and sixth of (A), when we find the following values for the difference of longitude between Greenwich and the observatory on Capitol Hill, Washington:

$$\begin{array}{lll}\text{Celæno} & w' = 5^h\ 8^m\ 3^s.42 - 1.712\Delta\pi;\\ \text{Taygeta} & w' = 5\ \ 8\ \ 2\ .33 - 1.681\Delta\pi;\\ \text{Maja} & w' = 5\ \ 8\ \ 1\ .14 - 1.665\Delta\pi.\\ \text{Mean} & w' = 5\ \ 8\ \ 2\ .30 - 1.686\Delta\pi.\end{array}$$

The Capitol Hill observatory is $10^s.25$ east of the Naval Observatory. The longitude of the latter, determined telegraphically, is $5^h\ 8^m\ 12^s.09$ west of Greenwich. Therefore the true value of w' is $5^h\ 8^m\ 1^s.84$, corresponding very closely with the above value if we neglect $\Delta\pi$ altogether.

With these values of γ and ϑ we may now determine the correction to the assumed right ascension and declination of the moon.

We have
$$\left.\begin{aligned}\sin N\cos D\Delta(A-\alpha) + \cos N\Delta(D-\delta) &= \gamma;\\ -\cos N\cos D\Delta(A-\alpha) + \sin N\Delta(D-\delta) &= \vartheta.\end{aligned}\right\} \quad \ldots \quad (460)$$

Substituting for the coefficients of $\Delta(A-\alpha)$ and $\Delta(D-\delta)$ the mean of the values for the three stars, we have the equations

$$\begin{aligned}879\Delta(A-\alpha) + 264\Delta(D-\delta) &= -\ 4582;\\ -\ 240\Delta(A-\alpha) + 965\Delta(D-\delta) &= -\ 1330.\end{aligned}$$

From which we find
$$\begin{aligned}\Delta(A-\alpha) &= -\ 4''.46;\\ \Delta(D-\delta) &= -\ 2\ .49.\end{aligned}$$

Assuming the errors of the star places to be inappreciable, these will represent the errors in the computed right ascension and declination of the moon at a time corresponding to the mean of the times of observation. These corrections

it will be seen are affected by any small outstanding error in the parallax, as they have been derived by assuming $\Delta\pi = 0$.

In the same way, assuming $\Delta\pi = 0$ and taking for π the mean of the values given above, viz., 3608″, we find from the above value of $\pi\Delta k$

	$\Delta k = + .0000717.$
We have assumed	$k = \quad .272270.$
Therefore	$k = \quad .272342,$

as shown from these observations. This result from so small a number of occultations has no value, however, as a determination of the moon's semi-diameter.

Observations of Different Weights.

269. In the solution of our equations we have supposed all to be of the same weight. Such will not in general be the case. Other things being equal, those occultations will be best for longitude determination which are most nearly central in reference to the moon's disk. When both immersion and emersion of the same star are observed, the observation at the dark limb of the moon is entitled to greater weight than that at the bright limb, except, perhaps, in case of the brighter stars.

In order to determine the proper manner of treating the equations when different weights are assigned, let us suppose, as in our example, three observations to have been made at one place whose true longitude is w; then for the present, considering only terms in γ and ϑ, we shall have three equations of this form:

$$\left.\begin{array}{l} \sqrt{p}w - \sqrt{p}a\vartheta - \sqrt{p}O = 0; \\ \sqrt{p'}w - \sqrt{p'}a'\vartheta - \sqrt{p'}O' = 0; \\ \sqrt{p''}w - \sqrt{p''}a''\vartheta - \sqrt{p''}O'' = 0. \end{array}\right\} \quad . \quad . \quad (466)$$

Where $O = \Omega - \nu\gamma$, and p, p', p'' are the respective weights. From these we derive the normal equations

$$\left.\begin{aligned}[p]w - [pa]\,\vartheta - [pO] &= 0;\\ [pa]w + [paa]\vartheta - [paO] &= 0.\end{aligned}\right\} \quad . \quad . \quad (467)$$

The solution of these equations in the usual manner gives

$$\left.\begin{aligned}[paa1] &= [paa] - \frac{[pa]}{[p]}[pa];\\ [paO1] &= [paO] - \frac{[pa]}{[p]}[pO];\\ [paa1]\,\vartheta &= [paO1].\end{aligned}\right\} \quad . \quad . \quad (468)$$

Which gives ϑ with the weight $[paa1]$.

But, as we have seen, this form of solution is inconvenient on account of the large quantities involved.

Let us write out in full the values of $[paa1]$ and $[paO1]$:

$$\left.\begin{aligned}[paa1] &= pa^2 + p'a'^2 + p''a''^2 - \frac{pa + p'a' + p''a''}{p + p' + p''}(pa + p'a' + p''a'')\\ &= \frac{pp'(a - a')^2 + pp''(a - a'')^2 + p'p''(a' - a'')^2}{p + p' + p''};\\ [paO1] &= paO + p'a'O' + p''a''O'' - \frac{pa + p'a' + p''a''}{p + p' + p''}(pO + p'O' + p''O'')\\ &= \frac{pp'(a-a')(O-O') + pp''(a-a'')(O-O'') + p'p''(a'-a'')(O'-O'')}{p + p' + p''}.\end{aligned}\right\} (469)$$

Comparing these expressions with our equations of condition (466), we see that the final equation for ϑ may be obtained as follows: Before multiplying the equations through by $\sqrt{p}$, $\sqrt{p'}$, and $\sqrt{p''}$, subtract the second from the first, the third from the first, and the third from the

second, then give to the three resulting equations the following weights respectively:

$$\frac{pp'}{p+p'+p''}; \quad \frac{pp''}{p+p'+p''}; \quad \frac{p'p''}{p+p'+p''}. \qquad (470)$$

We may apply the same reasoning to the equation in which all of the unknown quantities are retained, and may extend it to any number of equations of condition. Thus if the number of equations of condition were four, we find by combining them in a like manner, two and two, six equations with weights

$$\frac{pp'}{p+p'+p''+p'''}; \frac{pp''}{p+p'+p''+p'''}; \ldots \frac{p''p'''}{p+p'+p''+p'''}.$$

It is not possible to give a rule by which the proper weight can be assigned in every case, as it will depend upon a variety of circumstances, such as the skill and experience of the observer, the magnitude of the star, condition of the atmosphere, and various other causes. Evidently, if weights are to be assigned depending upon these circumstances, much must be left to the judgment of the observer and computer. If the conditions are otherwise the same in case of two stars, the weights may be assumed proportional to the numerical values of cos ψ; that is, proportional to the chord of the moon's disk traversed by the stars—a central occultation having the weight unity.

If we assign weights to our six equations (A) in accordance with this principle, we shall have for the weights, taken in order, $p = .49$; $p_1 = 1.00$; $p' = .94$; $p_1' = .84$; $p'' = .58$; $p_1'' = 1.00$.

The weights of equations (B) will then be in accordance with formulæ (470).

	p		p
[2] − [1]	.229	[5] − [4]	.296
[3] − [1]	.141	[6] − [4]	.352
[3] − [2]	.271	[6] − [5]	.296

Multiplying the equations by the square roots of the respective weights and proceeding in the usual way, we obtain the following normal equations:

$$2.7630\vartheta - 1.2391\pi\Delta k = -3.7605 + 1.5129\Delta\pi;$$
$$-1.2391\vartheta + 1.0174\pi\Delta k = 1.6907 - .6678\Delta\pi.$$

From these we find $\pi\Delta k = + .00931 + .0232\Delta\pi;$
$\vartheta = - 1.3570 + .5579\Delta\pi.$

Substituting these values in [1], [2], and [3] of equations (A), and taking the mean by weights, we find

$$1.648\gamma = - 8.161 + 1.221\Delta\pi.$$

Finally, substituting these values of ϑ, $\pi\Delta k$, and γ in [4], [5], and [6], we find the following values for w':

$$\begin{array}{llll}
[4] & w' = 5^h\ 8^m\ 3^s.61 - 1.706\Delta\pi; & \text{wt.} = 1.00. \\
[5] & w' = 5\ \ 8\ \ 2\ .43 - 1.675\Delta\pi; & \text{wt.} = \ .84. \\
[6] & w' = 5\ \ 8\ \ 1\ .34 - 1.660\Delta\pi; & \text{wt.} = 1.00.
\end{array}$$

From these we have

$$w = 5^h\ 8^m\ 2^s.46 - 1.681\Delta\pi$$

CHAPTER VIII.

THE ZENITH TELESCOPE.

270. This instrument is used in determining latitude, and is particularly useful when a high degree of accuracy is required, the precision being not inferior to that of the most refined instruments of a fixed observatory, while on account of its great simplicity it is especially adapted to use in the field.

We have already developed several methods for determining latitude: those of Chapter V. are very useful, but will not be employed in the field except in cases where an error of five or six seconds in the result is not considered objectionable. The prime vertical transit gives results of high precision, but not without the expenditure of much labor. The method by the zenith telescope is superior to the first of these in accuracy, and to the second in facility of application. On account of these advantages it has superseded all other methods on the Coast and other government surveys in cases where extreme accuracy is required.

The most common form of instrument is shown in Fig. 54. In general appearance, as will be seen, it is a telescope with an altitude and azimuth mounting. The essential characteristics are a very delicate level attached to the tube, like the level of the finding-circles in the transit instrument, and the eye-piece micrometer. The vertical axis is made very long to insure steadiness of motion in azimuth. The instrument is used in the meridian like the transit.

FIG. 54.—THE ZENITH TELESCOPE.

In the Coast Survey instrument the aperture of the telescope is $3\frac{1}{4}$ inches, focal length 45 inches, length of horizontal axis 7 inches, vertical axis 24 inches, diameter of horizontal circle 12 inches, vertical circle 6 inches (sometimes this is only a semicircle, the radius being 6 inches). The instrument rests on three foot-screws. The lamp at the end of the horizontal axis opposite the telescope illuminates the field; the weight seen at the same end of the axis acts as a counterpoise to the telescope. This weight is connected with the telescope by a bent metallic bar, shown in the figure, in such a way as to prevent to some extent the flexure of the axis.

The horizontal circle is read by means of two verniers. The level attached to the vertical circle is generally graduated so that the motion of the bubble over one millimetre corresponds to an angle of one second of arc. The accuracy of the instrument depends in a great degree on the delicacy of this level. In testing an instrument it may generally be assumed that if the level is a good one the performance of the instrument as a whole will be satisfactory. The striding-level shown on the horizontal axis is used for adjusting the instrument, and is not necessarily of so great accuracy.

The micrometer* is provided with one or more movable threads, the value of one revolution of the screw being from $45''$ to $60''$. The head of the screw is divided into 100 parts, of which tenths may be estimated; thus by estimation $\frac{1}{1000}$ of one revolution may be read, or about $0''.05$. The entire revolutions are read by means of a comb at one side of the field of view, the distance between two consecutive notches corresponding to one revolution. There are three, and sometimes five, vertical threads which may be used for observing transits. A rack and pinion is provided for sliding the eye-piece in the direction of the vertical so that the star may always be observed in the middle of the field.

* For description of the micrometer see Art. 97.

The instrument is mounted like the transit on a pier of masonry, or simply a solid wooden post planted three feet in the ground.

The dimensions given above are those of a large-sized instrument; much smaller ones are often used.

The transit instrument may be used as a zenith telescope if it is provided with the fine level and micrometer. A special appliance for reversing is convenient, but not essential. As we have seen in the descriptions of the different forms of portable transit instruments, the two are often combined. This arrangement is very advantageous on the ground of economy of first cost and of transportation; at the same time nothing is lost in accuracy and little in convenience.

Adjustments.

271. *First.* The vertical axis must be made truly vertical. In setting up the instrument it will be found advisable to place two of the foot-screws in an east and west direction, otherwise if it is found necessary to move the screws after the instrument has been brought into the plane of the meridian this last adjustment will be disturbed.

The axis is brought into the vertical position by the use of the striding-level, which should read the same while the instrument is turned completely around in azimuth. This adjustment will also be tested by means of the more delicate level attached to the telescope.

Second. The horizontal axis should be perpendicular to the vertical axis. This may be tested by reversing the striding-level after the vertical axis has been properly adjusted.

Third. The line of collimation may be adjusted by directing the telescope to some distant terrestrial mark, then turning the instrument 180° in azimuth by means of the horizontal

circle. Allowance must be made for the parallax of the instrument, unless the mark is so far away that it is not appreciable. This is necessary, since the line of collimation is not in the same vertical plane as the axis.

Let d = distance of the line of collimation from vertical axis;
D = distance of mark;
p = correction for parallax.

Then $$p = \frac{d}{D \sin 1''}. \quad \ldots \ldots \quad (471)$$

This method of adjustment depends entirely on the reading of the circle, and is therefore not capable of extreme accuracy. If considered desirable, a more accurate adjustment may be made by means of a pair of collimating telescopes* or by the mercury collimator.* The error may also be determined by transits of stars observed in both positions of the axis, as explained in connection with the transit instrument. If stars are chosen which culminate near the zenith, an error of azimuth will have but little influence on the result.

When used as a transit instrument a meridian mark is recommended, consisting of two lamps placed side by side and at a distance apart equal to twice the distance of the vertical from the collimation axis.

It is perhaps unnecessary to say that the instrument must be focused and the threads placed truly vertical and horizontal respectively, precisely as in the transit instrument.

Fourth. The instrument must be brought into the plane of the meridian. For this and other purposes we require the local time, a chronometer or clock being an essential part of

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* See Art. 168.

the outfit. The clock correction T may be determined by the sextant, transit instrument, or by transits observed with the zenith telescope itself. In the latter case the process of bringing the instrument into the meridian will be the same as that already described for the transit.

If ΔT is known within one second of its true value, that will be sufficient.

ΔT being supposed known,

Let $\alpha =$ the right ascension of a star near the pole.
Then $\alpha - \Delta T =$ the chronometer time of culmination.

At this instant, as shown by the chronometer, the middle thread is placed on the star, the horizontal circle being provided with a clamp and tangent-screw for this and similar purposes. The reading of the verniers now shows the true direction of the meridian. Two stops arranged for the purpose are now clamped to the horizontal circle so that the instrument may be turned freely in azimuth, but brought to a stop when it reaches the meridian. Care must be taken in turning the instrument in azimuth not to bring it up against these stops with a shock, as this will disturb the adjustment.

South stars may be used for adjusting in the meridian, provided they are sufficiently far from the zenith. In any case the adjustment should be tested by trying whether a south star crosses the middle thread at the proper time.

The stops should be placed so that in reversing the instrument in azimuth the object end of the telescope always turns towards the east. The observer can then turn it in azimuth a little, so as to find a star a moment before it enters the field; then knowing exactly where to look for the star, the eye-piece can be brought to the right place by the rack and pinion, and the micrometer-thread moved to nearly the proper place, so that when the star finally comes into view the bisection can be made with all necessary deliberation.

All of the above matters having been attended to, the instrument is ready for regular latitude observation.

The Observing List.

272. The stars are observed in pairs, one star culminating north of the zenith and the other south. The difference of zenith distance should not exceed 15′ or 20′.

Let φ, δ, and δ' = respectively the latitude of station and declination of south and north star;
z and z' = the zenith distances.

Then

$$\varphi = \delta + z;$$
$$\varphi = \delta' - z';$$
$$\varphi = \tfrac{1}{2}(\delta + \delta') + \tfrac{1}{2}(z - z'). \quad . \quad . \quad . \quad . \quad (472)$$

Thus the latitude is equal to one half the sum of the declinations plus one half the difference of zenith distance, which latter must be small enough to be capable of measurement by the micrometer.

The difference of right ascension of the two stars forming the pair should not exceed 15^m or 20^m, as changes may take place in the instrument if a longer time elapses. If care is used in the selection, it will seldom be necessary to use a pair with so long an interval as 15 minutes. The interval should not be less than one minute, as the instrument must be read and reversed in azimuth for the second star, which will require at least that amount of time.

Stars smaller than the 7th magnitude cannot be well observed with the instrument which has been described. With smaller instruments the 6th magnitude will be about the limit.

Stars at any zenith distance may be observed, but generally it will not be necessary or advisable to go beyond 30° or 35°.

The catalogues most suitable for the selection of stars are the Coast Survey catalogue,* the various Greenwich catalogues, and the British Association catalogue. The declinations of the latter are not sufficiently reliable for a good latitude determination; but as it contains nearly all the stars down to the 6th magnitude inclusive, it may very conveniently be used in selecting the list, the final declinations being afterwards taken from more reliable catalogues.

In selecting the stars we require an approximate value of the latitude, which may often be taken from a map with sufficient accuracy, or if suitable maps are not available it may be determined by a single altitude of the sun or a star at culmination measured with the sextant. An error of 1′ or 2′ in the assumed value will cause no inconvenience.

In selecting the list of stars we proceed as follows: First we must know with what right ascension to begin. If, for instance, we intend beginning our observations at 7^h P.M., this mean solar time converted into sidereal time will give the right ascension of a star which culminates at that instant. Starting with this right ascension, we take the first star whose zenith distance at culmination does not exceed 35° and look down the list to find whether there is another star which differs from this in right ascension between 1^m and 15^m, and which will unite with this to form a suitable pair. From (472) we have

$$\left.\begin{aligned} \delta &= 2\varphi - \delta' - (z - z'); \\ \delta' &= 2\varphi - \delta - (z - z'). \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (473)$$

Thus if δ' is the declination of the star, if we can find another

* Coast Survey Report 1876, Appendix No. 7.

whose declination δ does not differ from $2\varphi - \delta'$ more than 15′ or 20′, the two stars will form a pair suitable for our purpose. With the great majority of trials we shall find no second star fulfilling the above conditions. If we use the British Association catalogue we can generally find from one to three dozen pairs suitable for observation for any night in the year.

Having gone over the catalogue in this manner, writing down the catalogue numbers of the stars, the right ascensions, declinations, and magnitudes, it will often be found that some of the pairs interfere with others in reference to time of culmination. We may, if we choose, make out two lists for observation on alternate nights, or we may drop those pairs which are less suitable when they interfere with others.

The places of the stars must then be reduced to the date of observation by applying the corrections for precession, nutation, and aberration.* The declinations need only be reduced to the mean place for the year, but the apparent right ascensions for the date of observation will be required within the nearest second. The necessary reduction may be obtained very readily by comparing the stars with those of approximately the same right ascension and declination of the Nautical Almanac.

The following is an example of an observing list prepared for determining the latitude along the northern boundary of the United States. The first column contains the number of the star in the British Association catalogue, the second column the magnitude, the third and fourth the right ascension and declination, the fifth the zenith distance. The letter N. or S. in the next column shows whether the star culminates north or south of the zenith: the stars with the large

* For a full explanation of this subject see Art. 354 and following.

declinations culminate north, those with the small declination south. The setting, given in the last column, is the mean of the zenith distances.

U. S. Northern Boundary Survey.—Astronomical Station No. 4.
Observing List for Zenith Telescope. 1873, June 27. Approx. φ 49° 0′.

B. A. C.	Mag.	α	δ	z	N. or S.	Setting.
4937	6	14ʰ 52ᵐ 12ˢ	50° 9′	1° 9′	N.	1° 00′
4974	5	14 59 38	48 9	0 51	S.	
5026	6	15 8 47	38 44	10 16	S.	10 20
5097	3	15 22 8	59 24	10 24	N.	
5271	6	15 48 19	42 48	6 12	S.	6 9.5
5313	5.5	15 54 49	55 7	6 7	N.	
5415	6	16 6 36	58 16	9 16	N.	9 7.5
5460	6	16 15 36	40 1	8 59	S.	
5502	5	16 21 41	55 30	6 30	N.	6 40
5523	5	16 24 31	42 10	6 50	S.	
5545	4.5	16 28 17	69 3	20 3	N.	20 14
5624	7	16 40 4	28 35	20 25	S.	
5644	6	16 43 18	42 28	6 32	S.	6 35
5658	6	16 44 17	55 38	6 38	N.	

As will be seen, the selection of a good list of stars involves considerable labor. Where great accuracy is required especial care should be exercised in selecting the stars, and none should be employed whose declinations are not well determined. This part of the subject will be considered more in detail hereafte.

Directions for Observing.

273. A suitable list of stars having been prepared, the instrument adjusted, and the chronometer error determined, the observer sets the vertical circle at the proper reading, the telescope is directed towards that side of the zenith

where the first star will culminate, and the bubble brought to the middle of the level-tube by means of the tangent-screw connected with the horizontal axis. At the time of culmination, as shown by the chronometer, the star is bisected by the micrometer-thread, and the micrometer and level are read; the instrument is then reversed in azimuth and the second star observed in the same way: this forms a complete observation.

During the operations described the tangent-screw of the vertical circle must not be touched, but the tangent-screw which moves the telescope, and consequently the level, may be turned after reversing, in the exceptional case where the vertical axis is not well adjusted.

If for any reason the bisection is not obtained at the instant of culmination, the star may be observed off the meridian and the time of observation recorded, when a correction may be computed to reduce it to the meridian. Several bisections might be made while the star is crossing the field, and the observations reduced to the meridian in a similar manner; but experience shows that little or nothing is gained in this way. The accuracy with which a bisection can be made by a skilled observer being greater than that of the average declinations which will be employed, it is advisable to increase the number of stars observed rather than to multiply observations on the same star under the same circumstances.

Determination of Value of Micrometer-screw.

274. This value may be determined most advantageously by meams of a circumpolar star observed near elongation. One of the four close circumpolar stars whose places are given in the American Ephemeris will generally be selected for the purpose, viz., 51 Cephei, δ, α, or λ Ursæ Minoris.

The observations are made as follows: From 15 to 30 minutes before the star reaches elongation the telescope is pointed to the star, the micrometer-thread being near that end of the screw from which the star is moving. The telescope is set at such an elevation that the thread is a little in advance of the star, and the bubble of the level brought into the middle of the tube, without disturbing the position of the telescope. The time of transit of the star over the thread is then observed and the level read. The thread is then moved forward one revolution (or sometimes only half a revolution) and the transit of the star observed in the new position, and so on throughout the entire length of the screw.

It is well to time the work so that the elongation will occur near the middle of the series, though this is not essential. With this in view it may be borne in mind that the time required for Polaris to pass over a space equal to the range of an ordinary zenith telescope micrometer will be about 50^m, for λ Ursæ Minoris 70^m, for 51 Cephei 30^m.

The record of the observations will be kept according to the following or a similar schedule:

No.	Micrometer.	Chronom. Time.	Level.	
			N.	S.

To prepare for the observation, the chronometer time of elongation must be computed. It will facilitate setting the instrument on the star if the azimuth and zenith distance are also computed.

In the triangle formed by the arcs of great circles joining the zenith, the pole, and the star, the angle at the star S will be a right angle at the time of elongation. Then by Napier's rules,

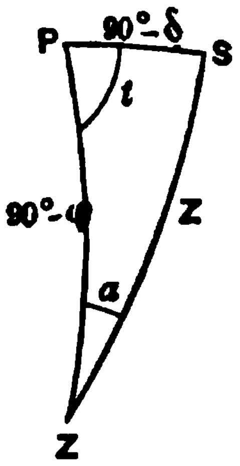

FIG. 55.

$$\left.\begin{aligned} \sin a &= \frac{\cos \delta}{\cos \varphi}; \\ \cos z &= \frac{\sin \varphi}{\sin \delta}; \\ \cos t &= \tan \varphi \cot \delta. \end{aligned}\right\} \quad . \quad . \quad . \quad (474)$$

Let $T =$ the chronometer time of elongation.

Then $$T = \alpha \pm t - \Delta T \left\{ \begin{matrix} \text{W.} \\ \text{E.} \end{matrix} \right\} \text{elongation.} \quad . \quad . \quad (474)_1$$

Method of Reduction.

275. We have by observation a series of times corresponding to observed transits of the star over the thread at successive equal distances. If now the star moved uniformly in a great circle the intervals between these observed times would be uniform, aside from errors of observation and the effect of change of level. The star, however, moves in a small circle which is tangent to the vertical circle at the point of elongation. We may, however, compute the correction necessary to convert this motion in the small circle to uniform motion in a great circle, as follows:

For any one of our observed transits let

τ = the interval of time between observation and elongation;

$z'' =$ the number of seconds of arc from elongation measured on the vertical circle $= SK$.

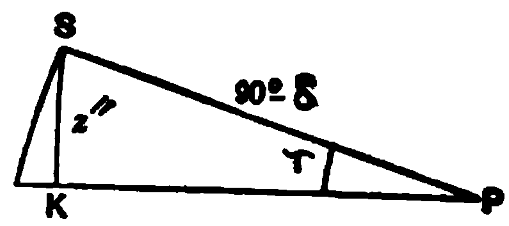

FIG. 56.

Then the angle $SPK = 15\tau$ expressed in arc, and

$$\sin z'' = \cos \delta \sin (15\tau),$$

or $$z'' = \cos \delta \frac{\sin (15\tau)}{\sin 1''}..$$

By expansion,

$$\sin(15\tau) = (15\tau)\sin 1'' - \tfrac{1}{6}(15\tau \sin 1'')^3 + \tfrac{1}{120}(15\tau \sin 1'')^5.$$

If the time of elongation falls anywhere within the series the last term is never likely to be appreciable, so we shall have with sufficient accuracy

$$s'' = 15 \cos \delta \, [\tau - \tfrac{1}{6}(15 \sin 1'')^2 \tau^3]. \quad . \quad . \quad (475)$$

In which $\log \tfrac{1}{6}(15 \sin 1'')^2 = 0.94518 - 10.$

This term may be readily computed from the formula, but the following table is more convenient, where its value is given for every minute of time from elongation to 65^m. It will seldom be advisable to extend the observations farther from elongation than this. For this interval, viz., 65^m, the term in τ^5 is $0^s.21$, and may very well be neglected, but it would soon become appreciable.

τ	Term.	τ	Term.	τ	Term.
m.	*s.*	*m.*	*s.*	*m.*	*s.*
6	0.0	26	3.3	46	18.5
7	0.1	27	3.7	47	19.7
8	0.1	28	4.2	48	21.0
9	0.1	29	4.6	49	22.3
10	0.2	30	5.1	50	23.7
11	0.2	31	5.7	51	25.2
12	0.3	32	6.2	52	26.7
13	0.4	33	6.8	53	28.3
14	0.5	34	7.5	54	29.9
15	0.6	35	8.2	55	31.6
16	0.8	36	8.9	56	33.3
17	0.9	37	9.6	57	35.1
18	1.1	38	10.4	58	37.0
19	1.3	39	11.3	59	39.0
20	1.5	40	12.2	60	41.0
21	1.8	41	13.1	61	43.1
22	2.0	42	14.1	62	45.2
23	2.3	43	15.1	63	47.4
24	2.6	44	16.2	64	49.7
25	3.0	45	17.3	65	52.1

Instead of applying this correction to τ (the difference between the time of elongation and observation) it is more convenient to apply it directly to the observed time. It will be plus before and minus after either elongation. We thus reduce the observed times to what they would have been if the star had moved uniformly in a vertical circle.

276. *Correction for Change of Level Reading.* A change in the level reading indicates a change in the angle which the line of collimation forms with the horizon. The correction necessary to apply to the observed times will be derived as follows:

Let n, s = any level reading;
n_0, s_0 = an assumed level reading to which all are to be reduced.

Then $$l = d[\tfrac{1}{2}(n - s) - \tfrac{1}{2}(n_0 - s_0)].$$

This quantity will be an increment to z'', and since it will always be very small it may be treated as a differential. To find the necessary correction to τ we differentiate equation (476):

$$\cos z''\, dz'' = \cos \delta \cos (15\tau)\, d(15\tau).$$

Writing $dz'' = l$, $\cos z'' = 1$, $\cos 15\tau = 1$, this gives

$$\delta\tau = \frac{l}{15 \cos \delta} = \pm \frac{d}{30 \cos \delta}[(n - s) - (n_0 - s_0)] \left\{ \begin{matrix} \text{W.} \\ \text{E.} \end{matrix} \right\} \text{elongation.} \qquad (476)$$

Applying this and the correction taken from the table Art. 275 to the observed times, we shall have in one column the readings of the micrometer, and in another the times reduced to what they would have been if the star had moved uniformly in vertical circle, and if no change had taken place in the position of the instrument. These may now be com-

bined by subtracting the first from the middle one, the second from the middle plus one, and so on.

If n is the number of revolutions of the micrometer between the first and middle observations, we thus have a series of values for the time required for the star to pass over this space; if all errors could be avoided, these times would consequently be the same. The mean of these values multiplied by $\frac{15 \cos \delta}{n}$, in accordance with formula (475), then gives the value of one revolution expressed in seconds of arc.

277. *Micrometer Value when Level Value is not known.* There is no more convenient or satisfactory method for determining the value of the micrometer-screw than that just explained, when the value of the level has been previously determined. This may be done by a level-trier, or by a finely-graduated circle, as already explained in Art. 164.

Circumstances sometimes make it necessary to determine the values of both micrometer and level when no special appliances are at hand for the latter. In such a case the value of the level must first be determined in terms of the micrometer, as follows:

The telescope is directed to a sharply-defined mark, as the threads of a collimating telescope, and the bubble brought near one end of the tube; the mark is carefully bisected by the thread of the micrometer, and both micrometer and level are read. The instrument is then moved through a small vertical angle so as to bring the bubble towards the other end of the tube, and the mark again bisected by the micrometer.

The difference between the two readings of the micrometer is the measure of the angle through which the instrument has been moved in terms of the micrometer, and the difference between the two level readings is the measure of the same angle in terms of the level.

Let M, M' = the two micrometer readings;
L, L' = the two level readings;
R, d = value of micrometer and level respectively.

Then $$d(L - L') = R(M - M'). \quad . \quad . \quad . \quad . \quad (477)$$

The value of both d and R may now be determined by a series of approximations, as follows: The value of R is determined by the method just explained, neglecting the level correction; then with this value of R, d is computed by (477), and the value used in a recomputation of R. This more accurate value of R gives a more accurate approximation to the value of d, and the operation may be again repeated if necessary. If the instrument is mounted on a good foundation, the change of level during the time of observation will generally be so small that a very close approximation to the true value of R is obtained by neglecting the level correction. It will seldom happen that the change will be great enough to render more than one repetition of the computation necessary.

A method theoretically more rigorous is as follows:

Let $\frac{d}{R} = \frac{M - M'}{L - L'} = D$ = the value of one division of the level expressed in terms of the micrometer;
z_0, T_0, M_0, L_0 = zenith distance, time, micrometer, and level of a circumpolar star observed at elongation;
z, T, M, L = the same quantities at time T.
$RD = d$ = value of one division of the level.

Then $$z = z_0 + (M - M_0)R - (L - L_0)RD,$$
$$z' = z_0 + (M' - M_0)R - (L' - L_0)RD,$$

for a second observation.

From these, $$R = \frac{(z' - z_0) - (z - z_0)}{(M' - M) - (L' - L)D}. \quad . \quad . \quad . \quad (477)$$

$z - z_0$, $z' - z_0$ are the same as the quantity which we have called z'' in the previous formula, and may be computed by (475). The correction $\frac{1}{8}(15 \sin 1'')^2 \tau^2$ may of course be taken from the table and applied directly to the time of observation as before. We shall then have in one column the readings of the micrometer, and in another the times reduced to the vertical circle. We combine as before by subtracting the first from the middle, the second from the middle plus one, and so on; then divide each by its value of $(M - M') - (L - L')D$. This gives the time required for the star to pass over a space equal to one revolution of the micrometer, which multiplied by $15 \cos \delta$ gives the value in seconds of arc.

We might compute $z - z_0$ directly for each observation by (475). This will involve a little more labor than the method outlined above, as each term must be multiplied by $15 \cos \delta$, while in the other case only one such multiplication is necessary.

Example.

278. Polaris was observed at eastern elongation, 1874, June 18, for determining the value of one revolution of the micrometer of zenith telescope Würdemann, No. 20.

Station: Fort Buford, Dakota. Observer: Captain J. F. Gregory.

The preliminary computation necessary to prepare for the observation is first given, viz., the computation of the azimuth, zenith distance, and time of elongation by formulæ (474).

For this purpose the right ascension and declination of *Polaris* are taken from the Nautical Almanac, viz.:

$$\alpha = 1^h\ 12^m\ 6^s.4;$$
$$\delta = 88°\ 38'\ 3''.3.$$

The latitude of station was $\varphi = 47°\ 59'\ 7''$.

The computation is as follows:

$\cos\delta = 8.37721$	$\sin\delta = 9.99988$	$\cot\delta = 8.37733$
$\cos\varphi = 9.82563$	$\sin\varphi = 9.87097$	$\tan\varphi = .04534$
$\sin a = 8.55158$	$\cos z = 9.87109$	$\cos t = 8.42267$

$$a = 2^\circ\ 2'\ 27'' \qquad z = 41^\circ\ 59'\ 50'' \qquad t = 88^\circ\ 29'\ 1''$$

$$\begin{aligned} t &= 5^h\ 53^m\ 56^s \\ \alpha &= 1\ \ 12\ \ 06 \\ \alpha - t &= 19\ \ 18\ \ 10 \\ \Delta T &= \quad\ \ -\ 2 \end{aligned}$$

$$\text{Chronometer time of elongation} = \alpha - t - \Delta T = 19^h\ 18^m\ 12^s$$

The transit of Polaris was observed over the micrometer-thread at every half turn, beginning with revolution 35 and ending with 5.5—sixty transits in all. In the example I have only used those observed at the even revolutions, as this will be sufficient for illustrating the method of reduction.

No.	Micrometer Reading.	Chronometer Time.	Level. N.	Level. S.	Time from Elongation.	Reduction to Vertical.	Reduction to Mean State of Level.	Correction for Level.	Reduced Times.
1	35	$18^h\ 38^m\ 40^s.0$	18.6	19.1	$-\ 39^m\ 32^s.0$	$+\ 11^s.8$	− .5	+ $0^s.6$	$18^h\ 38^m\ 52^s.4$
2	34	41 38 .0	18.5	19.1	36 34 .0	9 .3	− .6	+ .7	41 48 .0
3	33	44 32 .8	18.6	19.2	33 39 .2	7 .3	− .6	+ .7	44 40 .8
4	32	47 27 .6	18.7	19.2	30 44 .4	5 .5	− .5	+ .6	47 33 .7
5	31	50 24 .0	19.0	19.0	27 48 .0	4 .1	.0	.0	50 28 .1
6	30	53 20 .6			24 51 .4	2 .9	.0	.0	53 23 .5
7	29	56 13 .7	19.0	19.1	21 58 .3	2 .0	− .1	+ .1	56 15 .8
8	28	18 59 10 .0			19 2 .0	1 .3	− .1	+ .1	18 59 11 .4
9	27	19 2 4 .4	19.0	19.2	16 7 .6	.8	− .2	+ .2	19 2 5 .4
10	26	5 0 .0	19.5	19.1	13 12 .0	.4	+ .4	− .5	4 59 .9
11	25	7 52 .3	19.2	19.2	10 19 .7	.2	.0	.0	7 52 .5
12	24	10 49 .0	19.6	19.3	7 23 .0	+ .1	+ .3	− .4	10 48 .7
13	23	13 41 .9			4 30 .1	.0	+ .3	− .4	13 41 .5
14	22	16 35 .0	19.7	19.5	− 1 37 .0	.0	+ .2	− .2	16 34 .8
15	21	19 29 .0	19.6	19.5	+ 1 17 .0	.0	+ .1	− .1	19 28 .9
16	20	22 21 .9	20.0	19.4	4 9 .9	.0	+ .6	− .7	22 21. 2
17	19	25 16 .3	20.2	19.3	7 4 .3	− .1	+ .9	− 1 .1	25 15 .1
18	18	28 10 .6	20.3	19.4	9 58 .6	− .2	+ .9	− 1 .1	28 9 .3
19	17	31 3 .9	20.5	19.5	12 51 .9	.4	+ 1.0	− 1 .2	31 2 .3
20	16	33 59 .0			15 47 .0	.8	+ 1.0	− 1 .2	33 57 .0
21	15	36 52 .6	20.6	19.5	18 40 .6	1 .2	+ 1.1	− 1 .4	36 50 .0
22	14	39 46 .0			21 34 .0	1 .9	+ 1.1	− 1 .4	39 42 .7
23	13	42 40 .0			24 28 .0	2 .8	+ 1.1	− 1 .4	42 35 .8
24	12	45 35 .4	20.7	19.5	27 23 .4	3 .9	+ 1.2	− 1 .5	45 30 .0
25	11	48 29 .0	21.0	19.3	30 17 .0	5 .2	+ 1.7	− 2 .1	48 21 .7
26	10	51 25 .0	21.0	19.5	33 13 .0	6 .9	+ 1.5	− 1 .9	51 16 .2
27	9	54 19 .7	21.1	19.4	36 7 .7	9 .0	+ 1.7	− 2 .1	54 8 .6
28	8	19 57 14 7	21.0	19.6	39 2 .7	11 .3	+ 1.4	− 1 .7	57 1 .7
29	7	20 0 13 .6	21.0	19.7	42 1 .6	14 .1	+ 1.3	− 1 .6	19 59 57 .9
30	6	20 3 8 .6	21.0	19.8	+ 44 56 .6	− 17 .2	+ 1.2	− 1 .5	20 2 49 .9

The first five columns require no explanation. The sixth contains the quantities which we have called r. The "reduction to vertical" is taken from the table Art. 275. The "reduction to mean state of level" is $(n - s) - (n_0 - s_0)$, where $(n_0 - s_0) = 0$ in this case. The "correction for level" is this quantity multiplied by $\frac{d}{30 \cos \delta}$. The value of one division of the level, $d = ''.893$. Therefore this factor equals 1.25.

The elongation being *east*, the sign of the level reduction is *minus*.

The "reduction to vertical" and "correction for level" being applied to the observed time, we have the "reduced times" of the last column. We combine these quantities by subtracting No. 1 from 16, No. 2 from 17, . . . No. 15 from 30, thus obtaining a series of values for the time required for the star to pass over a space equal to 15 revolutions of the screw. The mean of these quantities multiplied by $\frac{15 \cos \delta}{15} = \cos \delta$ then will give the value of one revolution in seconds of arc.

The numerical work is as follows:

Nos.	Time of 15 Revolutions.	v.	vv.
16 — 1	$43^m\ 28^s.8$	3.9	15.21
17 — 2	43 27 .1	2.2	4.84
18 — 3	43 28 .5	3.6	12.96
19 — 4	43 28 .6	3.7	13.69
20 — 5	43 28 .9	4.0	16.00
21 — 6	43 26 .5	1.6	2.56
22 — 7	43 26 .9	2.0	4.00
23 — 8	43 24 .4	.5	.25
24 — 9	43 24 .6	.3	.09
25 — 10	43 21 .8	3.1	9.61
26 — 11	43 23 .7	1.2	1.44
27 — 12	43 19 .9	5.0	25.00
28 — 13	43 20 .2	4.7	22.09
29 — 14	43 23 .1	1.8	3.24
30 — 15	43 21 .0	3.9	15.21

$[vv] = 146^s.19$

Mean $43^m\ 24^s.93$

$= 2604^s.93$ log $= 3.4157961$

$\cos \delta = 8.3772074$

log one revolution $= 1.7930035$

One revolution $62''.0874$

Correction for refraction $-.0315$

Corrected value $62''.056$

The correction for differential refraction is computed by the last of formulæ (481), viz.,

$$r - r' = [6.44676] \sec^2 z(z - z') \qquad 6.4468$$
$$\log (z - z') = 1.7930$$
$$\sec^2 z = \ .2578$$
$$\log (r - r') = 8.4976 \qquad r - r' = ''.0315$$

The probable error is computed from the sum of the squares of the residuals in the last column by formula (27), viz.,

$$r_0 = .6745 \sqrt{\frac{[vv]}{m(m-1)}}.$$

m in this case being 15. Substituting in this formula, we find

$$r_0 = {}^s.563.$$

This is now the probable error of the determination of the time required for the star to pass over 15 revolutions of the screw. The probable error of the above determination of the value of one revolution of the screw will be obtained from this quantity by multiplying by the factor $\frac{15 \cos \delta}{15} = \cos \delta$, viz., $\pm$ ''.013.

From this series we therefore conclude the most probable value of one revolution of the screw to be

$$R = 62''.056 \pm ''.013.$$

Value of One Division of Level.

279. An example has been given (Art. 164) of the determination of the level value by means of the level-trier. Opposite is given an example of the determination of the level value of the above instrument by means of the micrometer. See equation (477).

1873, June 15. Observer, L. Boss. Mark cross-threads of transit telescope.

No.	Micrometer 1st position.	Micrometer 2d position.	Level 1st position.		Level 2d position.		Mean Change of Level.	Mean Change of Microm.	$\frac{d}{R'}$	v.	vv.
			N.	S.	N.	S.					
1	21.036	21.542	13.5	44.9	49.1	9.3	35.6	50.6	1.421	.019	.000361
2	21.538	22.131	8.2	49.7	51.8	5.9	43.7	59.3	1.357	.083	6889
3	27.097	27.650	7.6	49.8	44.7	12.4	37.25	55.3	1.485	.045	2025
4	26.752	27.387	5.1	51.9	49.2	7.1	45.45	63.5	1.429	.011	121
5	19.825	20.386	6.9	48.6	43.9	11.4	37.1	56.1	1.512	.072	5184
6	20.361	20.889	6.3	49.0	44.5	10.8	38.2	52.8	1.382	.058	3364
7	20.852	21.445	5.3	49.9	48.1	6.8	42.95	59.3	1.381	.059	3481
8	21.438	21.986	9.5	45.5	48.2	6.5	38.85	54.8	1.411	.029	841
9	21.992	22.555	5.7	48.9	44.1	10.3	38.5	56.3	1.462	.022	484
10	22.548	23.058	7.7	46.6	42.8	11.5	35.1	51.0	1.453	.013	169
11	14.532	13.910	49.3	5.1	6.1	48.2	43.15	62.2	1.441	.001	1
12	13.903	13.413	43.2	10.7	10.2	43.5	32.9	49.0	1.489	.049	2401
13	13.415	12.828	48.1	5.6	7.6	46.0	40.45	58.7	1.451	.011	121
14	17.146	17.670	8.1	45.5	44.3	8.9	36.4	52.4	1.440	.000	0
15	24.822	25.310	7.1	45.9	40.0	12.9	32.95	48.8	1.481	.041	1681
16	25.944	26.537	5.2	47.5	-46.3	6.1	41.25	59.3	1.438	.002	4

$[vv] = .027127$

$$\text{Mean value of } \frac{d}{R'} = 1.4396 \pm .0071.$$

The above value of R' is $''.62056$.

Therefore $$d = {}''.893 \pm .004.$$

If both the level and micrometer values were unknown, the above series of observations of *Polaris* would give for one division of the micrometer, by neglecting the level readings, $R' = {}''.6209$, which gives practically the same value of d as above.

With this value of d the level corrections would then be computed and the final value of the micrometer determined, no second approximation to the value of d being required.

280. For the purpose of illustrating the method of Art. 277 let us apply it to the example already solved. The first part of the computation will be precisely the same as before except the correction for level. Applying to the observed chronometer times the "reduction to vertical" already found, we have the "reduced times" of the following table:

No.	Micrometer.	Reduced Times.	Level.		Nos.	$L' - L$.	$(L' - L)D$.	$(M' - M) + (L' - L)D$.*	Times.	Time of one Revolution.	v.	vv.
			N.	S.								
1	35	$18^h 38^m 51^s.8$	18.6	19.1	16— 1	+ .55	.0079	15.0079	$2610^s.1$	$173^s.92$	25	625
2	34	18 41 47 .3	18.5	19.1	17— 2	+ .75	.0108	15.0108	2608 .9	173 .80	13	169
3	33	44 40 .1	18.6	19.2	18— 3	+ .75	.0108	15.0108	2610 .3	173 .90	23	529
4	32	47 33 .1	18.7	19.2	19— 4	+ .75	.0108	15.0108	2610 .4	173 .90	23	529
5	31	50 28 .1	19.0	19.0	20— 5	+ .50	.0072	15.0072	2610 .1	173 .92	25	625
6	30	53 23 .5			21— 6	+ .55	.0079	15.0079	2607 .9	173 .77	10	100
7	29	56 15 .7	19.0	19.1	22— 7	+ .60	.0086	15.0086	2608 .4	173 .88	21	441
8	28	18 59 11 .3			23— 8	+ .60	.0086	15.0086	2605 .9	173 .63	4	16
9	27	19 2 5 .2	19.0	19.2	24— 9	+ .70	.0101	15.0101	2606 .3	173 .64	3	9
10	26	5 0 .4	19.5	19.1	25—10	+ .65	.0094	15.0094	2603 .4	173 .45	23	529
11	25	7 52 .5	19.2	19.2	26—11	+ .75	.0108	15.0108	2605 .6	173 .58	9	81
12	24	10 49 .1	19.6	19.3	27—12	+ .70	.0101	15.0101	2601 .6	173 .32	35	1225
13	23	13 41 .9			28—13	+ .55	.0079	15.0079	2601 .5	173 .34	33	1089
14	22	16 35 .0	19.7	19.5	29—14	+ .55	.0079	15.0079	2604 .5	173 .54	13	169
15	21	19 29 .0	19.6	19.5	30—15	+ .55	.0079	15.0079	2602 .4	173 .40	27	729
16	20	22 21 .9	20.0	19.4								
17	19	25 16 .2	20.2	19.3								
18	18	28 10 .4	20.3	19.4								
19	17	31 3 .5	20.5	19.5								
20	16	33 58 .2										
21	15	36 51 .4	20.6	19.5								
22	14	39 44 .1										
23	13	42 37 .2										
24	12	45 31 .5	20.7	19.5								
25	11	48 23 .8	21.0	19.3								
26	10	51 18 .1	21.0	19.5								
27	9	54 10 .7	21.1	19.4								
28	8	57 3 .4	21.0	19.6								
29	7	19 59 59 .5	21.0	19.7								
30	6	20 2 51 .4	21.0	19.8								

$6865 = [vv]$.

Mean time of one revolution $= 173^s.666 \pm .0385$.

The value of one revolution is now found by multiplying this time by $15 \cos \delta$, viz.,

$$R = 62''.0888 \pm .0138.$$
$$\text{Refraction} = \quad .0315.$$
$$\text{Final value of } R = 62''.057 \pm .014.$$

If the chronometer employed has an appreciable rate the interval of time corresponding to one revolution of the screw will require a correction which may be determined as follows:

Let δT = the daily rate of the micrometer, + when losing;
I_1 = any interval expressed in terms of chronometer;
I = true value of interval.

Then $$I : I_1 = 24^h : 24^h - \delta T = 86400^s : 86400^s - \delta T;$$

$$I = I_1 \frac{1}{1 - \dfrac{\delta T}{86400}} = I_1 + I_1 \frac{\delta T}{86400} \text{ nearly.}$$

If, for example, the above observations had been made with a mean time chronometer, for δT we should have $3^m 56^s = 236^s$. Therefore

$$I = I_1 + I_1 \frac{236^s}{86400} = I_1 + I .002735 = 173^s.666 + .474 = 174^s.140.$$

* When the reduction is made in this manner the term $(L' - L)D$ will be $\pm$ for $\left\{ \begin{matrix} \text{E.} \\ \text{W.} \end{matrix} \right\}$ elongation.

General Formulæ for the Latitude.

281. Let m = the micrometer reading for the south star, expressed in seconds of arc;

m_0* = the micrometer reading for the zero-point of the micrometer;

l = the correction for level, plus when the north reading is large;

r = the correction for refraction.

Then $z = z_0 + (m - m_0) + l + r$ for south star.
Similarly $z' = z_0 + (m' - m_0) - l' + r'$ for north star.

$$z - z' = (m - m') + (l + l') + (r - r').$$

Substituting this value in equation (472),

$$\varphi = \tfrac{1}{2}(\delta + \delta') + \tfrac{1}{2}(m - m') + \tfrac{1}{2}(l + l') + \tfrac{1}{2}(r - r'). \quad (478)$$

It has been assumed in the foregoing that the readings of the micrometer increase with the zenith distance; but, whether they increase or diminish, practically a case will very seldom occur where the algebraic sign of the term $\frac{1}{2}(m - m')$ will be in doubt, as may be seen by referring to the numerical example.

Equation (478) shows that the value of the latitude is found by adding to the mean of the declinations of the two stars three corrections: *first*, the correction for micrometer; *second*, the correction for level; *third*, for refraction.

---

* Any point may be assumed arbitrarily as the zero-point, for by referring to equations (478) and (479) it will be seen that only the difference of micrometer readings on the two stars is required, and this will be the same wherever we assume the zero to be. It will be convenient to assume this point so far to one end of the scale that the readings will all be plus.

282. *The Correction for Micrometer.*

Let M and M' = the micrometer readings for the south and north stars respectively;
R = the value of one revolution of the screw expressed in seconds of arc.

Then
$$\tfrac{1}{2}(m - m') = \tfrac{1}{2}R(M - M'). \quad . \quad . \quad . \quad . \quad (479)$$

If the micrometer reads towards the zenith the algebraic sign will simply be reversed.

283. *The Correction for Level.* If the mean of the north readings in both positions of the instrument is greater than the mean of the south readings, it shows that the vertical axis produced pierces the celestial sphere south of the zenith; therefore the instrumental zenith distance of a south star is too small, and of a north star too large.

Let n and s = readings of north and south ends of bubble for south star;
n' and s' = readings of north and south ends of bubble for north star;
x = the error of the level;
d = the value of one division of the level in seconds of arc.

Then
$$l = \tfrac{1}{2}d(n - s) + x;$$
$$l' = \tfrac{1}{2}d(n' - s') - x;$$
$$\tfrac{1}{2}(l + l') = \tfrac{1}{4}d[(n + n') - (s + s')]. \quad . \quad . \quad (480)$$

284. *Correction for Refraction.* The difference of zenith distance is so small that nothing is gained by applying to the correction for refraction the terms depending on the barometer and thermometer.

Bessel's formula for mean refraction is

$$r = \alpha \tan z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

α for present purposes is considered constant and equal to $57''.7$.

The correction $r - r'$ being very small, we may use a differential formula, viz.,

$$r - r' = \frac{dr}{dz}(z - z'); \quad . \quad . \quad . \quad . \quad . \quad (b)$$

and from (a), $$\frac{dr}{dz} = 57''.7 \sec^2 z.$$

If $z - z'$ is given in minutes we may write (b) as follows:

$$\left.\begin{array}{ll} & r - r' = 57''.7 \sec^2 z . \sin 1' . (z - z'), \\ \text{or} & r - r' = [8.22491] \sec^2 z . (z - z'). \\ \multicolumn{2}{l}{\text{If } (z - z') \text{ is expressed in seconds,}} \\ & (r - r') = [6.44676] \sec^2 z . (z - z'). \end{array}\right\} \quad . \quad . \quad (481)$$

As usual the numerical quantities in brackets are logarithms.

The computation by either of these formulæ is quite simple, but as this correction must be applied to every pair of stars observed the following table has been added, being the same as that given by Schott, of the U. S. Coast Survey. The vertical argument is one half the difference of zenith distance, for which we may use $\frac{1}{2}(m - m')$. The horizontal argument is the zenith distance, the table being extended to 35°. In the exceptional cases where stars are observed at greater zenith distances the correction must be computed by the formula (481). The algebraic sign will always be the same as that of the micrometer correction.

TABLE B—DIFFERENTIAL REFRACTION.

Half Difference in Zenith Distance.	Zenith Distance.					
	0°	10°	20°	25°	30°	35°
′	″	″	″	″	″	″
0	.00	.00	.00	.00	.00	.00
0.5	.01	.01	.01	.01	.01	.01
1	.02	.02	.02	.02	.02	.02
1.5	.03	.03	.03	.03	.03	.03
2	.03	.03	.04	.04	.04	.05
2.5	.04	.04	.05	.05	.05	.06
3	.05	.05	.06	.06	.07	.08
3.5	.06	.06	.07	.07	.08	.09
4	.07	.07	.08	.08	.09	.10
4.5	.08	.08	.09	.09	.10	.11
5	.08	.09	.10	.10	.11	.13
5.5	.09	.10	.10	.11	.12	.14
6	.10	.10	.11	.12	.13	.15
6.5	.11	.11	.12	.13	.14	.16
7	.12	.12	.13	.14	.15	.18
7.5	.13	.13	.14	.15	.16	.19
8	.13	.14	.15	.16	.18	.21
8.5	.14	.15	.16	.17	.19	.22
9	.15	.16	.17	.18	.20	.23
9.5	.16	.17	.18	.20	.21	.24
10	.17	.18	.19	.21	.23	.26
10.5	.18	.19	.20	.22	.24	.27
11	.18	.19	.21	.23	.25	.28
11.5	.19	.20	.22	.24	.26	.30
12	.20	.21	.23	.25	.27	.31
12.5	.21	.21	.24	.26	.28	.32

285. *Reduction to the Meridian.* If the observation has been missed at the instant of the star's meridian passage, it may be observed off the meridian in either of two ways:

First. The instrument may be revolved in azimuth so as to bisect the star in the middle of the field; or

Second. The instrument may be allowed to remain in the meridian, and the star may be bisected off the line of collimation before it passes out of the field.

In the first case the correction to the zenith distance will be precisely the same as that already derived for reducing

circummeridian altitudes, viz.—see equations (XIII), Art. 149—

$$\frac{\cos\varphi\cos\delta}{\sin z}\,\frac{2\sin^2\frac{1}{2}t}{\sin 1''};$$

where t is the hour-angle of the star at the instant of observation.

The quantity given by this formula is to be subtracted from the zenith distance at the instant of observation; therefore by referring to (472) we see that the correction to the latitude will be

$$\varDelta\varphi = \pm\frac{1}{2}\,\frac{\cos\varphi\cos\delta}{\sin z}\,\frac{2\sin^2\frac{1}{2}t}{\sin 1''}. \quad . \quad . \quad . \quad (482)$$

$\varDelta\varphi$ will be plus for a north and minus for a south star. $\dfrac{2\sin^2\frac{1}{2}t}{\sin 1''}$ is taken from table VIII A at the end of this volume.

286. *When the star is observed off the line of collimation, the instrument remaining in the meridian.* In the figure, PK is the meridian, PS the hour-circle passing through the star. If the star is observed on the meridian, SK will be the position of the micrometer-thread. If observed off the meridian at S', this thread will have the position $S'K'$.

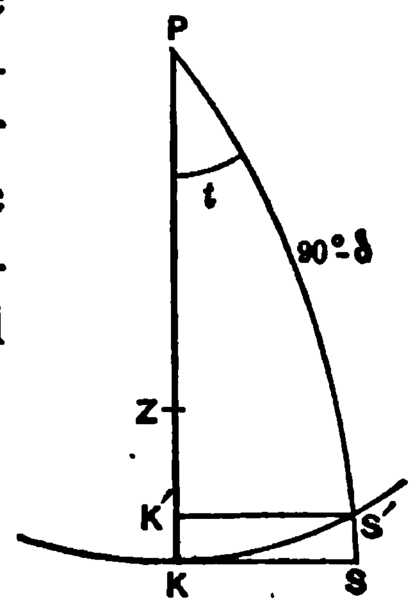

FIG. 57.

Let $KK' = x$.

Then $$pK' = 90° - (\delta + x),$$

and, by Napier's second rule,

$$\cos t = \tan\delta\cot(\delta + x).$$

This may be placed in the form

$$\tan \delta = (1 - 2 \sin^2 \tfrac{1}{2}t) \frac{\tan \delta + \tan x}{1 - \tan \delta \tan x}.$$

Clearing of fractions and neglecting the small term $\tan x . 2 \sin^2 \frac{1}{2}t$, we readily find

$$\tan x = \sin \delta \cos \delta \; 2 \sin^2 \tfrac{1}{2}t,$$

or, with sufficient accuracy,

$$x = \sin 2\delta \frac{\sin^2 \frac{1}{2}t}{\sin 1''} \quad . \quad . \quad . \quad . \quad . \quad . \quad (483),$$

As the apparent zenith distance is diminished for a south star and increased for a north star when observed in this manner, the correction to the latitude will always be plus and will be equal to $\frac{1}{2}x$. That is,

$$\Delta\varphi = \tfrac{1}{2} \sin 2\delta \frac{\sin^2 \frac{1}{2}t}{\sin 1''} \quad . \quad . \quad . \quad . \quad . \quad . \quad (483)$$

This method of proceeding will generally be preferred when the observation on the meridian is lost, as when the other method is used the stop must be unclamped, and where other stars follow in quick succession a pair may be lost in consequence. If the star cannot be observed before it gets beyond the field of view, the observer will generally prefer to let it go altogether.

The computation of $\Delta\varphi$ by the above formula is very simple, but a table is added from which the value of $x = 2\Delta\varphi$ may be taken at once. The horizontal argument is the hour-angle of the star, and the vertical argument the declination.

TABLE C—REDUCTION TO MERIDIAN.

	10s.	15s.	20s.	25s.	30s.	35s.	40s.	45s.	50s.	55s.	60s.	
δ	"	"	"	"	"	"	"	"	"	"	"	δ
5°	.00	.01	.02	.03	.04	.06	.08	.10	.12	.14	.17	85°
10	.01	.02	.04	.06	.08	.11	.15	.19	.23	.28	.34	80
15	.01	.03	.05	.09	.12	.17	.22	.28	.34	.41	.49	75
20	.02	.04	.07	.11	.16	.22	.28	.36	.44	.53	.63	70
25	.02	.05	.08	.13	.19	.26	.34	.42	.52	.63	.75	65
30	.02	.05	.09	.15	.21	.29	.38	.48	.59	.71	.85	60
35	.03	.06	.10	.16	.23	.31	.41	.53	.64	.77	.92	55
40	.03	.06	.11	.17	.24	.33	.43	.54	.67	.81	.97	50
45	.03	.06	.11	.17	.25	.33	.44	.55	.68	.82	.98	45

287. *Formulæ for Computation of Latitude from Observations with the Zenith Telescope.*

$$\left.\begin{aligned}
\varphi &= \tfrac{1}{2}(\delta+\delta')+\tfrac{1}{2}(m-m')+\tfrac{1}{2}(l+l')+\tfrac{1}{2}(r-r'); \\
\tfrac{1}{2}(m-m') &= \tfrac{1}{2}R(M-M'); \\
\tfrac{1}{2}(l+l') &= \frac{d}{4}[(n+n')-(s+s')]; \\
{}^*\tfrac{1}{2}(r-r') &= [8.22491]\sec^2 z\,\tfrac{1}{2}(z-z'). \\
&\text{Reduction to Meridian.} \\
\Delta\varphi &= \pm\frac{1}{2}\frac{\cos\varphi\cos\delta}{\sin z}\frac{2\sin^2\frac{1}{2}t}{\sin 1''}\cdot\left\{\begin{matrix}\text{N.}\\ \text{S.}\end{matrix}\right\}\text{star}; \\
\dagger\,\Delta\varphi &= +\frac{1}{4}\sin 2\delta\,\frac{2\sin^2\frac{1}{2}t}{\sin 1''}.
\end{aligned}\right\}\text{(XXIII)}$$

Combination of the Individual Values of the Latitude.

288. For many purposes a sufficient degree of accuracy will be given by simply taking the arithmetical mean of the individual values, giving all equal weight.

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* See table, p. 504. † See table above.

When a more rigorous procedure is demanded we must consider the weights of the separate values. This weight depends on the probable errors of the declinations of the stars observed, and on the probable error of observation.

Let p = the number of separate pairs employed in determining a latitude;

$n_1, n_2, n_3, \dots n_p$ = the number of observations on each pair respectively;

$n = n_1 + n_2 + \dots + n_p$ = the whole number of observations;

e = the probable error of a single observation.

Then, from (35),

$$(n_1 - 1)ee = (.6745)^2[v_1v_1];$$
$$(n_2 - 1)ee = (.6745)^2[v_2v_2];$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(n_p - 1)ee = (.6745)^2[v_pv_p].$$

The sum of these equations gives

$$(n - p)ee = (.6745)^2[vv];$$

therefore

$$e = .6745\sqrt{\frac{[vv]}{n-p}}. \quad . \quad . \quad . \quad . \quad . \quad (484)$$

$[v_1v_1]$ is the sum of the squares of the residuals formed by taking the differences between the mean of the observations on the first pair and each individual value; and similarly for $[v_2v_2], \dots [v_pv_p]$,

$$[vv] = [v_1v_1] + [v_2v_2] + \dots + [v_pv_p].$$

The determination of the probable errors of the declinations is a much more complicated problem. For a discussion of this subject the reader will refer to Articles 346 and 347.

In order to obtain the expression for the weight of the value of φ derived from a single pair,

Let ε_δ, $\varepsilon_{\delta'}$ = the probable errors of the declinations;

$$E_\delta{}^* = \tfrac{1}{2}\sqrt{\varepsilon_\delta{}^2 + \varepsilon_{\delta'}{}^2}.$$

Then if n, is the number of observations on this pair the probable error of the mean will be $\sqrt{\dfrac{e^2}{n}}$,

and $$E_\phi{}^* = \sqrt{E_\delta + \frac{e^2}{n}};$$

E_ϕ being the probable error of the resulting latitude.

The relative weights are proportional to the reciprocals of the squares of the probable errors; or, since the unit of weight is arbitrary, we may write

$$P = \frac{1}{4E_\phi{}^2} = \frac{1}{\varepsilon_\delta{}^2 + \varepsilon_{\delta'}{}^2 + \dfrac{4e^2}{n}}. \quad . \quad . \quad . \quad (485)$$

Value of Micrometer from the Latitude Observations.

289. If no special observations have been made for determining the value of the micrometer-screw, it may be derived from the latitude observations themselves.

---

* Equation (29).

Let $R =$ an assumed value of one revolution as near the true value as possible;

$\Delta R =$ the correction required.

Then $R + \Delta R =$ the true value of one revolution;

$\varphi' =$ the latitude computed with the assumed value of R from all of the observations;

$\varphi' + \Delta\varphi =$ true value of the latitude.

Then from (484),

$$\varphi' + \Delta\varphi = \tfrac{1}{2}(\delta + \delta') + \tfrac{1}{2}(R + \Delta R)(M - M') + \tfrac{1}{2}(l + l') + \tfrac{1}{2}(r - r').$$

Let $n =$ the sum of the known quantities of this equation;

that is, $n = \varphi' - \tfrac{1}{2}(\delta + \delta') - \tfrac{1}{2}R(M - M') - \tfrac{1}{2}(l + l') - \tfrac{1}{2}(r - r')$.

Then
$$\Delta\varphi - \tfrac{1}{2}(M - M')\Delta R = n. \quad . \quad . \quad . \quad . \quad (486)$$

Each pair of stars observed will give an equation of this form for determining $\Delta\varphi$ and ΔR.

This process is sometimes employed when there is reason to suspect that the adopted value of R is erroneous; but if the value has been carefully determined by the transits of circumpolar stars the result will generally be accepted as absolute.

290. The example which follows is taken from the report of the U. S. Northern Boundary Survey. The station is 47 miles west of Pembina, the approximate position being

Latitude 49° 00′, Longitude $1^h\ 24^m\ 52^s$ west of Washington.

Twenty-nine pairs of stars were observed from two to five times each, in all 81 observations.

The form in which the example is given will be found a convenient one for the record and preliminary reduction. For this purpose a book will be required with a page of about 7 inches in width. It will be ruled or printed in blank form as shown.

Example.

Astronomical Station No. 4.—West side of Pembina Mountain.
Observer, Lewis Boss.
Zenith Telescope, Würdemann No. 20. Chronometer, Negus Sidereal No. 1513.

Date.	Star. B. A. C.	N. or S.	Micrometer Reading.	$M - M'$.	Level. N.	Level. S.	N − S.	Meridian Distance.	δ.
1873. June 27.	4937	N.	28.191		30.9	25.7			50° 9′ 0″.77
	4974	S.	10.209	− 17.982	39.2	19.0	+ 25.4		48 9 4 .70
	5026	S.	18.927		31.0	27.2			38 44 32 .88
	5097	N.	28.265	− 9.338	29.2	32.0	+ 1.0		59 24 48 .13
	5271	S.	21.628		27.0	28.7			42 48 31 .36
	5313	N.	17.220	+ 4.408	28.3	27.1	− 0.5		55 6 37 .48
	5415	N.	27.762		29.5	25.6			58 16 13 .36
	5460	S.	11.010	− 16.752	27.0	28.3	+ 2.6		40 0 50 .34
	5502	N.	9.401		32.4	22.3			55 29 43 .30
	5523	S.	29.009	+ 19.608	21.5	34.0	− 2.4		42 9 45 .25
	5853	N.	25.158		31.0	26 2			49 49 41 .09
	5911	S.	13.555	− 11.603	24.3	33.2	− 4.1		48 22 1 .81
	6047	N.	26.368		34.0	23.4			72 12 35 .57
	6073	S.	9.814	− 16.554	22.0	35.4	− 2.8		26 4 15 .06
	6114	N.	12.071		28.5	28.3		59°	76 58 38 .19
	6157	S.	25.001	+ 12.930	27.1	30.4	− 3.1		20 47 41 .84
	6268	S.	14.417		26.9	31.3			39 26 16 .95
	6289	N.	24.251	− 9.834	30.5	27.4	− 1.3	16	58 43 35 .54
June 29.	5271	S.	20.694		25.5	30.9			42 48 32 .48
	5313	N.	16.306	+ 4.338	32.1	24.8	+ 1.9		55 6 37 .90
	5415	N.	27.413		30.5	26.3			58 16 13 .78
	5460	S.	10.648	− 16.765	27.8	29.6	+ 2.4		40 0 50 .83
	5502	N.	9.152		29.6	28.0			55 29 43 .76
	5523	S.	28.712	+ 19.560	27.8	30.2	− 0.8		42 9 45 .68

Left-hand page.

$(\delta + \delta')$ and $\frac{1}{2}(\delta + \delta')$.	CORRECTIONS.				Latitude.	Remarks.
	Micrometer.	Level.	Refraction.	Meridian.		
98° 18′ 5″.47 49 9 2 .74	− 9′ 17″.94	+5.69	−.16		48° 59′ 50″.33	
98 9 21 .01 49 4 40 .50	− 4 49 .73	+ .22	−.08		50 .91	
97 55 8 .84 48 57 34 .42	+ 2 16 .77	− .11	+.04		51 .12	
98 17 3 .70 49 8 31 .85	− 8 39 .77	+ .58	−.15		52 .51	
97 39 28 .55 48 49 44 .28	+10 8 .40	− .53	+.18		52 .33	
98 11 42 .90 49 5 51 .45	− 6 0 .02	− .91	−.10		50 .42	
98 16 50 .63 49 8 25 .32	− 8 33 .62	− .62	−.16		50 .92	
97 46 20 .03 48 53 10 .02	+ 6 41 .19	− .69	+.14	+.21	50 .87	Reduction to meridian taken from table C, Art. 286.
98 9 52 .49 49 4 56 .25	− 5 5 .13	− .29	−.09	+.03	50 .77	
97 55 10 .38 48 57 35 .19	+ 2 16 .15	+ .42	+.04		51 .80	
98 17 4 .61 49 8 32 .31	− 8 40 .17	+ .53	−.15		52 .52	
97 39 29 .44 48 49 44 .72	+10 6 .90	− .18	+.18		51 .62	

Right-hand page.

The above probably requires no further explanation than a reference to formulæ (XXIII), Art. 287.

The values of the micrometer-screw and level which we have employed are those derived in Articles 278 and 279, viz.,

$$R = 62''.056;$$
$$d = 0\ .893.$$

This will be sufficient for illustrating the method of reduction. In order to illustrate the combination of the individual values to determine the most probable value, the weights and probable errors, the results of the entire 81 observations will be employed. They are as follows:

Star. B. A. C.	Date. June.	Latitude. 48° 59′	Mean. 48° 59′	*v.*	*vv.*
4937 4974	26	50″.87		27	729
	27	50 .33	50″.60	27	729
5026 5097	26	51 .59		34	1156
	27	50 .91	51 .25	34	1156
5271 5313	25	51 .42		06	36
	26	51 .10		26	676
	27	51 .12		24	576
	29	51 .80	51 .36	44	1936
5415 5460	25	51 .58		48	2304
	26	51 .63		43	1849
	27	52 .51		45	2025
	29	52 .52	52 .06	46	2116
5502 5523	25	52 .56		47	2209
	26	51 .84		25	625
	27	52 .33		24	576
	29	51 .62	52 .09	47	2209
5545 5624	25	50 .95		32	1024
	26	51 .68		41	1681
	29	51 .18	51 .27	9	81
5644 5658	26	50 .78		44	1936
	29	51 .66	51 .22	44	1936
5693 5823	25	52 .44		46	2116
	26	52 .44		46	2116
	29	51 .07	51 .98	91	8281
5853 5911	25	51 .11		24	576
	26	51 .18		31	961
	27	50 .42		45	2025
	29	50 .77	50 .87	10	100

Star. B. A. C.	Date. June.	Latitude. 48° 59′	Mean. 48° 59′	*v.*	*vv.*
6047 6073	25	51″.33		2	4
	26	51 .53		22	484
	27	50 .92		39	1521
	30	51 .46	51″.31	15	225
6114 6157	25	51 .16		52	2704
	26	52 .44		76	5776
	27	50 .87		81	6561
	29	51 .64		4	16
	30	52 .31	51 .68	63	3969
6268 6289	25	50 .36		46	2116
	26	50 .53		29	841
	27	50 .77		5	25
	29	50 .72		10	100
	30	51 .71	50 .82	89	7921
6318 6365	26	51 .57		17	289
	30	51 .24	51 .40	16	256
6421 6476	25	52 .98		84	7056
	26	51 .89		25	625
	29	51 .86		28	784
	30	51 .85	52 .14	29	841
6553 6586	25	52 .01		26	676
	26	51 .51		24	576
	30	51 .74	51 .75	1	1
6624 6681	25	51 .13		25	625
	26	51 .28		10	100
	30	51 .73	51 .38	35	1225
6728 6748	25	51 .56		10	100
	26	51 .75	51 .66	9	81

Star. B. A. C.	Date. June.	Latitude. 48° 59′	Mean. 48° 59′	v.	vv.
6780 6817	25	51″.96		43	1849
	26	51 .19		34	1156
	30	51 .44	51″.53	9	81
6937 6970	26	51 .50		29	841
	30	50 .93	51 .21	28	784
7024 7073	26	52 .23		61	3721
	30	51 .02	51 .62	60	3600
7100 7166	26	51 .31		13	169
	30	51 .57	51 .44	13	169
7215 7277	26	53 .27		99	9801
	30	51 .29	52 .28	99	9801
7320	26	53 .18		92	8464
	30	51 .33	52 .26	93	8649

Star. B. A. C.	Date. June.	Latitude. 48° 59′	Mean. 48° 59′	v.	vv.
7377 7398	26	51″.98		1	1
	30	51 .99	51″.99	0	0
7416 7453	26	51 .56		9	81
	30	51 .39	51 .47	8	64
7480 7489	26	53 .02		3	9
	30	52 .97	52 .99	2	4
7505 7605	26	51 .57		10	100
	30	51 .77	51 .67	10	100
7627 7686	26	52 .55		76	5776
	30	51 .03	51 .79	76	5776
7755 7765	26	52 .19		9	81
	30	52 .01	52 .10	9	81
			[vv]	= 15	.0396

If we take the arithmetical mean of the 81 determinations, giving equal weights to all, we find as the result

$$\varphi = 48^\circ\ 59'\ 51''.60 \pm .048.$$

291. If we desire the highest degree of precision, we must combine the values obtained from the individual pairs of stars according to their respective weights. The probable error of observation is determined from the quantity $[vv]$ above by means of formula (484), viz.,

$$\epsilon = .6745\sqrt{\frac{[vv]}{n-p}}.$$

In this case $n = 81$, $p = 29$; therefore $\epsilon = ''.363$.

We shall assume $\epsilon = 0''.4$ in computing the weights by formula (486).

This computation immediately follows. The values of e_δ are those given by Boss in his *Catalogue of* 500 *Stars*. In case of a few stars where Boss assigns no value to the probable error, it has been assumed to be $0''.75$.

Referring to formula (485), the following computation will be clearly understood

	Star. B. A. C.	No. of Observations.	ε_δ	ε_δ^2	$\frac{4\varepsilon^2}{n}$	p	ϕ 48° 59′	$p\phi$*	v	pvv
1	4937 4984	2	.25 .23	.0625 529	.3200	2.30	50″.60	1.38	− .96	2.1197
2	5026 5097	2	.27 .09	729 81	.3200	2.49	51 .25	3.11	− .31	.2393
3	5271 5313	4	.26 .22	676 484	.1600	3.62	51 .36	4.92	− .20	.1448
4	5415 5460	4	.35 .49	1225 2401	.1600	1.91	52 .06	3.96	+ .50	.4775
5	5502 5523	4	.25 .27	625 729	.1600	3.39	52 .09	7.09	+ .53	.9523
6	5545 5624	3	.13 .30	169 900	.2133	3.12	51 .27	3.96	− .29	.2624
7	5644 5658	2	.29 .29	841 841	.3200	2.05	51 .22	2.50	− .34	.2370
8	5693 5823	3	.19 .11	361 121	.2133	3.82	51 .98	7.56	+ .42	.6738
9	5853 5911	4	.30 .18	900 324	.1600	3.54	50 .87	3.08	− .69	1.6854
10	6047 6073	4	.11 .14	121 196	.1600	5.22	51 .31	6.84	− .25	.3263
11	6114 6157	5	.18 .25	324 625	.1280	4.49	51 .68	7.54	+ .12	.0647
12	6268 6289	5	.22 .23	484 529	.1280	4.36	50 .82	3.58	− .74	2.3875
13	6318 6365	2	.21 .25	441 625	.3200	2.34	51 .40	3.28	− .16	.0599
14	6421 6476	4	.30 .56	900 3136	.1600	1.77	52 .14	3.79	+ .58	.5954
15	6553 6586	3	.23 .19	529 361	.2133	3.31	51 .75	5.79	+ .19	.1195
16	6624 6681	3	.75 .40	5625 1600	.2133	1 07	51 .38	1.48	− .18	.0347
17	6728 6748	2	.75 .35	5625 1225	.3200	.99	51 .66	1.64	+ .10	.0099
18	6780 6817	3	.28 .41	784 1681	.2133	2.17	51 .53	3.32	− .03	.0020
19	6930 6970	2	.22 .26	484 676	.3200	2.29	51 .21	2.77	− .35	.2805
20	7024 7073	2	.37 .19	1369 361	.3200	2.03	51 .62	3.29	+ .06	[illegible]
21	7100 7166	2	.75 .75	5625 5625	.3200	.69	51 .44	.99	− .12	[illegible]
22	7215 7377	2	.30 .12	900 144	.3200	2.36	52 .28	5.38	+ .72	1.2234
23	7320 ——	2	.24 .75	0576 5625	.3200	1.06	52 .26	2.40	+ .70	.5194
24	7377 7398	2	.29 .13	841 169	.3200	2.38	51 .99	4.74	+ .43	.4401
25	7416 7453	2	.07 .25	49 625	.3200	2.58	51 .47	3.79	− .09	.0209
26	7480 7489	2	.19 .75	361 5625	.3200	1.09	52 .99	3.26	+ 1.43	2.2289
27	7505 7605	2	.14 .30	196 900	.3200	2.33	51 .67	3.89	+ .11	.0282
28	7627 7686	2	.08 .16	64 256	.3200	2.84	51 .79	5.08	+ .23	.1502
29	7755 7765	2	.25 .20	625 400	.3200	2.37	52 .10	4.98	+ .54	.6911

$[p] = 73.98 \qquad [p\phi] = 115.39 \qquad [pvv] = 15.9920$

$$\varphi = \frac{[p\varphi]}{[p]} = 48^\circ\ 59'\ 51''.56.$$

* In this column only the last three figures of $p\phi$ are given.

The probable error of φ is $r_0 = .6745\sqrt{\frac{[pvv]}{[p]\,(m-1)}}$. . . . . . . (35)

Substituting in this formula, $r_0 = .059$.

The smaller value of the probable error when the mean of the 81 determinations is formed directly is fallacious, since it rests on the assumption that these 81 values are independently determined. If each value were derived from a separate pair of stars this would be correct, but since the 81 values depend on only 29 separate pairs the error of the assumption is obvious.

It might be a question whether No. 26 should not be rejected, this value differing from the mean by a quantity so much larger than any of the others. There appears to be no reason for its rejection aside from this rather large discrepancy. If we reject it we find from the remaining 28 pairs

$$\varphi = 48^\circ\ 59'\ 51''.54 \pm .056.$$

292. For an illustration of the method of Art. 289, let us form the equations for determining the correction to the adopted value of R and to the above value of φ. We shall have 29 equations of the form (486); the above values of v will be the absolute terms. If we refer to the observations given in Art. 290, we have for the first pair $\frac{1}{2}(M - M') = -8.99$. We have from this pair the equation

$$\Delta\varphi + 8.99\,\Delta R = n.$$

This star was observed on two nights, so taking the mean of the values of $\frac{1}{2}(M - M')$ and multiplying the resulting equation through by the square root of the weight determined for this star, we have the following equation:

$$1.52\Delta\varphi + 13.57\Delta R = -1.46.$$

Proceeding in a similar manner, we derive the following 29 equations of condition for determining $\Delta\varphi$ and ΔR, for which we shall write x and y:

$$\begin{aligned}
1.52x + 13.57y &= -1.46;\\
1.58x + 7.36y &= -\ .49;\\
1.90x - 4.22y &= -\ .38;\\
1.38x + 11.56y &= +\ .69;\\
1.84x - 18.03y &= +\ .98;\\
1.77x - 18.53y &= -\ .51;\\
1.43x + 4.45y &= -\ .49;\\
1.95x - 11.97y &= +\ .82;\\
1.88x + 10.88y &= -1.30;
\end{aligned}$$

$$
\begin{aligned}
2.28x + 18.86y &= - \ \ .57; \\
2.12x - 13.72y &= + \ \ .25; \\
2.09x + 10.30y &= - 1.55 \\
1.53x - 12.51y &= - \ \ .24; \\
1.33x - \quad .20y &= + \ \ .77; \\
1.82x + \ \ 3.69y &= + \ \ .35; \\
1.03x - \ \ 2.99y &= - \ \ .19; \\
1.00x + \ \ 2.88y &= + \ \ .10; \\
1.47x - \quad .35y &= - \ \ .04; \\
1.51x + \ \ 7.07y &= - \ \ .53; \\
1.42x - \ \ 4.69y &= + \ \ .09; \\
.83x + \ \ 7.63y &= - \ \ .10; \\
1.54x - \ \ 8.81y &= + 1.11; \\
1.03x - \ \ 2.81y &= + \ \ .72; \\
1.54x + 14.60y &= + \ \ .66; \\
1.61x + \ \ 7.78y &= - \ \ .14; \\
1.04x + \ \ 1.21y &= + 1.49; \\
1.53x + \ \ 3.01y &= + \ \ .17; \\
1.69x - \ \ 4.77y &= + \ \ .39; \\
1.54x - \ \ 5.70y &= + \ \ .83.
\end{aligned}
$$

Proceeding in the usual manner, we derive from these the two normal equations

$$
\begin{aligned}
73.98x + \quad 17.65y &= - \ \ 0.04; \\
17.65x + 2732.35y &= - 85.80.
\end{aligned}
$$

From these,

$$
\begin{aligned}
x &= + .007 \pm .054; \\
y &= - .031 \pm .009.
\end{aligned}
$$

The most probable values of the latitude and micrometer-screw as indicated by this series of observations are therefore

$$
\begin{aligned}
\varphi &= 48^\circ \ 59' \ 51''.567 \pm .054; \\
R &= \qquad\quad 62''.025 \pm .009.
\end{aligned}
$$

In order to have the value of R determined in this way of any value in comparison with that determined by transits of circumpolar stars, the declinations of the stars employed must be well determined.

293. There are various ways in which the observation of stars in pairs at equal or nearly equal altitudes by means of the zenith telescope may be employed for the determination

of latitude and time. As may be seen, the instrument is adapted to the solution of any problem of Spherical Astronomy which depends upon the observation of two or more bodies at the same altitude. The most favorable condition for latitude determination is when the two stars are on the meridian, one north, the other south, while time is best determined by observing two stars on the prime vertical, one east, the other west.

On account of the facility with which the latitude is determined in the manner already explained, and the ease with which the instrument may be converted into a transit when it is necessary to employ it for determining the approximate time, other solutions of the problem depending on observations out of the meridian have never met with much favor.

Some of these methods are interesting from a theoretical point of view, but for the reasons stated the subject will not be developed further in this connection.

CHAPTER IX.

DETERMINATION OF AZIMUTH.

294. *The Azimuth* of a point on the earth's surface is the angle between the plane of the meridian and the vertical plane which passes through this point and the eye of the observer.

Since the vertical plane is determined by the direction of the plumb-line, and this line may deviate from the true normal to the earth's surface, a corresponding deviation in the azimuth must exist. We must therefore distinguish between the *Astronomical Azimuth* and the *Geodetic Azimuth.*

The Astronomical Azimuth of a point is the angle between two planes drawn through the plumb-line at the point of observation, the first plane parallel to the earth's axis, and the second passing through the point.

The Geodetic Azimuth is the angle between two planes drawn through the normal to the earth's surface at the point of observation, the first plane passing through the earth's axis, and the second through the point.

It is with the *Astronomical Azimuth* only that we are at present concerned. The azimuth may be reckoned from either the north or south point of the horizon. For astronomical purposes it is usually reckoned from the south point towards the west from zero to 360°. In determining the azimuth of a point on the earth's surface it is more convenient to use stars near the north pole of the heavens; consequently for geodetic purposes the azimuth is generally

FIG. 58a.

reckoned from the north point. For the sake of uniformity we shall in this chapter always suppose the azimuth reckoned from the north in the direction N., E., S., W. A minus azimuth will be reckoned from north towards west.

Extreme accuracy in the determination of azimuth is required in connection with the geodetic operations of primary triangulation. The principal methods employed in such cases will be given, when it will be shown how they may be abridged where a less degree of accuracy is demanded. There are a variety of these methods, depending on the form of instrument employed and the position of the stars observed. The instrument will be either the theodolite, used for measuring horizontal angles, or the astronomical transit. In any case the azimuth of the point is determined by measuring instrumentally the difference between the azimuth of the point and a star. The azimuth of the star is computed by its known right ascension and declination, and the local time and latitude, which have been previously determined; from these data we have the azimuth of the point.

295. *The Theodolite.* Figures 58*a* and 58*b* show two forms of instruments used on the U. S. Coast Survey. The older form, Fig. 58*a*, has a horizontal circle from 20 to 30 inches in diameter. With the newer instruments, circles from 12 to 20 inches are considered sufficiently large, as such circles can now be graduated much more accurately than formerly; the instrument can therefore be made more compact and portable, a matter of some importance in the field.

The horizontal circle is commonly divided directly to 5′, these spaces being subdivided by reading microscopes directly to single seconds, and by estimation to tenths of a second. Two or three microscopes are used. The essential features of the instruments will be understood from the plates without further description.

For secondary azimuths a less perfect instrument will often

FIG. 286.

be used. For magnetic work or ordinary land-surveying a common surveyor's transit with 5- or 6-inch circle will frequently be employed. It is perhaps unnecessary to say that the instrument must be carefully adjusted in every particular.

296. *The Signal.* For observing at night an illuminated mark is required. A convenient mark is a square wooden box firmly mounted on a post or other support, the light of

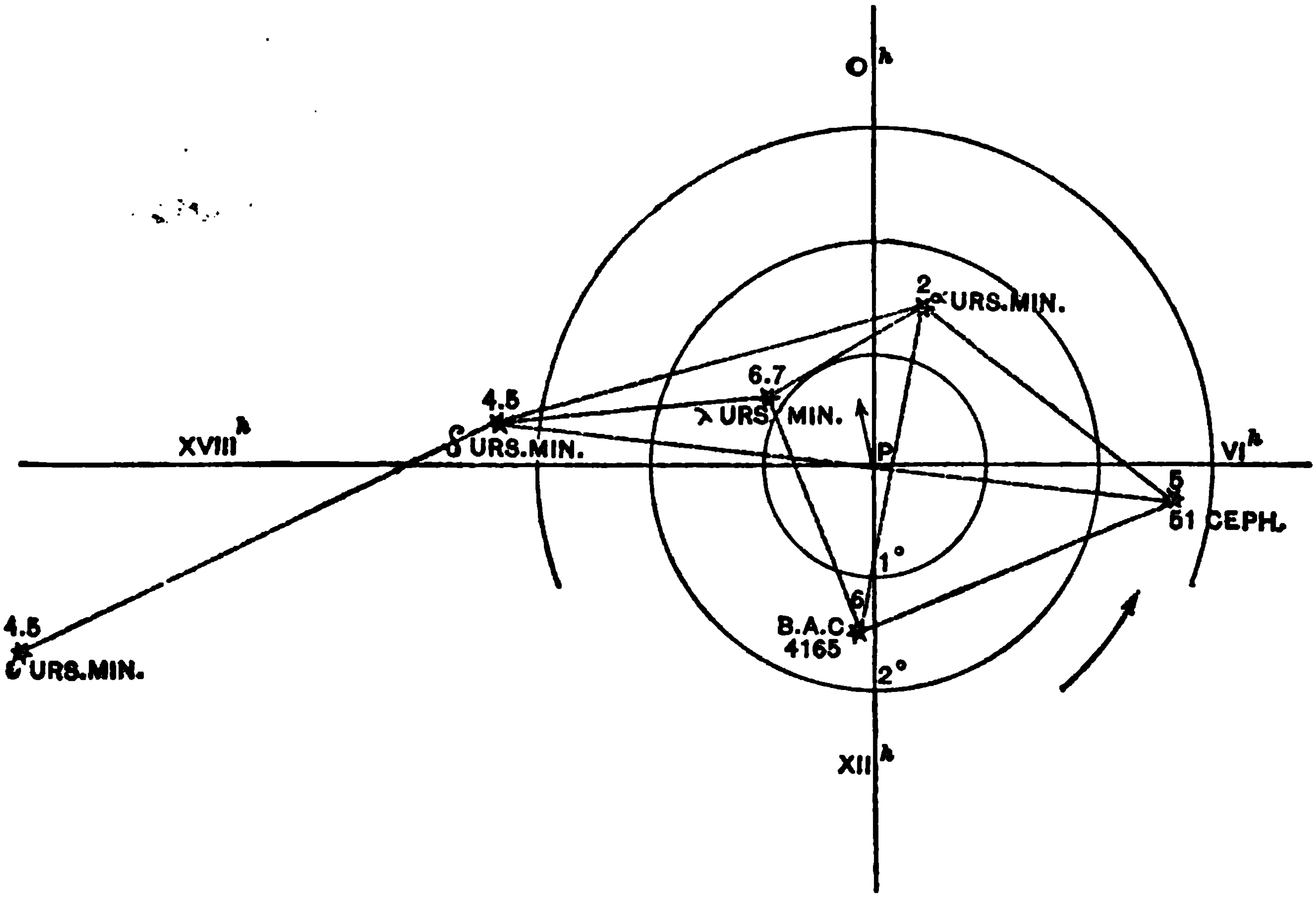

FIG. 59.

a bull's-eye lantern being thrown through a small hole in the front. The box itself may be painted so as to form a convenient target for day observation. This mark must be placed far enough from the station so that no change will be required in the sidereal focus of the telescope: about one mile will generally be sufficient. When from any cause a distant mark is not practicable a collimating telescope may be used; but the greatest care must be exercised in mount-

ing both the instrument and collimator firmly, piers of solid masonry being used for both.

297. *Choice of Stars.* For first-class azimuths only close circumpolar stars will be used. Preference will be given to the four circumpolar stars whose places are given in the ephemeris, viz., α, δ, and λ Ursæ Minoris, and 51 Cephei. Fig. 59 shows their relative positions, and will assist in finding the smaller ones which are not readily distinguished with the naked eye unless the position is previously known.

298. *Method of Observing.* A complete series of observations on one star will consist of ten or twelve readings on the mark and about the same number on the star, the instrument being reversed about the middle of the series. The following order of observation is recommended:

1st. 6 readings on the mark.
2d. 6 readings on the star.
3d. Read the level.
4th. Reverse.
5th. Read level.
6th. 6 readings on the star.
7th. 6 readings on the mark.

If more than one series is taken it is advisable to change the position of the horizontal circle so as to bring the readings in another place, in order to eliminate to some extent the errors of graduation.

Readings are sometimes taken on the star directly, and on its image reflected from a basin of mercury. When this is done reading the level may be dispensed with.

By the process above described we have a carefully-executed measurement of the difference in azimuth between the star and mark. It only remains to compute the azimuth of the star, when we shall have the azimuth of the mark.

Let m = reading of circle on mark;
s = reading of circle on star;
A = azimuth of mark measured from north towards east;
a = azimuth of star measured from north towards east.

Then
$$A = a + (m - s). \quad . \quad . \quad . \quad . \quad . \quad (487)$$

Different methods of computing a will be employed, depending on the position of the star when observed.

Errors of Collimation and Level.

299. The mark and star being at different altitudes above the horizon, the measured difference of azimuth will be affected by an error of collimation, also by a want of parallelism between the horizontal axis and the horizon.

Other theoretical errors of the instrument we need not consider, since their effect may be made inappreciable by careful adjustment.

In the figure let $NWSE$ represent the horizon, z the zenith, s any star, w' the point where the horizontal axis produced pierces the celestial sphere.

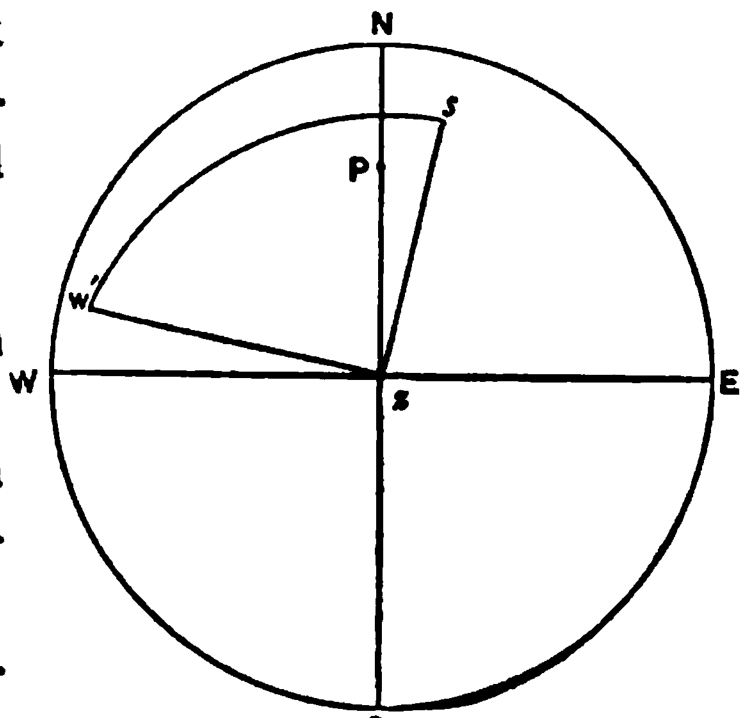

FIG. 60.

*b is the inclination, + when west end of axis is high;
*c, error of collimation, + when thread is east of collimation axis;
x, error in reading of horizontal circle due to b and c.

* This designation is sufficiently general for our purpose, since we shall only have occasion to apply it to stars observed near the pole.

Then in the triangle $sw'z$, $sz = z =$ zenith distance of star;

$$zw' = 90^\circ - b; \qquad w's = 90^\circ + c; \qquad w'zs = 90^\circ + x.$$

Therefore $-\sin c = \sin b \cos z - \cos b \sin z \sin x$.

Or, since c, b, and x will be very small, the above may be written

$$-c = b \cos z - x \sin z;$$

from which $$x = \frac{c}{\sin z} + \frac{b}{\tan z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (488).$$

It will seldom be necessary to apply the correction for collimation, since it may be eliminated by observing in both positions of the axis.

If the mark is not in the horizon a similar correction to readings on mark will be required, where, of course, for z we shall have the zenith distance of the mark.

Azimuth by a Circumpolar Star near Elongation.

300. When the star is within a short distance of elongation, either east or west, the position is especially favorable, since the motion in azimuth then is very slow. Only one reading can be taken at elongation, but we may apply a correction to the readings near elongation to reduce them to the reading at elongation.

The azimuth and hour-angle of the star at elongation are

computed by considering the right-angle triangle formed at this instant by the zenith, pole, and star.

Let $-a_e$* and t_e be the azimuth and hour-angle at elongation;
α, δ, and θ, the right ascension, declination, and sidereal time.

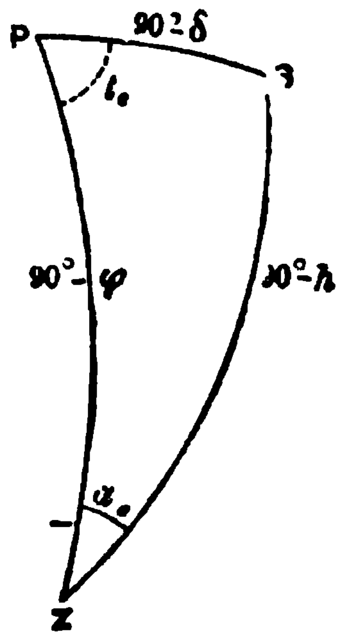

FIG. 61.

Then †

$$-\sin a_e = \cos\delta\sec\varphi;$$
$$\cos t_e = \cot\delta\tan\varphi;$$
$$\theta = \alpha \pm t_e \left\{ \begin{matrix}\text{western}\\ \text{eastern}\end{matrix} \right\} \text{elongation.} \quad (488)$$

Chronometer time of elongation $= \theta - \varDelta\theta$.

The chronometer correction should be known within about one second, and may be determined by any of the methods previously given; or the theodolite itself may be used for the purpose, either as a transit or by measuring altitudes as with the sextant, provided it has a good vertical circle.

301. The formulæ for reducing the readings to elongation will now be developed.

Formulæ (121) give the values of h and a in terms of δ and t for a star at any hour-angle. Recollecting that we now measure the azimuth from the north instead of the south point, these equations are

$$(a) \qquad \cos h\cos a = \sin\delta\cos\varphi - \cos\delta\sin\varphi\cos t;$$
$$(b) \qquad \cos h\sin a = -\cos\delta\sin t.$$

* $-a_e$, since a plus value of the hour-angle t_e corresponds to a minus azimuth.

† If many observations of the same star are to be made, it will be convenient to prepare in advance a table of the values of a_e and θ extending over the time during which it is intended to observe.

At elongation we have

$$(c) \qquad -\sin a_e = \frac{\cos\delta}{\cos\varphi} = \frac{\sin\delta\cos t_e}{\sin\varphi};$$

$$(d) \qquad \cos a_e = \sin\delta\sin t_e.$$

Multiplying together first (*a*) and (*c*), then (*b*) and (*d*), we have

$$(e) \quad -\cos h\cos a\sin a_e = \sin\delta\cos\delta - \sin\delta\cos\delta\cos t\cos t_e;$$

$$(f) \quad \cos h\sin a\cos a_e = \qquad\qquad -\sin\delta\cos\delta\sin t\sin t_e.$$

Subtract (*f*) from (*e*),

$$-\cos h\sin(a_e - a) = \sin\delta\cos\delta - \sin\delta\cos\delta\cos(t_e - t).$$

From this, $\sin(a_e - a) = -\dfrac{\sin\delta\cos\delta}{\cos h}\, 2\sin^2\tfrac{1}{2}(t_e - t).$

The computation will be more convenient if for $\cos h$ we substitute its value in terms of a_e and δ, viz.,

$$\cos h = -\cot a_e\cot\delta;$$

and therefore $\sin(a_e - a) = \tan a_e\sin^2\delta\, 2\sin^2\tfrac{1}{2}(t_e - t)$. (489)

We now have an equation which gives the difference between the azimuth at elongation and at any hour-angle t.

As this will only be used for stars near elongation, and consequently $t_e - t$, a small quantity, it will be convenient to expand it into a series, viz.,

$$a_e - a = \tan a_e\sin^2\delta\,\frac{2\sin^2\frac{1}{2}(t_e - t)}{\sin 1''} + \frac{1}{6}(\tan a_e\sin^2\delta)^3\,\frac{(2\sin^2\frac{1}{2}t)^3}{\sin 1''}.^* \qquad (490)$$

---

* $y = \sin^{-1}x = x + \dfrac{1}{6}\dfrac{x^3}{\sin 1''} + \text{etc.}$

In this case $(a^e - a) = \sin^{-1}[\tan a_e\sin^2\delta\ 2\sin^2\frac{1}{2}(t_e - t)]$.

When this formula is applied to the close circumpolar stars, $\sin^2 \delta$ differs but little from unity, and the last term will in all practical cases be inappreciable.

We have therefore the simple formula

$$a_e - a = \tan a_e \frac{2 \sin^2 \frac{1}{2}(t_e - t)}{\sin 1''} \quad . \quad . \quad . \quad . \quad (491)$$

302. *Correction for Inclination of Axis.* When the *west* end of the axis is high the reading of the horizontal circle will be small; therefore the correction will be *plus.*

The inclination will be given by the formula derived for transit instrument, (289):

$$b = \frac{d}{4}[(w + w') - (e + e')]. \quad . \quad . \quad . \quad (492)$$

Or if the level is reversed more than once,

$$b = \frac{d}{2}[W - E]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (493)$$

Where W and E are the means of the readings of the east and west ends respectively.

The effect upon the reading of the horizontal circle we have by equation (488), viz.,

$$x = \frac{b}{\tan z} = b \tan h.$$

Where h is the altitude of the star.

Such a correction must also be applied to the reading on mark when appreciable.

With the circumpolar stars observed at elongation we may write tan φ for tan h. Then we have

$$\text{Correction for level} = \delta a = \frac{d}{2}[W - E] \tan \varphi. \quad . \quad . \quad . \quad (494)$$

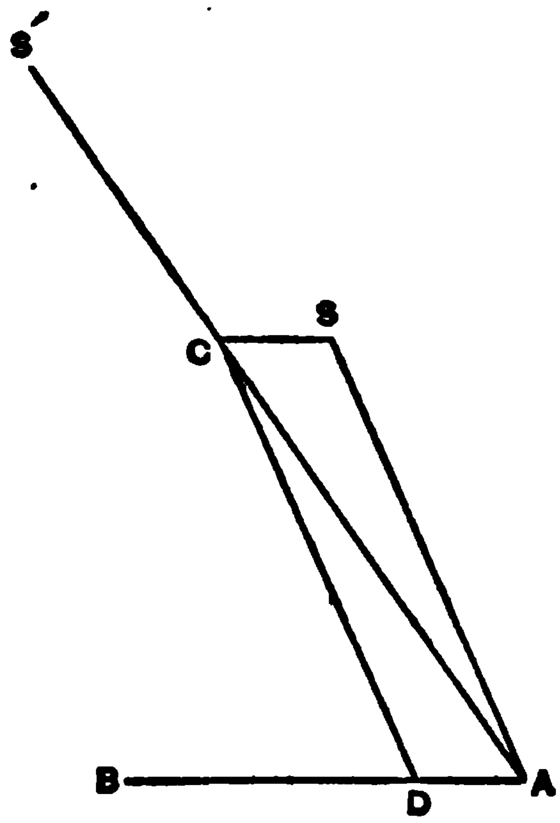

FIG. 62.

303. *Correction for Diurnal Aberration.* Suppose at the instant of observation the point from which observation is made to be moving in the direction AB.

Let SA be the true direction of a ray of light coming from a star; then in consequence of aberration the star will appear in the direction AS'.

Let AC be drawn equal to the distance traversed by the ray of light in one second $= V$;

AD, the distance traversed by the point on the earth's surface in one second $= v$.

Let angle $SAB = \vartheta$; $S'AB = \vartheta'$. Then $ACD = \vartheta - \vartheta' = \Delta\vartheta$.

$$\text{Then} \quad \frac{\sin \Delta\vartheta}{\sin \vartheta} = \frac{v}{V}; \quad \text{or} \quad \Delta\vartheta = \frac{v}{V} \sin \vartheta.$$

We have found, equation (286), $\frac{v}{V} = 0''.319 \cos \varphi$.

$$\text{Therefore} \quad \Delta\vartheta = ''.319 \cos \varphi \sin \vartheta. \quad . \quad . \quad . \quad . \quad (495)$$

This gives the displacement in the plane determined by the direction of the ray of light and the direction of motion of the point of observation. It remains to determine its effect on the star's azimuth.

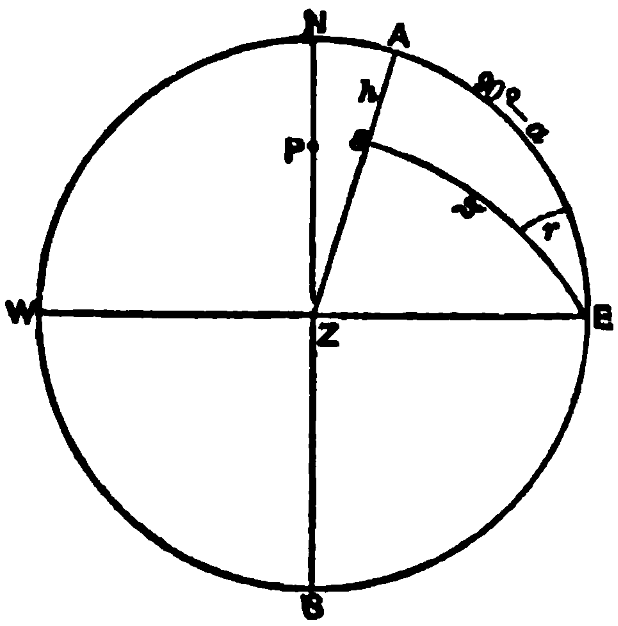

FIG. 63.

In Fig. 63 let s be any star, NS the meridian, $NESW$ the horizon. sA is drawn perpendicular to the horizon, and therefore equals the altitude. NA equals the azimuth. The angle at E is called γ.

Since the point occupied by the observer is moving directly towards the east point of the horizon at the instant of observation, sE will be equal to ϑ.

Then the right triangle sEA gives the equations

$$(a) \qquad \cos h \cos a = \sin \vartheta \cos \gamma;$$
$$(b) \qquad \cos h \sin a = \cos \vartheta.$$

We require the effect produced on a by a small change in ϑ; therefore we differentiate with respect to h, a, and ϑ.

$$-\cos h \sin a \, da - \sin h \cos a \, dh = \cos \vartheta \cos \gamma d\vartheta;$$
$$\cos h \cos a \, da - \sin h \sin a \, dh = -\sin \vartheta \, d\vartheta.$$

Multiply the first of these by $\sin a$, the second by $\cos a$, subtract to eliminate dh, and reduce by (a) and (b); we readily find

$$da = -\frac{\cos a}{\sin \vartheta \cos h} d\vartheta.$$

Substitute for $d\vartheta$ the value of $\Delta\vartheta$ given by (495), and recollect that the azimuth is reckoned from the north; we have

$$\delta a = \frac{''.319 \cos \varphi \cos a}{\cos h}. \quad . \quad . \quad . \quad . \quad (496)$$

For a close circumpolar star this will not differ appreciably from

$$\delta a = ''.319 \cos a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (497)$$

This will be added algebraically to the computed azimuth of the star.

304. *Formulæ for Azimuth by a Circumpolar Star near Elongation.*

$$\left.\begin{aligned}
\sin a_e &= \cos \delta \sec \varphi; \\
\cos t_e &= \cot \delta \tan \varphi; \\
\text{Chron. time} &= \alpha \pm t_e - \Delta\theta \begin{Bmatrix} \text{western} \\ \text{eastern} \end{Bmatrix}; \\
a_e - a &= \tan a_e \frac{2 \sin^2 \frac{1}{2}(t_e - t)}{\sin 1''}; \\
\text{Level} &= \frac{d}{2}[W - E] \tan \varphi; \\
\text{Aberration} &= ''.319 \cos a; \\
A &= a_e + (m - s)^* - \text{level} + \text{aberration}.
\end{aligned}\right\} \quad (\text{XXIV})$$

* m = reading of circle on mark; s = reading on star.

Example.

1847, October 17th, Polaris was observed near western elongation at Agamenticus, York County, Maine, with one of the 30-inch theodolites of the Coast Survey, as follows:

No.	Object	Tel.	Time by Sidereal Chronometer.	Azimuth Circle.	A		B		C		Level.		
			h. m. s.	° ′	d.	d.	d.	d.	d.	d.	1 div.=0″.97		
1	Mark.	R.	6 30	63 55	39.7	39.0	27.5	27.0	27.7	26.5	Correction for run + 0.1		
2			33	63 55	41.0	39.7	27.0	28.0	26.0	24.3			
3			34	63 55	41.0	41.0	29.8	29.0	26.4	26.3			
4		D.	37	243 55	26.2	28.2	16.8	17.0	16.8	13.3			
5			39	243 55	25.5	28.0	17.0	17.0	16.4	15.2			
6			42	243 55	27.0	29.0	19.0	19.0	16.2	14.0		Level.	
1	Star.	D.	6 47 12	127 42	68.0	67.0	61.5	63.0	64.5	64.3	Correction for run 0.0	E.	W.
2			49 06	127 42	65.0	65.0	63.5	63.2	63.1	60.5		44	62
3			51 38	127 42	62.8	62.8	57.0	59.8	60.0	58.2		63	44
4			52 12 .5	127 42	58.0	58.0	54.0	52.5	55.3	53.5		43	63
5			53 55 .5	127 42	56.0	57.0	51.1	52.0	53.0	52.0		64	43
6		R.	7 00 54	307 42	48.2	48.7	45.2	45.0	47.7	45.8			
7			2 25 .5	307 42	48.0	49.2	43.2	44.2	45.0	44.8		46	62
8			4 01 .5	307 42	48.0	48.7	43.0	44.7	46.8	45.0		62	46
9			5 51	307 42	49.0	49.0	44.7	45.0	47.9	46.9			
10			7 14 .5	307 42	49.2	50.5	44.8	44.8	47.2	46.2		43	63
7	Mark.	R.	7 16	63 55	40.0	40.0	23.0	25.0	26.8	25.2	Correction for run + 0.1	63	43
8			17	63 55	39.7	39.7	23.0	23.0	25.7	24.8			
9			18	63 55	38.0	39.0	21.5	22.7	25.0	23.8			
10		D.	23	243 55	26.0	26.5	13.7	14.0	15.0	14.6			
11			24	243 55	26.8	26.8	14.5	14.8	15.2	14.0			
12			26	243 55	26.7	27.3	14.0	13.0	14.5	13.9			

The horizontal circle was read by means of three microscopes designated A, B, C respectively; the value of one division of the micrometer-head corresponding to one second of arc, subject to the correction for run. The circle being graduated directly to 5′, if five revolutions of the screw exactly cover this space there is no correction for run; otherwise it represents the excess or deficiency.

For reducing these observations we have:

$$\begin{aligned}
&\text{Right ascension of Polaris} &&= \alpha &&= 1^h\ 5^m\ 32^s.96\\
&\text{Declination of Polaris} &&= \delta &&= 88°\ 29'\ 54''.27\\
&\text{Latitude of station} &&= \varphi &&= 43\ \ 13\ \ 25\ .0\\
&\text{Chronometer correction} &&= \Delta\theta &&= -\ 1^m\ 51^s.8
\end{aligned}$$

We first compute the azimuth and time of elongation:

$$\cos\delta = 8.4183795 \qquad \cot\delta = 8.4185287$$
$$\cos\varphi = 9.8625407 \qquad \tan\varphi = 9.9730531$$
$$\sin a_e = 8.5558388 \qquad \cos t_e = 8.3915818$$
$$a_e = -\,2^\circ\ 3'\ 39''.21 \qquad t_e = 88^\circ\ 35'\ 17''.8$$

(a_e is minus, since elongation is west.)

$$t_e = 5^h\ 54^m\ 21^s.2$$
$$\alpha = 1\quad 5\quad 33\ .0$$
$$\theta = 6\quad 59\quad 54\ .2$$
$$\Delta\theta = \quad -1\quad 51\ .8$$
$$\text{Chronometer time of elongation} = 7^h\ 1^m\ 46^s.0$$

In the table which follows, the column marked *corrected readings* is the mean of the readings of the three microscopes corrected for run when necessary; the remaining columns will be explained by referring to formulæ (XXIV).

No.	Position.	Corrected Readings.	$t_e - t$.	$\dfrac{2\sin^2\frac{1}{2}(t_e-t)}{\sin 1''}$.	$a_e - a$.	Reduced Readings.	Means.
1	R.	63°55′31″.3					
2		31 .1					
3		32 .3					
4	D.	243 55 19 .8					
5		19 .9					
6		20 .8					
1	D.	127 42 64 .7	$+14^m\ 34^s$	416″.5	− 15″.0	127°42′49″.7	
2		63 .4	12 40	315 .0	11 .3	52 .1	
3		60 .1	10 8	201 .6	7 .3	52 .8	
4		55 .2	9 33 .5	179 .4	6 .5	48 .7	
5		53 .5	7 50 .5	120 .7	4 .3	49 .2	127°42′50″.50
6	R.	307 42 46 .8	+ 52	1 .5	.1	307 42 46 .7	Level − .23
7		45 .7	− 39 .5	.8	.0	45 .7	
8		46 .0	2 15 .5	10 .0	.3	45 .7	
9		47 .1	4 5	32 .7	1 .2	45 .9	
10		47 .1	− 5 28 .5	58 .9	− 2 .1	45 .0	307 42 45 .80 Level .00
7	R.	63 55 30 .0					
8		29 .4					
9		28 .4					
10	D.	243 55 18 .3					
11		18 .7					
12		18 .3					

Mean of readings on mark	$= m =$	243° 55′ 24″.86
Mean of readings on star	$= s =$	127 42 48 .03
	$m - s =$	116 12 36 .83
Azimuth of star	$= a_e =$	− 2 39 39 .21
Azimuth of mark	$A =$	114 8 57 .62
Diurnal aberration		+ .32
Final value of azimuth,		114° 8′ 57″.94

From the level readings we have—

	Direct.	Reverse.
	$E = 53.50$	53.50
	$W = 53.00$	53.50

$$\frac{d}{2}[W - E] = -.24 \qquad d = ''.97$$

Azimuth by a Circumpolar Star observed at any Hour-angle.

305. This method differs from the preceding in the manner of computing the azimuth of the star, which may be conveniently done by either of three methods.

First. By the fundamental equations (a) and (b), Art. (301), we readily find

$$\tan a = -\frac{\sin t}{\cos \varphi \tan \delta - \sin \varphi \cos t}. \quad . \quad . \quad (498)$$

Second. We may apply Napier's analogies to the triangle formed by the zenith, pole, and star, viz.,

$$\left.\begin{aligned} \tan \tfrac{1}{2}(q + a) &= \frac{\sin \frac{1}{2}(\delta - \varphi)}{\cos \frac{1}{2}(\delta + \varphi)} \cot \tfrac{1}{2}t; \\ \tan \tfrac{1}{2}(q - a) &= \frac{\cos \frac{1}{2}(\delta - \varphi)}{\sin \frac{1}{2}(\delta + \varphi)} \cot \tfrac{1}{2}t; \\ a &= \tfrac{1}{2}(q + a) - \tfrac{1}{2}(q - a). \end{aligned}\right\} \quad . \quad . \quad (499)$$

Third. By expansion into series.

In equation (498) write $p = 90° - \delta$. Then

$$\tan a = -\frac{\sin t \sin p}{\cos \varphi \cos p - \sin \varphi \cos t \sin p}.$$

a and p being small, we may expand $\tan a$, $\sin p$, $\cos p$ into

series, when the equation becomes, to terms of the third order inclusive,

$$a + \tfrac{1}{3}a^3 = -\frac{\sin t(p - \frac{1}{6}p^3)}{\cos \varphi(1 - \frac{1}{2}p^2) - \sin \varphi \cos t(p - \frac{1}{6}p^3)},$$

or

$$a \cos \varphi = -p \sin t + ap \sin \varphi \cos t + \tfrac{1}{2}ap^2 \cos \varphi - \tfrac{1}{3}a^3 \cos \varphi + \tfrac{1}{6}p^3 \sin t.$$

Solving this equation for a by approximations, we have for the first approximation

$$a = -\frac{\sin t}{\cos \varphi}.p.$$

This value substituted in the second term of the above equation gives for a second approximation

$$a = -\frac{\sin t}{\cos \varphi}\Big[p + p^2 \tan \varphi \cos t\Big].$$

This value substituted in the second, third, and fourth terms of the above gives finally

$$a = -\frac{\sin t}{\cos \varphi}\Big[p + p^2 \sin 1'' \tan \varphi \cos t + \tfrac{1}{3}p^3 \sin^2 1''[(1 + 4 \tan^2 \varphi) \cos^2 t - \tan^2 \varphi]\Big]. \quad (500)$$

For *Polaris* within the limits of the United States the term in p^3 will not exceed $2''$, while the terms neglected will not be greater than $0''.1$.

For a close circumpolar star observed near culmination this formula may be written

$$a = -\frac{\sin t}{\cos \varphi}\Big[p + p^2 \sin 1'' \tan \varphi \cos t + \tfrac{1}{3}p^3 \sin^2 1''(1 + 3 \tan^2 \varphi)\Big]. \quad (501)$$

The corrections for level reading and aberration will be computed by the same formulæ as in the previous case.

Correction of the Mean Azimuth for Second Differences.

306. In applying the foregoing method to a series of ten or more readings on a star we may proceed in either of two ways: *first*, we may reduce each reading separately, computing the azimuth of the star for each time of observation; or *second*, we may take the mean of the readings and compute the azimuth for the mean of the corresponding times, applying to this computed azimuth a small correction for second differences.

The first method involves considerable labor, but at the same time the individual values furnish a rough check on the accuracy of the work. When the second method is preferred we may derive the expression for the correction as follows:

Let $t_1, t_2, t_3, \ldots t_n$ = the observed times;

$a_1, a_2, a_3, \ldots a_n$ = the corresponding azimuths of the star;

$\dfrac{t_1 + t_2 + \ldots + t_n}{n} = t_0$ = the mean of the observed times;

a_0 = the azimuth corresponding to t_0.

Let $\Delta t_1 = t_1 - t_0;\quad \Delta t_2 = t_2 - t_0;\ \ldots\ \Delta t_n = t_n - t_0$.

Then we have $\Delta t_1 + \Delta t_2 + \ldots + \Delta t_n = 0$.

We may now write

$$a_1 = f(t_1) = f(t_0 + \Delta t_1) = a_0 + \frac{da}{dt}\Delta t_1 + \frac{d^2a}{dt^2}\tfrac{1}{2}\Delta t_1^2;$$

$$a_2 = f(t_2) = f(t_0 + \Delta t_2) = a_0 + \frac{da}{dt}\Delta t_2 + \frac{d^2a}{dt^2}\tfrac{1}{2}\Delta t_2^2;$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_n = f(t_n) = f(t_0 + \Delta t_n) = a_0 + \frac{da}{dt}\Delta t_n + \frac{d^2a}{dt^2}\tfrac{1}{2}\Delta t_n^2.$$

The mean of these expressions will be

$$\frac{a_1 + a_2 + \ldots + a_n}{n} = a_0 + \frac{d^2a}{dt^2}\frac{1}{2}\frac{\Delta t_1^2 + \Delta t_2^2 + \ldots + \Delta t_n^2}{n}.$$

The quantities Δt will be expressed in time: multiplying by 15 to reduce to arc, and also multiplying each quantity of the form $(15\Delta t)^2$ by sin 1″, the term multiplied by $\frac{d^2a}{dt^2}$ will be

$$\frac{(15)^2}{2}\sin 1'' \frac{\Delta t_1^2 + \Delta t_2^2 + \ldots + \Delta t_n^2}{n} = [6.73672]\frac{1}{n}\Sigma\Delta t^2. \quad (502)$$

Or, if preferred, this term may be computed by table VIII A, for, since the quantities Δt will be small, we shall have practically

$$\tfrac{1}{2}\Delta t^2 \sin 1'' = \frac{2\sin^2\frac{1}{2}t}{\sin 1''},$$

and the above term becomes

$$\frac{1}{n}\Sigma\frac{2\sin^2\frac{1}{2}t^*}{\sin 1''}. \quad \ldots \ldots \quad (503)$$

It remains to determine a convenient expression for $\frac{d^2a}{dt^2}$.

Differentiating equation (b), Art. 301, with respect to a and t, we find

$$\frac{d^2a}{dt^2} = + \frac{\tan a}{\sin^2 t}\left(\frac{\cos^2 t - \cos^2 a}{\cos^2 a}\right). \quad \ldots \quad (504)$$

For a close circumpolar star $\cos^2 a$ differs but little from unity, so that we shall have very nearly

$$\frac{d^2a}{dt^2} = -\tan a. \quad \ldots \ldots \quad (505)$$

* It will be seen that the expression which we have derived for reducing the reading taken near elongation to the reading at elongation is a special case of this same form.

We therefore have for the mean of the azimuths

$$\frac{a_1 + a_2 + \ldots + a_n}{n} = a_0 - \tan a_0\,[6.73672]\,\frac{1}{n}\Sigma\Delta t^2, \qquad (506)$$

where, as usual, the quantity in brackets is a logarithm, and the quantities Δt are expressed in seconds of time.

Example.

307. 1848, April 5. Observations on *Polaris* at Dollar Point, Galveston Bay, Texas. Instrument, 18-inch Troughton & Simms theodolite.

One division of level $= 0''.82$.
$\varphi = 29°\ 26'\ 2''.6$;
$\alpha = 1^h\ 4^m\ 4^s.7$;
$\delta = 88°\ 29'\ 57''.83$;
$\Delta T = \quad +1^s.8$.

Object.	Position.	Chronometer Time.	Azimuth Circle.			Level.	
			A	B	C	E.	W.
Mark.	D.		158° 50′ 55″	65″	50″	129	71.5
	R.		51 20	20	00	81	119
					.	126	74
Star.	D.	$9^h\ 3^m\ 33^s.5$	337 18 40	35	20	83	117
		4 47 .5	18 55	55	35		
		6 7 .0	18 75	70	55		
	R.	9 8 6 .5	19 45	55	40		
		9 24 .0	19 65	75	55		
		10 23 .5	20 20	30	10		
						121.5	79
Mark.	D.		158 50 55	65	50	80	120
	R.		51 20	15	00	121.5	78
						77.5	122

The reduction is now as follows :

Object.	Position.	Reduced Reading.	Mean of Readings.	Chronometer.	Δt.	Δt^2.
Mark.	D.	158° 50′ 56″.7				
	R.	51 13 .3				
Star.	D.	337 18 31 .7		$9^h\ 3^m\ 33^s.5$	$210^s.2$	44184
		18 48 .3		4 47 .5	136 .2	18523
		18 66 .7		6 7 .0	56 .7	3215
	R.	19 46 .7		8 6 .5	62 .8	3944
		19 65 .0		9 24 .0	140 .3	19684
		337 20 20 .0	337° 19′ 26″.4	9 10 23 .5	199 .8	39920
Mark.	D.	158 50 56 .7				
	R.	51 11 .7	158 51 4 .6			

Formula (506):

$\Sigma \Delta t^2 = 129470 \quad \log = 5.1122$

$\log \frac{1}{n} = 9.2218$

Constant $\log = 6.7367$

$\tan a = 8.4092_n$

log correction $= 9.4800_n$

Correction $= -0''.3$

Mean of times $= 9^h\ 7^m\ 3^s.7$

$\Delta T = \quad +1\ .8$

$\alpha = 1\ 4\ 4\ .7$

$t = 8^h\ 2^m\ 57^s.2 = 120°44'18''.0$

The azimuth of the star may now be computed either by equation (498), (499), or (500). We shall compute it by each method for illustration.

Formula (498) is $\tan a = -\dfrac{\sin t}{\cos \varphi \tan \delta - \sin \varphi \cos t}$.

$\varphi = 29°\ 26'\ 2''.6$
$\delta = 88\ 29\ 57\ .83$

$\cos \varphi = 9.9399792$
$\tan \delta = 1.5817575$
$\text{Sum}_1 = 1.5217367$
* Zech $\quad .0032688$
log denom. $= 1.5250055$
$\sin t = 9.9342512$
$\tan a = 8.4092457$

$\sin \varphi = 9.6914542$
$\cos t = 9.7085212_n$
$\text{Sum}_2 = 9.3999754_n$
$s_1 - s_2 = 2.1217613$

$a = -1°\ 28'\ 11''.5$

Formulæ (499):
$$\tan \tfrac{1}{2}(q + a) = \frac{\sin \frac{1}{2}(\delta - \varphi)}{\cos \frac{1}{2}(\delta + \varphi)} \cot \tfrac{1}{2}t;$$
$$\tan \tfrac{1}{2}(q - a) = \frac{\cos \frac{1}{2}(\delta - \varphi)}{\sin \frac{1}{2}(\delta + \varphi)} \cot \tfrac{1}{2}t.$$

$\delta =$	$88°\ 29'\ 57''.83$		
$\varphi =$	$29\ 26\ 2\ .6$		
$\delta - \varphi =$	$59\ 3\ 55\ .23$		
$\frac{1}{2}(\delta - \varphi) =$	$29\ 31\ 57\ .61$	$\sin = 9.6927762$	$\cos = 9.9395566$
$(\delta + \varphi) =$	$117\ 56\ 0\ .43$		
$\frac{1}{2}(\delta + \varphi) =$	$58\ 58\ 0\ .21$	$\cos = 9.7122589$	$\sin = 9.9329140$
$\frac{1}{2}t =$	$60\ 22\ 9\ .0$	$\cot = 9.7549528$	$\cot = 9.7549528$
		$\tan \frac{1}{2}(q + a) = 9.7354701$	$\tan \frac{1}{2}(q - a) = 9.7615954$
$\frac{1}{2}(q + a) =$	$28\ 32\ 20\ .60$		
$\frac{1}{2}(q - a) =$	$30\ 0\ 32\ .09$		
$a = -$	$1\ 28\ 11\ .5$		

* Addition and subtraction logarithms.

Formula (500):

$$a = -\frac{\sin t}{\cos \varphi}\left[p + p^2 \sin 1'' \tan \varphi \cos t + \tfrac{1}{3} p^3 \sin^2 1'' \left\{(1 + 4\tan^2\varphi)\cos^2 t - \tan^2\varphi\right\}\right].$$

$$p = 1^\circ\ 30'\ 2''.17$$
$$= 5402''.17$$

$\log p = 3.73257$		
$\log p^2 = 7.46514$	$\log p^3 = 11.1977$	$\tan^2\varphi = 9.5029$
$\sin 1'' = 4.68557$	$\sin^2 1'' = 9.3711$	$\log 4 = .6021$
$\tan\varphi = 9.75147$	$\log \tfrac{1}{3} = 9.5229$	Sum $= 0.1050$
$\cos t = 9.70852_n$		$\log(1 + 4\tan^2\varphi) = .3567$
		$\cos^2 t = 9.4170$
log 2d term $= 1.61070_n$	factor $= 9.4401$	Sum $= 9.7737$
	log 3d term $= 9.5318$	$\tan^2\varphi = 9.5029$
		Zech $= 9.9372$
		log factor $= 9.4401$

2d term $= -\ 40''.80$
3d term $= +\ \ .34$
Sum $= 5361\ .71$
log sum $= 3.72930$
$\sin t = 9\ 93425$
$\log \sec\varphi = .06002$
$\log a = 3.72357_n$ $\qquad a = -\ 5291''.4$
$\qquad a = -\ 1^\circ\ 28'\ 11''.4$

For computing a single azimuth, as in the present case, formula (498) will be preferred. For other cases, where a larger number of values are required, (499) and (500) will sometimes be found more convenient.

For the level correction

$$\frac{d}{2}[W - E]\tan\varphi = \frac{''.82}{2}[97.56 - 102.44]\tan\varphi = -\ 2.00 \times \tan\varphi = -\ 1''.13.$$

Mean reading on star + level correction	$= 337^\circ\ 19'\ 25''.3 = s.$
Mean reading on mark	$= 158\ \ 51\ \ 4\ .6 = m,$
Azimuth of star + correction for $\Sigma \Delta t^2$ + aberration	$= -1\ \ 28\ \ 10\ .9 = a.$
Azimuth of mark $= a + (m - s)$	$= 180\ \ 3\ \ 28\ .4 = A.$

The aberration, as before, is given by the formula $''.32 \cos a$.

Conditions favorable to Accuracy.

308. In order to determine the effect of small errors in φ, δ, and t upon the azimuth let us resume equations (121), viz.,

$$\begin{aligned} &(a) \quad \cos h \cos a = \sin \delta \cos \varphi - \cos \delta \sin \varphi \cos t; \\ &(b) \quad \cos h \sin a = - \cos \delta \sin t; \\ &(c) \quad \sin h = \sin \delta \sin \varphi + \cos \delta \cos \varphi \cos t; \end{aligned}$$

the azimuth being reckoned from the north point.

Differentiating (b) with respect to a and t, we have

$$da = - \frac{\cos \delta \cos t}{\cos h \cos a} dt, \quad \ldots \ldots \ldots \quad (507)$$

or

$$da = - \frac{\cos \delta \cos t}{\sin \delta \cos \varphi - \cos \delta \sin \varphi \cos t} dt. \quad \ldots \quad (508)$$

These equations show that small errors in t will produce the least error in the azimuth when the hour-angle is near 90°, i.e., when the star is at elongation, if a circumpolar star. If it is not a circumpolar star, (507) shows that h should be small, and that an error in t will have least effect on a if stars are used in which t can be near 90° and a near zero, without at the same time making $\cos h$ small. These requirements are best fulfilled by stars near the pole.

Differentiating (b) with respect to a and δ, we have

$$da = \frac{\sin \delta \sin t}{\cos h \cos a} d\delta. \quad \ldots \ldots \ldots \quad (509)$$

This shows that when the star is near elongation an error in δ produces the maximum effect on a; but if the star is observed at the same hour-angle east and west the effect of an error in δ will disappear from the mean.

Differentiating (a) with respect to a and φ, and reducing by (b) and (c), we have

$$da = - \frac{\sin h}{\cos \delta \sin t} d\varphi. \quad \ldots \ldots \ldots \quad (510)$$

This shows that an error in φ also produces the minimum effect on a when the star is at elongation; and like the effect of $d\delta$ it disappears from the mean of two determinations made at the same hour-angle east and west.

The azimuth will therefore be least affected by small errors in δ, t, and φ, if it is determined from circumpolar stars observed a like number of times at both eastern and western elongation.

Azimuth by the Sun or a Star at any Hour-angle, the Time not being Known.

309. In determining azimuths for the ordinary purposes of land-surveying or for magnetic work extreme accuracy is not required. In such cases it may be derived without a knowledge of the local time by using a theodolite and reading both horizontal and vertical circles.

Either a star or the sun may be employed; in the latter case the threads are placed tangent to the limbs and a correction for semidiameter applied. The vertical thread is placed alternately tangent to the first and second limbs, and the horizontal thread tangent to the upper and lower limbs. If the observations are arranged symmetrically with respect to the limbs the semidiameter will disappear from the mean.

The azimuth of the star is computed as follows:

The last of equations (113), substituting $90° - z$ for h, and recollecting that the azimuth is reckoned from the north point, is

$$\sin \delta = \cos z \sin \varphi + \sin z \cos \varphi \cos a.$$

δ and φ are known; z is the zenith distance measured as indicated, and corrected for refraction, and, when the sun is employed, for parallax. We therefore solve the equation for a.

Writing $\cos a = 1 - 2 \sin^2 \frac{1}{2}a$, then $\cos a = -1 + 2 \cos^2 \frac{1}{2}a$, we find by a familiar reduction

$$\left.\begin{aligned}
\sin \tfrac{1}{2}a &= \sqrt{\frac{\cos \frac{1}{2}(z + \varphi + \delta) \sin \frac{1}{2}(z + \varphi - \delta)}{\sin z \cos \varphi}}; \\
\cos \tfrac{1}{2}a &= \sqrt{\frac{\sin \frac{1}{2}(z - \varphi + \delta) \cos \frac{1}{2}(z - \varphi - \delta)}{\sin z \cos \varphi}}; \\
\tan \tfrac{1}{2}a &= \sqrt{\frac{\cos \frac{1}{2}(z + \varphi + \delta) \sin \frac{1}{2}(z + \varphi - \delta)}{\cos \frac{1}{2}(z - \varphi - \delta) \sin \frac{1}{2}(z - \varphi + \delta)}}.
\end{aligned}\right\} \quad (511)$$

The azimuth of the star may be computed by either of these formulæ, the last being most accurate. As this method will not be employed when extreme accuracy is required this consideration will have less weight than in other cases.

When the sun is employed the correction for semidiameter is obtained as follows:

Let $S =$ the sun's semidiameter taken from the ephemeris.

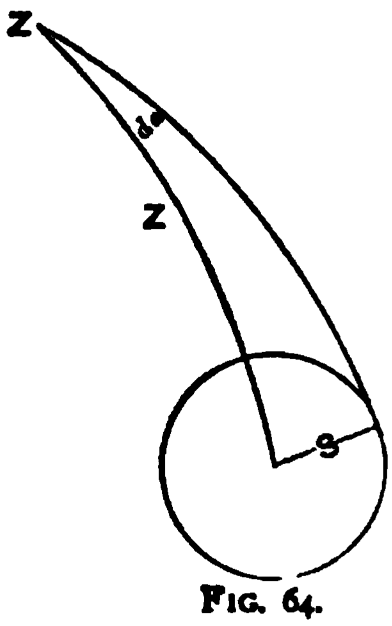

Fig. 64.

Then from the right-angle triangle formed by the great circles joining the zenith, centre, and limb of the sun we have, calling the angle at the zenith δa,

$$\sin S = \sin z . \sin \delta a,$$

or

$$\delta a = \pm \frac{S}{\sin z}, \quad . \quad . \quad . \qquad (512)$$

the proper algebraic sign being obvious.

If the time is also required, we derive it from the measured altitudes by the method of Articles (124) and (125).

Conditions favorable to Accuracy.

310. In order to investigate the effect upon the azimuth of small errors in assumed latitude and zenith distance we resume the fundamental equation

$$\sin \delta = \cos z \sin \varphi + \sin z \cos \varphi \cos a.$$

Differentiating first with respect to a and z, then with respect to a and φ, we have

$$\left.\begin{aligned} d_z a &= [-\tan \varphi \operatorname{cosec} a + \cot z \cot a] dz ; \\ d_\varphi a &= [-\tan \varphi \cot a + \cot z \operatorname{cosec} a] d\varphi . \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (513)$$

The coefficients of both dz and $d\varphi$ diminish as a and z approach 90°; also the coefficients have opposite signs for $a = 90°$ and $a = 270°$. Therefore by selecting stars which cross the prime vertical at as low altitudes as may be consistent with good definition, and observing at about the same distance from the meridian east and west, the best results will be obtained.

When the sun is used it should be observed as near the prime vertical as possible, east and west.

When an ordinary surveyor's theodolite is used there will be no provision for

illuminating the field; this may, however, be done by a bull's-eye lantern held in front and a little to one side of the object-glass.

Example.

311. Station, Capital, Washington, D. C.

Sun near prime vertical, August 15, A.M., 1856. Observer, Charles A. Schott.
Instrument, 5-inch theodolite. Longitude $5^h\ 8^m\ 1^s$ west of Greenwich.

Chronometer* Time.	Horizontal Circle. A	Horizontal Circle. B	Vertical Circle. A	Vertical Circle. B
	☉'s upper and first limb. Telescope D.			
$5^h\ 2^m\ 53^s$	25° 24′ 30″	205° 24′ 30″	61° 56′ 0″	61° 56′ 0″
5 34	25 50 45	205 51 30	61 24 30	61 25 0
6 55 .5	26 4 30	206 5 15	61 8 45	61 9 30
	☉'s lower and second limb. Telescope R.			
5 9 12	205 54 15	25 54 00	61 19 30	61 18 30
10 32	206 7 15	26 6 45	61 4 00	61 3 0
11 42	206 18 30	26 18 15	60 50 00	60 49 45

Thermometer 73°.

Barometer 30 inches.

We also have $\varphi = 38°\ 53'\ 18''$ Mean chronometer time* $= 5^h\ 7^m\ 48^s.1$
$\delta = 13\ 55\ 33$ Horizontal circle $= 25°56'40''$
Sun's eq. parallax $\pi = 8''.5$ Vertical circle $= 61\ 17\ 02$
Refraction $= r = +\ 1\ 41\ .7$
Parallax $-\ 7\ .4$
Corrected zenith dist. $= 61°18'36''$

We compute azimuth of star by the last of (511):

$$\tfrac{1}{2}(z + \varphi + \delta) = 57°\ 3'\ 44'' \qquad \cos = 9.73538$$
$$\tfrac{1}{2}(z + \varphi - \delta) = 43\ \ 8\ 11 \qquad \sin = 9.83489$$
$$\tfrac{1}{2}(z - \varphi - \delta) = 4\ 14\ 53 \qquad \sec = .00120$$
$$\tfrac{1}{2}(z - \varphi + \delta) = 18\ 10\ 26 \qquad \text{cosec} = .50598$$
$$.07745$$
$$\tfrac{1}{2}a = 47°\ 33'\ 3''.0 \qquad \tan \tfrac{1}{2}a = .03872.5$$
$$a = 95\ \ 6\ \ 7$$
$$\text{Hor. circle} = 25\ 56\ 40$$
$$290\ 50\ 33 = \text{Reading of circle for north point.}$$

* A sidereal chronometer was used. The time is only required for taking δ from the ephemeris and need not be very exact. When a star is used no record of the time is required.

Azimuth by the Transit Instrument.

312. It has already been shown, in connection with the general theory of the transit instrument, how the azimuth of the line of collimation is determined, either by special observations made for this purpose or from a series of transits reduced by least squares. If now the direction of this line is fixed by a meridian mark, we have the azimuth of the mark. Such a determination, though not of the highest order of accuracy, is sufficient for many purposes.

When the greatest precision is required, the telescope must be provided with an eye-piece micrometer moving a vertical thread. The instrument will generally be mounted either in the meridian or in the vertical plane of a circumpolar star at elongation.

313. *Azimuth by a Close Circumpolar Star near Culmination.* The instrument is set up and adjusted as already explained in Articles 166–9. The mark whose azimuth is to be determined must be placed so near the meridian that it may be well observed without changing the azimuth of the instrument. In positions where a distant meridian mark is not available a collimating telescope may be used, in which case the firmest possible mounting will be required for both transit and collimator.

The observations will be made as follows: A short time before the star's culmination the telescope is directed to the mark and a series of readings taken with the micrometer, both in direct and reverse position of the instrument. The level is then read and a series of transits observed over the micrometer-thread, which is moved forward successively one turn or less. The instrument may be reversed or not at the middle of the series. The level is again read and a series

of readings on the mark taken. Transits of zenith and equatorial stars will also be observed for determining the clock correction.

314. *Method of Reduction.* The value of one revolution of the micrometer-screw is required. If not previously known this may be derived from the observed transits of the star, by the same method used for determining the equatorial intervals of the transit-threads, viz.:

Let $I =$ the interval of time required for the star to pass over the space corresponding to one revolution of the screw.

Then, eq. (291), $$R = 15 I \cos\delta \sqrt[3]{\cos I}. \quad . \quad . \quad . \quad . \quad (514)$$

$\sqrt[3]{\cos I}$ being taken from table Art. 174 when it differs appreciably from unity. R, the value of one revolution, will be expressed in seconds of arc.

The collimation constant may be derived either from the transits of the star, the instrument being reversed at the middle of the series, or by means of the readings on the mark in the two positions as explained in Art. 182.

When the transits of the star are used for the purpose the formula for c is (see Art. 185)

$$c = \tfrac{1}{2}(T' - T)\cos\delta + \tfrac{1}{4}(T' - T)\delta T \cos\delta + \tfrac{1}{2}(b' - b)\cos(\varphi - \delta).$$

It is well to derive c from both the star and mark, the two determinations mutually checking each other.

315. The mean of the observed times must next be reduced to the time over the line of collimation of the telescope.

Let $r_1, r_2, \ldots r_m$ = the successive readings of the micrometer;

$t_1, t_2, \ldots t_m$ = chronometer times of observation;

r_o and t_o = micrometer reading and time for line of collimation.

$$r_0 = \frac{r_1 + r_2 + \ldots + r_m}{m}; \qquad t_0 = \frac{t_1 + t_2 + \ldots + t_m}{m}.$$

Then, from (295), $t_c - t_0 = R\frac{r_c - r_0}{15} \sec\delta \sqrt[2]{\sec(t_c - t_0)}$. (515)

The factor $\sqrt[2]{\sec(t_c - t_0)}$ is taken from the table Art. 174 if it differs appreciably from unity. We thus have T, the chronometer time of transit over the line of collimation.

Then, equations (284), (285), (287),

$$\alpha = T + \varDelta T + Aa + Bb + C(c - {}^s.021 \cos\varphi);^*$$

in which $A = \sin(\varphi - \delta)\sec\delta$, $B = \cos(\varphi - \delta)\sec\delta$, $C = \sec\delta$.

Let $\tau = \alpha - [T + \varDelta T + Bb + C(c - {}^s.021 \cos\varphi)]$; (516)

that is, the algebraic sum of the known terms.

Then $$a = \frac{15\tau}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (517)$$

is the expression for the azimuth of the star in seconds of arc. It will, however, be remembered that in the theory of the

---

* If the mean of the times has been reduced to the line of collimation as supposed above, c will be zero: if not, $c = t_c - t_0$.

transit instrument where the above formula is derived, a is considered plus when the south end of the telescope deviates to the east. For present purposes, therefore, the algebraic sign must be reversed, giving for azimuth of star

$$a' = -\frac{15\tau}{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (518)$$

The azimuth of the mark then follows at once from the difference between the micrometer readings on the mark and star.

By observing the same star at both upper and lower culmination the effect of any constant error in the right ascension or clock correction will be eliminated from the mean.

EXAMPLE.

δ Ursæ Minoris at Lower Culmination. *51 Cephei at Upper Culmination.*
1882, March 20. Instrument, Simms Transit C. S. No. 8.

Chronometer.	Mark.		Chronometer.	δ Ursæ Minoris.		Level.		51 Cephei.		Level.	
	Lamp E.	Lamp W.		Micrometer.	Chronometer.	E.	W.	Micrometer.	Chronometer.	E.	W.
$5^h\ 20^m$	18.760	12.670	$5^h\ 40^m$	18.22	$6^h 18^m 44^s$			13.22	$6^h 27^m 34^s$		
	18.760	12.665		17.72	19 11			13.72	28 8		
	18.750	12.670		17.22	19 37 .5	49.8	48.1	14.22	28 40 .5	38.0	63.0
	18.760	12.665		16.72	20 4 .5	35.0	63.0	14.72	29 15	53.5	49.0
	18.750	12.675		16.22	20 31 .5	50.8	48.2	15.22	29 48	39.0	63.5
	18.750	12.675		15.72	20 59	36.3	63.0	15.72	30 22	53.5	49.0
	18.751	12.670		15.22	21 25 .5			16.22	30 54		
	18.751	12.672		14.72	21 52			16.72	31 29		
	18.758	12.670		14.22	22 19			17.22	32 1 .5		
$5^h\ 30^m$	18.750	12.665	$5^h\ 50^m$	13.72	22 46			17.72	32 36		
				13.22	$6^h 23^m 13^s$			18.22	$6^h 33^m 10^s$		
Means	18.754	12.670		15.72	$6^h 20^m 58^s.46$	42.98	55.58	15.72	$6^h 30^m 21^s.64$	46.00	56.12

One division of level = 1''.

	δ Ursæ Minoris.	51 Cephei.
$\varphi = 29°\ 7'\ 30''$	$\delta = 93°\ 24'\ 24''$	$\delta = 87°\ 15'\ 33''$
$\Delta T = \quad -51^s.30$	$\alpha = \ 6^h\ 20^m\ 5^s.61$	$\alpha = \ 6^h\ 29^m 33^s.15$

By the foregoing formulæ we compute—

δ Ursæ Minoris.	51 Cephei.
$A = + 15.16$	$A = - 17.76$
$B = - 7.30$	$B = + 11.04$
$C = - 16.83$	$C = + 20.91$
$b = + 6''.30 = 0^s.42$	$b = + 5''.06 = 0^s.337$

We now derive the value of the micrometer-screw from the observed transits of each star, as follows: Subtracting in each case the first time from the seventh, the second from the eighth, etc., we have the following values:

δ Ursæ Minoris.

Nos.	Interval.
7 — 1	$2^m\ 41^s.5$
8 — 2	2 41.0
9 — 3	2 41.5
10 — 4	2 41.5
11 — 5	2 41.5

$\log I = 1.73078$
$\log 15 = 1.17609$
$\cos\delta = 8.77395$
$\log R = 1.68082$
$R = 47.95$

3 turns $= 2^m\ 41^s.4$
1 turn $= 53.80 = I$

51 Cephei.

Nos.	Interval.
7 — 1	$3^m\ 20^s$
8 — 2	3 21
9 — 3	3 21
10 — 4	3 21
11 — 5	3 22

$\log I = 1.82607$
$\log 15 = 1.17609$
$\cos\delta = 8.67961$
$\log R = 1.68177$
$R = 48.06$

3 turns $= 3^m\ 21^s$
1 turn $= 67.0 = I$ Mean $R = 48''.00$

The mean of the readings on the mark E. and W. gives $r_c = 15.712$. Therefore, by formulæ (515), (516), and (518)—

	δ Ursæ Minoris.	51 Cephei.
Observed time	$= 6^h\ 20^m\ 58^s.46$	$6^h\ 30^m\ 21^s.64$
$t_c - t_0$	$= +$ 42	$-$ .54
T	$= 6^h\ 20^m\ 58^s.88$	$6^h\ 30^m\ 21^s.10$
ΔT	$= -$ 51.30	$-$ 51.30
bB	$= -$ 3.07	$+$ 3.73
*Cc'	$= +$ .31	$-$ .38
α	$= 6\ 20\ 5.61$	$6\ 29\ 33.15$
τ	$= + {}^s.79$	$\tau =$ 0.00
a'	$= - 0''.78$	$a' =$ .00

* c is $= 0$, since we have reduced the times to the axis of collimation. Therefore

$$c' = - {}^s.021 \cos\phi$$
$$= - .018.$$

Mark west of collimation axis 3.042 revolutions	=	$2' \ 26''.02$
Mean value of a'	= −	$.39$
Azimuth of mark	= −	$2 \ 25 \ .63$

316. If the telescope is not provided with an eye-piece micrometer, the azimuth-screw at the end of the axis may be employed (see description of instrument, Art. 158). The mark in this case must be quite near the meridian, as the range of the screw is small. The method of observing is the same as that described in the last article.

Determination of the Value of the Screw. For this purpose a series of transits of a circumpolar star near culmination will be observed, extending over the entire available range of the screw. It will be as well not to extend it to the extreme limit in either direction.

Let $M =$ the micrometer reading at any observed time t;
$M_0 =$ the micrometer reading at time of culmination t_0;
$R =$ the value of one revolution of screw.

Then since the screw moves the instrument in azimuth, we have, by (517),

$$R(M - M_0) = \frac{1}{A}(15\tau),$$

where $\tau = t - t_0$.

This is a little more accurately written

$$R(M - M_0)\sin 1'' = \frac{1}{A}\sin(15\tau),$$

or

$$R(M - M_0) = \frac{1}{A}[15\tau - \tfrac{1}{6}(15\tau)^3 \sin^2 1''];$$

$$R(M - M_0) = \frac{15}{A}[\tau - \tfrac{1}{6}(15 \sin 1'')^2\tau^3]. \quad . \quad . \quad . \quad . \quad (519)$$

Where the log $\frac{1}{6}(15 \sin 1'')^2 = 0.94518 - 10$, and the quantity $\frac{1}{6}(15 \sin 1'')^2\tau^3$ may be taken from the table Art. 275. When this correction is appreciable it will be convenient to apply it directly to the observed times, when we shall have these times reduced to what they would have been if the star had moved uniformly in a great circle. The method of combining these reduced times is the same as that illustrated in the preceding article.

EXAMPLE.

δ Ursæ Minoris near lower culmination, February 5, 1869.
Chronometer time of lower culmination, $6^h\ 15^m\ 48^s$.

Micr.	Chron. time	Time from culmination.	Red'n.	Red'd time.	Time of 3 turns.		
t.	*h. m. s.*	*m.*	*s.*	*h. m. s.*	*t.*	*t.*	*m. s.*
21.0	5 55 57	19.9	+ 1.5	5 55 58.5	21	to 18	9 62.7
20.5	57 40	18.1	1.1	57 41.1	20.5	17.5	9 59.5
20.0	59 23.5	16.4	0.8	59 24.3	20	17	9 55.7
19 5	6 01 02.5	14.8	0.6	6 01 03.1	19.5	16.5	9 56.9
19.0	02 41.5	13.1	0.4	02 41.9	19	16	9 57.1
18.5	04 21.0	11.5	0.3	04 21.3	18.5	15.5	9 51.7
18.0	06 01.0	9.8	0.2	06 01.2	18	15	9 55.8
17.5	07 40.5	8.1	0.1	07 40.6			
17.0	09 20.0	6.5	0.0	09 20.0		Mean	9 57.07
16.5	11 00.0	4.8	0.0	11 00.0			
16.0	12 39.0	3.2	0.0	12 39.0			
15.5	14 13.0	1.6	0.0	14 13.0			
15.0	15 57.0	0.1	0.0	15 57.0			

Time of three revolutions, $597^s.07$

One revolution $= r = 199^s.0$ $\qquad \log = 2.29885$

$$\log 15 = 1.17609$$

$$\log \frac{1}{A} = 8.82216$$

$$\log R = 2.29710$$

$$R = 198''.2$$

Star's declination $= \delta = 93°\ 23'\ 48''$

Latitude $= \varphi = 30\ \ 13\ \ 54$

The computation of the azimuth of the star at the mean of the observed times, and the determination of the azimuth of the mark from the combination of the readings on star and on mark, will require no further illustration.

Azimuth by Circumpolar Star at any Hour-angle.

317. When extreme accuracy is required the instrument must be provided with an eye-piece micrometer. The mark, of course, must be near the line of collimation. The method of observing will be the same as with the theodolite, Art. 298, except that the readings are made with the micrometer.

If there is no eye-piece micrometer the azimuth-screw may

be used, in which case the reduction will be precisely the same as that given for the theodolite, formulæ (XXIV), Art. 304.

When the micrometer is employed the reduction will be as follows:

In the figure *NESW* represents the horizon, *P* the pole, *s* the star, *Z* the zenith, μ the mark, *CZ* the direction of the line of collimation, w' the point where the west end of axis pierces the celestial sphere.

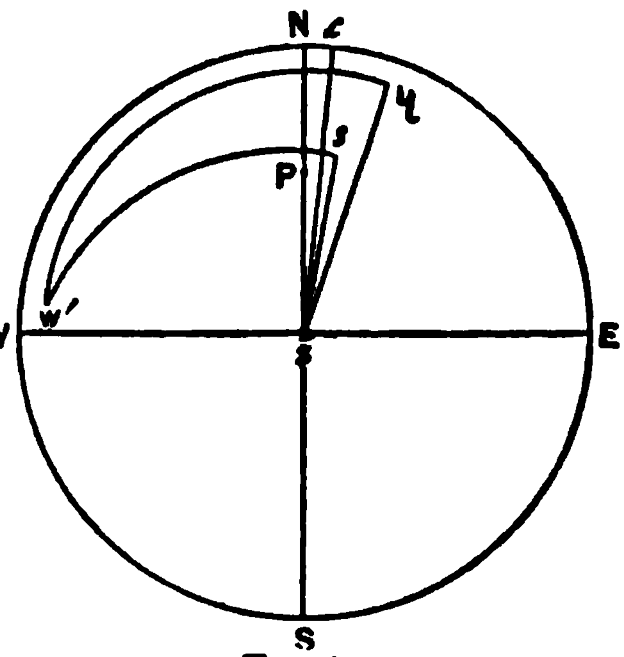

Fig. 65.

Let M_0 = micrometer reading on line of collimation;
M = micrometer reading on star;
M' = micrometer reading on mark;
R = value of one revolution of screw;
b = elevation of west end of axis.

Then from the micrometer and level readings we require the expression for the difference in azimuth of s and μ.

Let
$$R(M - M_0) = m;$$
$$R(M' - M_0) = m'.$$

Then from figure,

$$\mu z = z'; \quad sz = z; \quad zw' = 90^\circ - b;$$
$$w's = 90^\circ + m; \quad w'\mu = 90^\circ + m'; \quad w'zs = 90^\circ + a_1; \quad w'z\mu = 90^\circ + a_1'.$$

Then if a = azimuth of star, a' = azimuth of mark,

$$a - a' = a_1 - a_1' = \text{required difference of azimuth.}$$

From triangle $w'zs$,

$$-\sin m = \sin b \cos z - \cos b \sin z \sin a_1.$$

From triangle $w'z\mu$,

$$-\sin m' = \sin b \cos z' - \cos b \sin z' \sin a_1'.$$

m, m', b, a_1, and a_1' will always be small quantities; therefore the above equations may be written

$$\begin{aligned} -m &= b \cos z - a_1 \sin z; \\ -m' &= b \cos z' - a_1' \sin z'. \end{aligned}$$

From these equations we obtain

$$a_1 - a_1' = \frac{m}{\sin z} - \frac{m'}{\sin z'} + b\frac{\sin (z' - z)}{\sin z \sin z'}. \quad . \quad (520)$$

The micrometer reading is supposed to increase with the azimuth; if the opposite is the case the signs of m and m' will be changed.

b includes the correction for inequality of pivots; also for flexure, if the instrument is of the form shown in Fig. 28. (See Art. 192.) Thus the complete expression for b is

$$b = \frac{d}{2}(W - E) + p + f. \quad . \quad . \quad . \quad . \quad (521)$$

p is the correction for inequality of pivots, and f the flexure.

The azimuth of the star being computed by any of the methods before given, we have by (520) the required azimuth of the mark.

318. *A Circumpolar Star near Elongation.* It will be best when practicable to observe the stars near the time of elon-

gation. The readings on the star may then be reduced to the reading at elongation as follows: In the figure let

s_e = position of the star at time T_e = elongation;
s = position of the star at time T.

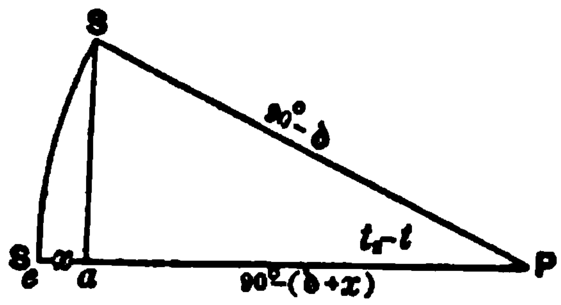

FIG. 68.

Then $s_e a = x$ is the correction required to reduce the reading at s to the reading at elongation.

From the right-angle triangle sPA, we have

$$\cos (t_e - t) = \tan \delta \cot (\delta + x).$$

From this, by the process given for deriving equation (483)

$$x = \tfrac{1}{2} \sin 2\delta \frac{2 \sin^2 \tfrac{1}{2}(t_e - t)}{\sin 1''}. \quad . \quad . \quad . \quad . \quad (522)$$

On account of the rapidity and accuracy with which the micrometer readings may be made several sets may be taken at one elongation if thought desirable.

Example.

319. In Vol. XXXVII, *Memoirs Royal Astronomical Society*, Captain Clarke gives among others the following observation of Polaris:

Station Findlay Seat, 1868, October 23.

Position E

$W - E$ =	$-$ 1.30	Latitude φ =	57° 34′ 50″.0
$M - M_0$ =	580.19	Declination δ =	88 36 34 4
$M' - M_0$ =	$-$ 77.01	Right ascension α =	$1^h\ 11^m\ 57^s.46$
Sidereal time =	$18^h\ 41^m\ 30^s.11$	Hour-angle t =	17 29 32 .65
		t =	262° 23′ 9″.75
		Zenith dist. of mark z' =	93 2

We also have	One division of level	$= d =$	1″.810
	One division of microm. screw	$= R =$	″.8345
	Inequality of pivots	$p =$	″.650
	Flexure	$f =$	3″.171

The observations given are the means of a series taken in the following order:

1st. Level.
2d. Mark.
3d. Direct telescope to star and read level.
4th. Three readings on star.
5th. Level.
6th. Mark.
7th. Level.

The instrument is then reversed and another series taken in the same order. The level reading given is the mean of the four above indicated.

We shall first reduce the observation by computing the azimuth of the star at the instant of observation.

As both zenith distance and azimuth are required, equations (II), Art. (65), may be employed. These equations are rewritten here for convenience.

$$\tan M = \frac{\tan\delta}{\cos t};$$

$$\tan a = \frac{\cos M}{\sin(\varphi - M)} \tan t;$$

$$\tan h = \frac{\cos a}{\tan(\varphi - M)}.$$

Proof: $$\frac{\cos M}{\sin(\varphi - M)} = \frac{\cos\delta \cos t}{\cos h \cos a}.$$

By means of these formulæ we readily find

$$\begin{aligned} a &= 2^\circ\ 33'\ 23''.58 \\ h &= 57\ \ 22\ \ 13\ .38 \\ z &= 32\ \ 37\ \ 47 \end{aligned}$$

By formula (521),

$$\begin{aligned} b &= -\ .65 \times 1''.81 + 3''.171 + ''.650 = 2''.645 \\ m &= 580.19 \times .8345 \log = 2.68500 \\ m' &= -\ 77.01 \times .8345 \log = 1.80798_n \end{aligned}$$

$$\begin{aligned} \frac{m}{\sin z} &= +\ 14'\ 57''.92 \\ -\frac{m'}{\sin z'} &= +\ \ 1\ \ 4\ .36 \\ b\frac{\sin(z' - z)}{\sin z \sin z'} &= +\ \ \ \ 4\ .27 \\ a - a' &= 16\ \ 6\ .55 \end{aligned}$$

Azimuth of star $= a = 2\ \ 33\ \ 23\ .58$
Azimuth of mark $a' = 2^\circ\ 17'\ 17''.03$

This still requires the correction for diurnal aberration, viz., $+\ 0''.32 \cos a$.

320. The observations of the foregoing example are taken too far from elongation for reduction by formula (522), but they will serve to illustrate the method. We compute the azimuth and time of elongation by the formulæ

$$\sin a_e = \cos\delta \sec\varphi$$
$$\cos t_e = \cot\delta \tan\varphi$$
$$\text{Time of elongation } T_e = \alpha - t_e$$

We readily find

$$a_e = 2^\circ\ 35'\ 39''.11$$
$$T_e = 19^h\ 20^m\ 43^s.13$$
$$\text{Time of observation } T = 18\quad 41\quad 30\ .11$$
$$T_e - T = t_e - t = \quad 39\quad 13\ .02$$

Then by (522), $\log \dfrac{2\sin^2 \frac{1}{2}(t_e - t)}{\sin 1''} = 3.47892$

$2\delta = 177^\circ\ 13'\ 8''.8$ $\qquad \sin = 8.68589$

$$\log \tfrac{1}{2} = 9.69897$$
$$\log x = 1.86378$$

$$\text{Reduction to elongation} = x = 73''.08$$
$$\text{Micrometer reading on star } m = 484\ .18$$
$$\text{Reading at elongation} = m + x = 557\ .26$$

$m + x$ now takes the place of m in equation (520). When the observation is within a few minutes of elongation we take for z the zenith distance at time of elongation; but in the present example this will not be admissible. Using for z the value derived in the previous reduction, we have

$$\frac{m+x}{\sin z} = \quad 17'\ 13''.48$$
$$-\frac{m'}{\sin z'} = \quad 1\quad 4\ .36$$
$$b\,\frac{\sin(z'-z)}{\sin z \sin z'} = \quad 4\ .27$$
$$a - a' = \quad 18\quad 22\ .11$$
$$a = 2\quad 35\quad 39\ .11$$
$$a' = 2\quad 17\quad 17\ .00$$
$$\text{Aberration} = \quad 0\ .32$$
$$\text{Azimuth of mark} = 2^\circ\ 17'\ 17''.32$$

CHAPTER X.

PRECESSION.—NUTATION.—ABERRATION.—PROPER MOTION.

321. The heavenly bodies which are employed for any of the purposes treated of in the foregoing pages are, first, the sun, moon, and planets; and second, the fixed stars.

In solving the problems of practical astronomy, we have in most cases supposed the position of the object observed to be accurately known. The co-ordinates which we have in most cases employed are the right ascension and declination.

The motions of the sun, moon, and planets are of a complicated character, and the prediction of their places for any given instant belongs to another department of astronomy. When their co-ordinates are required for any of the foregoing purposes they will simply be taken from the American Ephemeris or a similar publication.

With the fixed stars the case is different; their relative positions change very slightly from age to age. In most cases no change at all has been discovered.

The apparent co-ordinates of all stars, however, are varying slowly but continuously, owing to two causes which are independent of the star's motion, viz.: first, a shifting of the planes of reference, giving rest to precession and nutation; and second, an apparent motion of the star, due to the earth's motion combined with the progressive motion of light, called aberration.

Secular and Periodic Changes.

322. The small changes to which many of the quantities employed in astronomical operations are subject are divided into two classes, viz., secular and periodic.

Secular changes are those which are progressive in the same direction from year to year, requiring long periods of time—*seculæ*—to complete a cycle, so that during short periods the changes may be considered as proportional to the time.

Periodic changes are those which complete their cycle in a comparatively short time, and where the motion from maximum to minimum, or the reverse, is so rapid that the change cannot be considered proportional to the time, except for very short intervals.

The *precession of the equinoxes* produces a secular change in the co-ordinates of all stars referred either to the equator or ecliptic. It will be remembered that this is the name given to the slow motion which takes place in the line of intersection of the ecliptic and equator, causing the pole of the equator to describe a circle about the pole of the ecliptic in a period of about 25,000 years. This motion is due to the spheroidal form of the earth, in consequence of which one component of the attractive force of the sun and moon tends to draw the equator into coincidence with the ecliptic. This component of the attraction is not uniform. It is a maximum when the sun and moon are farthest from the plane of the equator, and a minimum when they are in the equator.

Nutation. The want of uniformity in the forces producing precession gives rise to small changes of short period which together are called nutation. There are a number of small changes embraced under this head, but the principal one causes the actual pole of the earth's equator to describe a

small ellipse about the mean pole; the major axis of this ellipse is directed to the pole of the ecliptic and embraces about 18″ of arc. The length of the conjugate axis is about 14″. The period is about 18 years.

Mean, Apparent, and True Place of a Star.

323. Suppose the right ascension and declination of a star to be accurately observed with a suitable instrument: the place of the star so determined will be the *apparent place.*

The apparent direction of the star is affected by aberration, the effect of which will be considered more fully hereafter. If we apply to the apparent right ascension and declination the corrections necessary to free them from the effect of aberration, we have the *true place.*

If now we apply to this true place the small periodic corrections called nutation, we have as the result the *mean place.*

In catalogues of stars the right ascensions and declinations are given, referred to the mean equator and equinox for the beginning of the year of the catalogue. If then the apparent place of the star is required for any given date, the precession must be applied to reduce the mean place of the catalogue to the mean place at the given date; the nutation and aberration must then be applied to reduce the mean place to apparent place. The determination of these reductions will be the immediate object of the present chapter.

Precession.

324. The change in the position of the equinoxes is due to two causes: first, the action of the sun and moon; and second, that of the planets. The first gives rise to luni-solar precession, and the second to planetary precession.

By the processes of physical astronomy it is shown that the attractions of the sun and moon upon the matter accumulated about the earth's equator, which gives it its spheroidal form, produce a slow retrograde motion in the line of intersection of the equator and ecliptic, without changing the angle between these planes. As the celestial longitudes are measured from this line, or rather from one of the points where it pierces the celestial sphere, the effect is a constant increase in the longitudes, with no change in the latitudes.

This is *luni-solar precession*, and is due simply to a motion of the equator.

The attractions exerted upon the earth by the other planets of the solar system tend to change the plane in which it revolves about the sun, without changing the position of the equator; this change is relatively small and tends to diminish the right ascensions without affecting the declinations.

The latter is called *planetary precession* and is due to a motion of the ecliptic.

The combined effect of the luni-solar and planetary precession is to produce small secular changes in the right ascensions and declinations, also of the longitudes and latitudes of all stars, and in the obliquity of the ecliptic.

325. In order to be able to determine the position of the equator or the ecliptic at any given instant it will be necessary to select the positions of those circles at some given epoch as fixed circles to which all motions may be referred. Let these fundamental circles be the mean equator and ecliptic for 1800.0.

In Fig. 67, let AA_0 be the mean equator for 1800.0;

$A'A''$, the mean equator for $1800 + t$.

Let EE_0 and EE' be the mean ecliptic for 1800.0 and $1800 + t$ respectively.

Then BD, the part of the fixed ecliptic over which the

point of intersection has moved, is the luni-solar precession in t years $= \psi$.

Let D' be the point on the movable ecliptic which coincided with D when the ecliptic had the position EE_0.

Then CD' is the general precession for t years $= \psi_1$.

Since B is the point of the equator which at the instant 1800.0 was at D, BC is the arc of the equator over which

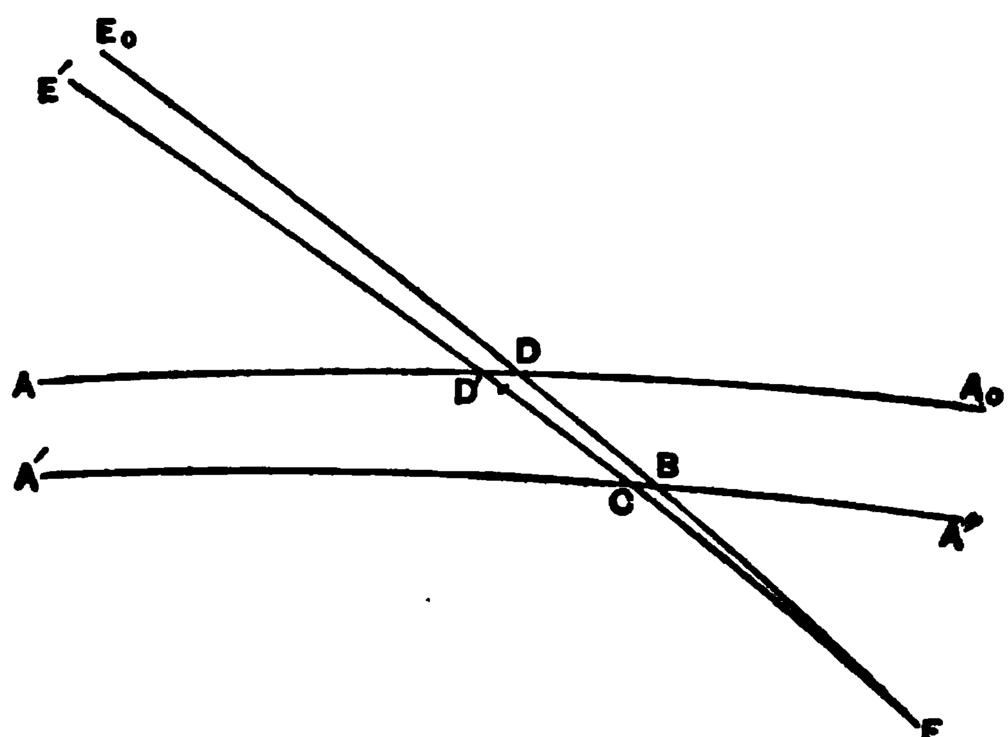

FIG. 67.

the intersection with the ecliptic has moved in a forward direction.

BC is therefore the planetary precession in the interval t years $= \vartheta$.

Let $\omega_0 =$ the mean obliquity of the ecliptic for 1800.0 $= A_0DE$;

$\omega_1 =$ the obliquity of the fixed ecliptic for $1800 + t$ $= A''BE$;

$\omega =$ the mean obliquity of the movable ecliptic for $1800 + t = A''CE$;

$\pi =$ the inclination of the mean ecliptic for $1800 + t$ to the fixed ecliptic $= BEC$.

D is the mean equinox of 1800; C is the mean equinox of $1800 + t$.

Since longitudes are reckoned in the direction DE_0, E will be the descending node of the movable on the fixed ecliptic.

Let Π = the longitude of the ascending node of the movable on the fixed ecliptic, reckoned from the mean equinox of 1800.

Then $\Pi = 180° - DE$.

326. The determination of the values of the above constants, by means of which the position of the mean ecliptic and equator at any time $1800 + t$ can be determined in reference to the fixed ecliptic and equator of 1800.0, belongs to the department of physical astronomy. Three different series of values have been quite extensively employed, viz., those of Bessel, Struve and Peters, and Leverrier. Bessel's values are given for the mean ecliptic and equinox of 1750, those of Struve and Peters for 1800, and Leverrier's for 1850.0. The values which we shall employ are those of Struve and Peters, being those which are more extensively used at present than either of the others. If, however, it is preferred to use other values, it will be a simple matter to make the necessary changes in the formulæ which will be derived. The values are as follows:*

$$\left.\begin{aligned}
\psi &= 50''.3798t - 0.000\,1084t^2;\\
\psi_1 &= 50''.2411t + 0.000\,1134t^2;\\
\omega_0 &= 23°\,27'\,54''.22;\\
\omega_1 &= \omega_0 + .000\,00735t^2;\\
\dagger\ \omega &= \omega_0 - ''.4738t - .000\,0014t^2;\\
\Pi &= 172°\,45'\,31'' - 8''.505t;\\
\pi &= ''.4776t - ''.000\,0035t^2;\\
\vartheta &= 0''.15119t - .000\,24186t^2.
\end{aligned}\right\} \quad . \quad . \quad . \quad (523)$$

* Dr. C. A. F. Peters' Numerus Constans Nutationis, p. 66 et 71.

† In the American Ephemeris the value of the annual diminution employed is 0''.4645, instead of ''.4738. The difference is so small as to be practically almost inappreciable.

Bessel gives the following values for the epoch 1750:*

$$\left.\begin{aligned}
\psi &= 50''.37572t - ''.000\,1217945t^2;\\
\psi_1 &= 50\ .21129t + .000\,1221483t^2;\\
\omega_1 &= 23°\ 28'\ 18''.0 + .000\,00984233t^2;\\
\omega &= 23\ \ 28\ \ 18\ .0 - .48368t - .000\,00272295t^2;\\
\Pi &= 171°\ 36'\ 10'' - 5''.21t;\\
\pi &= 0''.48892t - .000\,0030719t^2;\\
\vartheta &= 0\ .17926t - .000\,2660394t^2.
\end{aligned}\right\}\quad(524)$$

The following are Leverrier's values, the epoch being 1850:

$$\left.\begin{aligned}
\psi &= 50''.36924t - ''.000\,10881t^2;\\
\psi_1 &= 50\ .23465t + .000\,11288t^2;\\
\omega_1 &= 23°\ 27'\ 31''.83 + .000\,00719t^2;\\
\omega &= 23\ \ 27\ \ 31\ .83 - .47593t - ''.000\,00149t^2;\\
\Pi &= 173°\ 0'\ 12'' - 8''.694\,t;\\
\pi &= 0''.47950t - .000\,00312\,t^2;\\
\vartheta &= 0\ .14672t - .000\,24174\,t^2.
\end{aligned}\right\}\quad(525)$$

Assuming the values of the above quantities to be known, we may now solve the following problems.

327. *Problem First.* To find the precession in longitude and latitude for any star between 1800.0 and $1800 + t$.

Let the star be referred to a system of rectangular axes, the fixed ecliptic for 1800 being the plane of XY, the positive axis of X being directed to the ascending node of the ecliptic of $1800 + t$ on the fixed ecliptic, the positive axis of Z being directed to the pole of the fixed ecliptic.

Let L and B = the longitude and latitude for 1800. Then

$$x = \cos B \cos (L - \Pi);\quad y = \cos B \sin (L - \Pi);\quad z = \sin B. \quad (a)$$

Next, let the plane of XY be the mean ecliptic of $1800 + t$,

* Tabulæ Regiomontanæ, p. v, Introduction.

the new axis of X coinciding with the old, and the new axis of Z directed to the pole of the ecliptic of $1800 + t$.

Let λ and β = the longitude and latitude for $1800 + t$. Then

$$x' = \cos\beta\cos(\lambda - \Pi - \psi_1);\ y' = \cos\beta\sin(\lambda - \Pi - \psi_1);\ z' = \sin\beta. \quad (b)$$

Π is the same in both (a) and (b), being the value for 1800.0.

The new axes of Y and Z make the angle π with the old. Therefore

$$x' = x; \quad y' = y\cos\pi + z\sin\pi; \quad z' = -y\sin\pi + z\cos\pi. \quad (c)$$

From (a), (b), and (c),

$$\left.\begin{array}{ll} (d) & \cos\beta\cos(\lambda - \Pi - \psi_1) = \cos B\cos(L - \Pi); \\ (e) & \cos\beta\sin(\lambda - \Pi - \psi_1) = \cos B\sin(L - \Pi)\cos\pi + \sin B\sin\pi; \\ (f) & \sin\beta = -\cos B\sin(L - \Pi)\sin\pi + \sin B\cos\pi. \end{array}\right\} (526)$$

These equations are rigorous, but in practice they may be much abridged.

π is so small that no appreciable error will be involved in writing $\cos\pi = 1$, even when the interval t is several centuries.

Making $\cos\pi = 1$, and multiplying (d) by $\sin(L - \Pi)$, (e) by $\cos(L - \Pi)$, and subtracting, we have

$$\cos\beta\sin(\lambda - L - \psi_1) = \sin\pi\sin B\cos(L - \Pi).$$

Then multiplying by $\cos(L - \Pi)$ and $\sin(L - \Pi)$, and adding, we find

$$\cos\beta\cos(\lambda - L - \psi_1) = \cos B + \sin\pi\sin B\sin(L - \Pi);$$

and by division,

$$\tan(\lambda - L - \psi_1) = \frac{\sin\pi\tan B\cos(L - \Pi)}{1 + \sin\Pi\tan B\sin(L - \Pi)}.$$

Developing this into a series and writing $\sin \pi = \pi$, we have*

$$\lambda - L - \psi_1 = \pi \tan B \cos (L - \Pi) - \tfrac{1}{2}\pi^2 \tan^2 B \sin 2(L - \Pi) - \text{etc.}, \quad (527)$$

where the term in π^2 may always be omitted.

The last of (526) may be written

$$\sin \beta = \sin B - \sin \pi \cos B \sin (L - \Pi).$$

β is a function of π. Developing by Maclaurin's formula, we have

$$\beta - B = - \pi \sin (L - \Pi) + \tfrac{1}{2}\pi^2 \tan B \sin^2 (L - \Pi), \text{etc.} \quad (528)$$

Formulæ (527) and (528) solve the problem, where, as before remarked, the terms in π^2 may always be dropped.

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* This expansion, which is of frequent application, is obtained as follows:

Writing $\quad (\lambda - L - \psi_1) = x, \qquad \pi \tan B = m,$
$\quad 90° - (L - \Pi) = y,$

the above formula becomes $\tan x = \dfrac{m \sin y}{1 + m \cos y} = \dfrac{\sin x}{\cos x}.$

From this we have $\qquad \sin x = m \sin (y - x).$

Adding both members to $m \sin x$, then subtracting both members from $m \sin x$ and dividing,

$$\frac{m + 1}{m - 1} = \frac{\sin x + \sin (y - x)}{\sin x - \sin (y - x)} = \frac{\tan \frac{1}{2}y}{\tan (x - \frac{1}{2}y)}.$$

Now write $\quad \dfrac{m - 1}{m + 1} = p; \quad x - \tfrac{1}{2}y = u; \quad \tfrac{1}{2}y = v. \quad \tan u = p \tan v;$

and by Moivre's formula, equation (135),

$$\frac{e^{2u\sqrt{-1}} - 1}{e^{2u\sqrt{-1}} + 1} = p\,\frac{e^{2v\sqrt{-1}} - 1}{e^{2v\sqrt{-1}} + 1}.$$

328. *Problem Second.* To find the precession in longitude and latitude between two given dates $1800 + t$ and $1800 + t'$.

Let λ and β be the longitude and latitude for $1800 + t$;
λ' and β' be the longitude and latitude for $1800 + t'$.

Then by (527),
$$\lambda - L = \psi_1 + \pi \tan B \cos(L - \Pi);$$
$$\lambda' - L = \psi_1' + \pi' \tan B \cos(L - \Pi').$$

Subtracting,

$$\lambda' - \lambda = (\psi_1' - \psi_1) + \pi' \tan B \cos(L - \Pi') - \pi \tan B \cos(L - \Pi). \quad (529)$$

This may be placed in a better form by assuming the auxiliary equations

$$\left.\begin{aligned} a \sin A &= (\pi' + \pi) \sin \tfrac{1}{2}(\Pi' - \Pi); \\ a \cos A &= (\pi' - \pi) \cos \tfrac{1}{2}(\Pi' - \Pi). \end{aligned}\right\} \quad . \quad . \quad (530)$$

From this we find $$e^{2u\sqrt{-1}} = e^{-2v\sqrt{-1}} \frac{(p+1)e^{2v\sqrt{-1}} - (p-1)}{(p+1)e^{-2v\sqrt{-1}} - (p-1)};$$

$$e^{2(u+v)\sqrt{-1}} = \frac{1 + me^{2v\sqrt{-1}}}{1 + me^{-2v\sqrt{-1}}};$$

since $\dfrac{p+1}{p-1} = -m$.

Taking the logarithms of both members of the above and expanding,

$$2(u+v)\sqrt{-1} = me^{2v\sqrt{-1}} - \tfrac{1}{2}m^2 e^{4v\sqrt{-1}} + \tfrac{1}{3}m^3 e^{6v\sqrt{-1}}, \text{ etc.}$$
$$- me^{-2v\sqrt{-1}} + \tfrac{1}{2}m^2 e^{-4v\sqrt{-1}} - \tfrac{1}{3}m^3 e^{-6v\sqrt{-1}}, \text{ etc.}$$

Or $$u + v = m \sin 2v - \tfrac{1}{2}m^2 \sin 4v + \tfrac{1}{3}m^3 \sin 6v, \text{ etc.}$$

Writing for u, v, and m their values, we have

$$\lambda - L - \psi_1 = \pi \tan B \cos(L - \Pi) - \tfrac{1}{2}\pi^2 \tan^2 B \sin 2(L - \Pi)$$
$$- \tfrac{1}{3}\pi^3 \tan^3 B \sin 3(L - \Pi), \text{ etc.}$$

Combining these with (529), and eliminating π and π', we find

$$\lambda' - \lambda = (\psi_1' - \psi_1) + a \cos\left(L - \frac{\Pi' + \Pi}{2} - A\right)\tan B. \quad (531)$$

Similarly from (528) we have for $1800 + t$ and $1800 + t'$

$$\begin{aligned} \beta - B &= -\pi \sin(L - \Pi); \\ \beta' - B &= -\pi' \sin(L - \Pi'). \end{aligned}$$

Subtracting and eliminating π and π' by the auxiliary equations (530), we find

$$\beta' - \beta = -a \sin\left(L - \frac{\Pi' + \Pi}{2} - A\right). \quad . \quad (532)$$

For the auxiliary quantities a and A we find, from (530),

$$\tan A = \frac{\pi' + \pi}{\pi' - \pi} \tan \tfrac{1}{2}(\Pi' - \Pi).$$

If we substitute for π and π' their values from (523), neglecting the term in t^2, and recollecting that $\frac{1}{2}(\Pi' - \Pi)$ is very small, this equation may be written

$$A = \frac{t' + t}{2} \frac{\Pi' - \Pi}{t' - t} = -8''.505 \frac{t' + t}{2} \quad . \quad (533)$$

A being therefore very small even for large values of t and t', we may write $\cos A = 1$ in (530), when

$$a = \pi' - \pi = (t' - t)\,''.4776 - (t'^2 - t^2)\,''.000\,0035. \quad (534)$$

In equations (531) and (532) we may write $\lambda - \psi_1$ for L, and β for B. Introducing the auxiliary angle M such that

$$L - \frac{\Pi' + \Pi}{2} - A = \lambda - M, \quad . \quad . \quad . \quad (535)$$

and substituting in (531), (532), and (534) for ψ_1', ψ' π, Π, Struve and Peters' values—equation (523)—we have finally the following practical formulæ for computing the precession in longitude and latitude between any two intervals $1800 + t$ and $1800 + t'$:

$$\left.\begin{aligned} M &= 172^\circ\ 45'\ 31'' + t\ 50''.241 - (t' + t)\ 8''.505; \\ \lambda' - \lambda &= \quad (t' - t)\,[50''.2411 + (t' + t)\ 0''.000\ 1134] \\ &\quad + (t' - t)\,[\ 0''.4776 - (t' + t)\ 0''.000\ 0035] \cos(\lambda - M) \tan\beta; \\ \beta' - \beta &= -(t' - t)\,[\ 0''.4776 - (t' + t)\ 0''.000\ 0035] \sin(\lambda - M). \end{aligned}\right\} (536)$$

329. If we divide the expressions for $(\lambda' - \lambda)$ and $(\beta' - \beta)$ by $(t' - t)$, and then make $t = t'$, we shall have the values of $\frac{d\lambda}{dt}$ and $\frac{d\beta}{dt}$, or the cxpressions for the precession in longitude and latitude respectively at the instant t, viz.:

$$\left.\begin{aligned} M &= \quad 172^\circ\ 45'\ 31'' + 33''.231t; \\ \frac{d\lambda}{dt} &= \quad 50''.2411 + 0.000\ 2268t; \\ &\quad + [0''.4776 - 0.000\ 0070t] \cos(\lambda - M) \tan\beta; \\ \frac{d\beta}{dt} &= -\ [0''.4776 - 0.000\ 0070t] \sin(\lambda - M). \end{aligned}\right\} (537)$$

These formulæ may be used to compute the entire precession between two dates $1800 + t$ and $1800 + t'$, if we compute the values of the differential coefficients for the middle interval, viz., $1800 + \frac{1}{2}(t + t')$. The result will be accurate to terms of the second order inclusive.

We have developed these formulæ (536) and (537) (which are those of Bessel, except that we have employed other constants) for the sake of completeness, although they will not be used in connection with the problems of the present treatise, the co-ordinates commonly employed being the right ascension and declination.

Example. The mean longitude and latitude of *α Lyræ* for 1850.0 are as follows:

$$\lambda = 283^\circ\ 12'\ 48''.12;$$
$$\beta = 61\ \ 44\ \ 25\ .45.$$

Required the mean longitude and latitude for 1884.0.

Here $t = 50$; $t' = 84$; $t' - t = 34$; $t' + t = 134$.

Therefore we find, by (536),

$$\begin{aligned} M &= 173^\circ\ 8'\ 23''; \\ \lambda - M &= 110\ \ 4\ \ 25; \\ \lambda' - \lambda &= (t' - t) \times 50''.2563 + (t' - t) \times .4771 \cos(\lambda - M) \tan\beta; \\ \beta' - \beta &= -(t' - t) \times .4771 \sin(\lambda - M). \end{aligned}$$

$$\begin{array}{rr} \lambda' - \lambda = & 28'\ 18''.36 \\ \lambda = & 283^\circ\ 12'\ 48''.12 \\ \hline \lambda' = & 283^\circ\ 41'\ \ 6''.48 \end{array} \qquad \begin{array}{rr} \beta' - \beta = - & 15''.24 \\ \beta = & 61^\circ\ 44'\ 25''.45 \\ \hline \beta' = & 61^\circ\ 44'\ 10''.21 \end{array}$$

If we wish to employ (537), we shall have for t the middle of the interval between 1850 and 1884, viz., $t = 67$. For λ in the second member we require the longitude for 1867, which we shall have with all necessary accuracy by adding to the longitude for 1850 the general precession for 17 years and neglecting the smaller terms. Calling this value λ_0, we have

$$\begin{aligned} \lambda_0 &= 283^\circ\ 12'\ 48'' + 50''.24\ \ \times 17 = 283^\circ\ 27'\ \ 2''; \\ M &= 172\ \ 45\ \ 31\ \ + 33\ .231 \times 67 = 173\ \ 22\ \ 37; \\ \lambda_0 - M & \qquad\qquad\qquad\qquad\qquad\quad = 110\ \ \ 4\ \ 25; \end{aligned}$$

$$\frac{d\lambda}{dt} = 50''.2563 + .4771 \cos(\lambda_0 - M) \tan\beta = 49''.9517;$$

$$\frac{d\beta}{dt} = \qquad - .4771 \sin(\lambda_0 - M) \qquad = -''.4481.$$

Therefore $$\lambda' - \lambda = \frac{d\lambda}{dt}(t' - t) = 28'\ 18''.36;$$

$$\beta' - \beta = \frac{d\beta}{dt}(t' - t) = -\ 15''.24;$$

agreeing with the values obtained by the other formulæ.

330. *Problem Third.* Given the mean right ascension and declination of a star for the date $1800 + t$, required the right ascension and declination for $1800 + t'$.

We first require the values of certain auxiliary constants similar to those employed in solving the corresponding problem for the ecliptic.

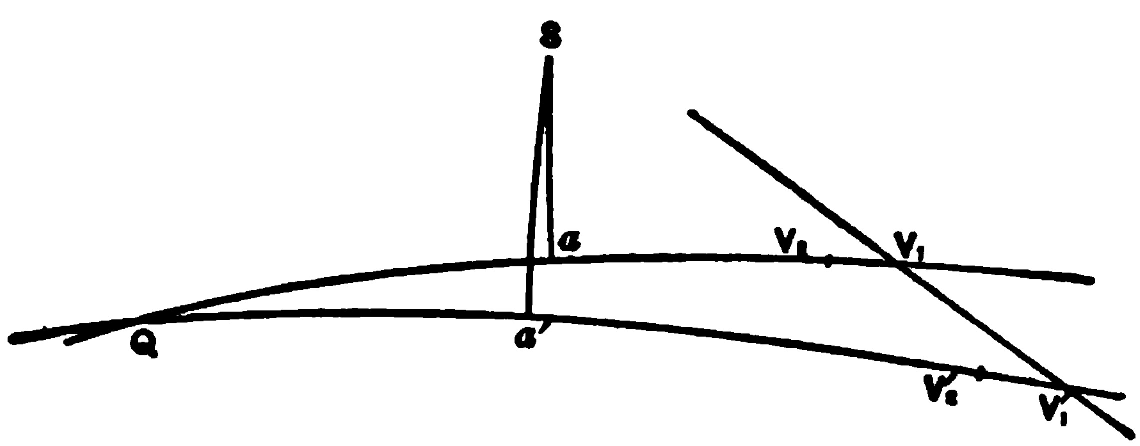

FIG. 68.

In Fig. 68 let V_1V_1' = the fixed ecliptic for 1800;
QV_1 = the equator for $1800 + t$;
QV_1' = the equator for $1800 + t'$;
V_1V_1' = the luni-solar precession in the interval $(t' - t)$.

Therefore $V_1V_1' = \psi' - \psi$.

Let $QV_1 = 90° - s$; $QV_1' = 90° + s'$; $V_1QV_1' = \theta$.

s, s', and θ will be quite small quantities, even when the interval $(t' - t)$ is considerable.

In accordance with our notation, angle $QV_1V_1' = 180° - \omega_1$, $QV_1'V_1 = \ _1'$.

Then in the triangle QV_1V_1' the quantities ω_1', ω_1, and

$\psi' - \psi$ are given by (523); we can therefore determine z, z', and θ.

By Napier's analogies, we readily find

$$\left.\begin{aligned}
\tan \tfrac{1}{2}(z' + z) &= \frac{\cos \frac{1}{2}(\omega_1' + \omega_1)}{\cos \frac{1}{2}(\omega_1' - \omega_1)} \tan \tfrac{1}{2}(\psi' - \psi); \\
\tan \tfrac{1}{2}(z' - z) &= \frac{\sin \frac{1}{2}(\omega_1' - \omega_1)}{\sin \frac{1}{2}(\omega_1' + \omega_1)} \cot \tfrac{1}{2}(\psi' - \psi); \\
\tan \tfrac{1}{2}\theta &= \frac{\sin \frac{1}{2}(z' + z)}{\cos \frac{1}{2}(z' - z)} \tan \tfrac{1}{2}(\omega_1' + \omega_1);
\end{aligned}\right\} \quad . \quad (538)$$

The second of these may be written

$$\tfrac{1}{2}(z' - z) = \frac{\cot \frac{1}{2}(\psi' - \psi)}{\sin \frac{1}{2}(\omega_1' + \omega_1)} \tfrac{1}{2}(\omega_1' - \omega_1).$$

In the first and third the denominator may be written equal to unity.

331. We can now solve our problem, viz., to determine the right ascension and declination for $1800 + t'$, having given those quantities for $1800 + t$.

In Fig. 68, S being any star, $Sa = \delta$, $Sa' = \delta'$.

If V_2 and V_2' represent the position of the mean equinox for $1800 + t$ and $1800 + t'$ respectively, then

The planetary precession in the interval $t = V_1V_2 = \vartheta$;
The planetary precession in the interval $t' = V_1'V_2' = \vartheta'$.

The right ascension $V_2a = \alpha$; $\quad V_2Q = 90° - z - \vartheta$;
$V_2'a' = \alpha'$; $\quad V_2'Q = 90° + z' - \vartheta'$.

Considering now the rectangular co-ordinates of the star,

the mean equator of $1800 + t$ being the plane of XY, the positive axis of X being directed to the point Q, we have

$$\begin{aligned} x &= \cos\delta \sin(\alpha + z + \vartheta); \\ y &= \cos\delta \cos(\alpha + z + \vartheta); \\ z &= \sin\delta. \end{aligned}$$

Similarly for the equator of $1800 + t'$,

$$\begin{aligned} x' &= \cos\delta' \sin(\alpha' - z' + \vartheta'); \\ y' &= \cos\delta' \cos(\alpha' - z' + \vartheta'); \\ z' &= \sin\delta'. \end{aligned}$$

The formulæ for x', y', and z', in terms of x, y, and z, are

$$\begin{aligned} x' &= x; \\ y' &= y\cos\theta - z\sin\theta; \\ z' &= y\sin\theta + z\cos\theta. \end{aligned}$$

Therefore

$$\left.\begin{aligned} \cos\delta' \sin(\alpha' - z' + \vartheta') &= \cos\delta \sin(\alpha + z + \vartheta); \\ \cos\delta' \cos(\alpha' - z' + \vartheta') &= \cos\delta \cos(\alpha + z + \vartheta)\cos\theta - \sin\delta\sin\theta; \\ \sin\delta' &= \cos\delta\cos(\alpha + z + \vartheta)\sin\theta + \sin\delta\cos\theta. \end{aligned}\right\} \quad (539)$$

We might have derived these equations by applying the formulæ of spherical trigonometry to the triangle formed by joining the place of the star with the pole of the equator in the two positions.

Thus in Fig. 69, S being the star, and P and P' the pole of the equator at the time $1800+t$ and $1800+t'$ respectively, we have the following for the sides and angles of the triangle. Calling the angle at the star C,

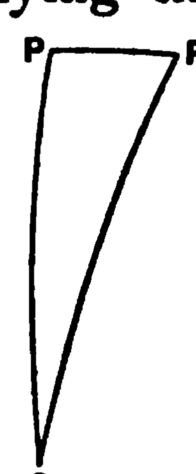

Fig. 69.

$$\begin{aligned} PP' &= \theta; & PS &= 90^\circ - \delta; \quad P'S = 90^\circ - \delta'; \\ SPP' &= \alpha + z + \vartheta & &= A, \text{ say, for convenience;} \\ SP'P &= 180^\circ - (\alpha' - z' + \vartheta') & &= 180^\circ - A'. \end{aligned}$$

Another solution of the problem is obtained by applying Gauss' equations to this triangle, viz.:

$$\left.\begin{aligned}
\cos\tfrac{1}{2}(90°+\delta')\cos\tfrac{1}{2}(A'+C)&=\cos\tfrac{1}{2}(90°+\delta+\theta)\cos\tfrac{1}{2}A;\\
\cos\tfrac{1}{2}(90°+\delta')\sin\tfrac{1}{2}(A'+C)&=\cos\tfrac{1}{2}(90°+\delta-\theta)\sin\tfrac{1}{2}A;\\
\sin\tfrac{1}{2}(90°+\delta')\cos\tfrac{1}{2}(A'-C)&=\sin\tfrac{1}{2}(90°+\delta+\theta)\cos\tfrac{1}{2}A;\\
\sin\tfrac{1}{2}(90°+\delta')\sin\tfrac{1}{2}(A'-C)&=\sin\tfrac{1}{2}(90°+\delta-\theta)\sin\tfrac{1}{2}A.
\end{aligned}\right\} \quad (540)$$

The auxiliary quantities z, z', and θ being computed by (538), either (539) or (540) give the required solution of our problem; these equations being solved in the usual manner.

332. Practically it is more convenient to compute the differences, $(\alpha' - \alpha)$ and $(\delta' - \delta)$. A formula for $(\alpha' - \alpha)$ is conveniently derived from the first and second of (539), which we write as follows:

$$\cos\delta'\sin A' = \cos\delta\sin A;$$
$$\cos\delta'\cos A' = \cos\delta\cos A\cos\theta - \sin\delta\sin\theta.$$

Multiply the first of these by $\cos A$, the second by $\sin A$, and subtract; then multiply the first by $\sin A$, the second by $\cos A$, and add. We readily find

$$\left.\begin{aligned}
\cos\delta'\sin(A'-A) &= \cos\delta\sin A\sin\theta\,[\tan\delta+\cos A\tan\tfrac{1}{2}\theta];\\
\cos\delta'\cos(A'-A) &= \cos\delta-\cos\delta\cos A\sin\theta[\tan\delta+\cos A\tan\tfrac{1}{2}\theta].
\end{aligned}\right\} \quad (541)$$

$$\left.\begin{aligned}
&\text{Let} \qquad p = \sin\theta[\tan\delta+\cos A\tan\tfrac{1}{2}\theta].\\
&\text{Then } \tan(A'-A) = \frac{p\sin A}{1-p\cos A};\\
&\qquad(\alpha'-\alpha) = (A'-A)+(z'+z)-(\vartheta'-\vartheta).\\
&\text{By the first of Napier's analogies,}\\
&\qquad\tan\tfrac{1}{2}(\delta'-\delta) = \tan\tfrac{1}{2}\theta\frac{\cos\tfrac{1}{2}(A'+A)}{\cos\tfrac{1}{2}(A'-A)}.
\end{aligned}\right\} \quad . \quad . \quad (542)$$

It will be necessary to make the computation in this complete form for circumpolar stars when the interval $(t' - t)$ is large. When the star is not too near the pole the computation will be much simpler, as we shall see.

Example. The mean place of Polaris for 1825.0 is as follows:

$$\begin{array}{lrl} \text{Right ascension} & \alpha = & 0^h\ 58^m\ 15^s.32; \\ & = & 14^\circ\ 33'\ 49''.8. \\ \text{Declination} & \delta = & 88^\circ\ 22'\ 31''.47. \end{array}$$

Required the precession in right ascension and declination between 1825 and 1900.

We have here $t = 25$, $t' = 100$. We therefore find, from formulæ (523),

$$\begin{array}{lll} \omega_1 = 23^\circ\ 27'\ 54''.22459; & \psi = 1259''.43; & \vartheta = 3''.628; \\ \omega_1' = 23\ \ 27\ \ 54\ .29350; & \psi' = 5036\ .90; & \vartheta' = 12\ .700. \end{array}$$

Then by formulæ (538), which we may write

$$\tan \tfrac{1}{2}(s' + s) = \cos \tfrac{1}{2}(\omega_1' + \omega_1) \tan \tfrac{1}{2}(\psi' - \psi),$$
$$\tfrac{1}{2}(s' - s) = \tfrac{1}{2}(\omega_1' - \omega_1) \cot \tfrac{1}{2}(\psi' - \psi) \operatorname{cosec} \tfrac{1}{2}(\omega_1' + \omega_1),$$

$$\tan \tfrac{1}{2}\theta = \sin \tfrac{1}{2}(s' + s) \tan \tfrac{1}{2}(\omega_1' + \omega_1).$$

$$\begin{array}{rll} \tfrac{1}{2}(\psi' - \psi) = \quad\ \ 31'\ 28''.74 & \tan = 7.9617592 & \cot = 2.03824 \\ \tfrac{1}{2}(\omega_1 + \omega_1') = 23^\circ\ 27'\ 54''.26 & \cos = 9.9625128 & \operatorname{cosec} = .39991 \\ \tfrac{1}{2}(s' + s) = \ \ 0^\circ\ 28\ \ 52\ .55 & \tan = 7.9242720 & \\ \hline \tfrac{1}{2}(\omega_1' - \omega_1) = 0''.03446 & & \log = 8.53732 \\ \hline \tfrac{1}{2}(s' - s) = 9.45 & & \log \tfrac{1}{2}(s' - s) = 0.97547 \\ \hline s' = 0^\circ\ 29'\ \ 2''.00 & & \\ s = 0\ \ 28\ \ 43\ .10 & & \end{array}$$

$$\begin{array}{r} \tan \tfrac{1}{2}(\omega_1 + \omega_1') = 9.6375775 \\ \sin \tfrac{1}{2}(s' + s) = 7.9242567 \\ \hline \tan \tfrac{1}{2}\theta = 7.5618342 \\ \tfrac{1}{2}\theta = 0^\circ 12' 32''.07 \\ \theta = 0\ 25\ \ 4\ .14 \end{array}$$

We now compute $(\alpha' - \alpha)$ and $(\delta' - \delta)$ by formulæ (542), viz.:

$$
\begin{array}{rlrl}
\alpha = & 14^\circ\ 33'\ 49''.8 & \tan \tfrac{1}{2}\theta = & 7.56183 \\
z = & \ 28\ \ 43\ .10 & \cos A = & 9.98486 \\
\vartheta = & \ \ 3\ .63 & & \\
\hline
A = & 15^\circ\ \ 2'\ 36''.53 & \text{Sum} = & 7.54669 \\
 & & \text{Zech} = & 434 \\
 & & \tan \delta = & 1.5472620 \\
 & & \sin \theta = & 7.8628593 \\
\hline
 & & \log p = & 9.4101647
\end{array}
$$

$$
\begin{array}{rlrlrl}
\sin A = & 9.4142243 & & & & \\
\log p = & 9.4101647 & & & & \\
\cos A = & 9.9848553 & \tfrac{1}{2}(A'-A) = & 2^\circ\ 32'\ 13''.06 & \sec = & 0.0004259 \\
\hline
p \cos A = & 9.3950200 & \tfrac{1}{2}(A'+A) = & 17\ \ 34\ \ 49\ .60 & \cos = & 9.9792268 \\
\text{Zech} = & .1239697 & & & \tan \tfrac{1}{2}\theta = & 7.5618342 \\
\hline
\text{log denominator} = & 9.8760303 & \tfrac{1}{2}(\delta'-\delta) = & 0\ \ 11\ \ 57\ .65 & \tan = & 7.5414869 \\
\text{log numerator} = & 8.8243890 & \delta' - \delta = & 0\ \ 23\ \ 55\ .30 & & \\
\hline
\tan (A' - A) = & 8.9483587 & & & & \\
A' - A = & 5^\circ\ \ 4'\ 26''.13 & & & & \\
A = & 15\ \ \ \ 2\ \ 36\ .53 & & & & \\
\hline
A' = & 20^\circ\ \ 7'\ \ 2''.66 & & & & \\
\hline
(A' - A) = & 5^\circ\ \ 4'\ 26''.13 & & & & \\
+ (z' + z) = & \ 57\ \ 45\ .10 & & & & \\
- (\vartheta' - \vartheta) = - & \ \ \ 9\ .07 & & & & \\
\hline
\alpha' - \alpha = & 6^\circ\ \ 2'\ \ 2''.16 & & & & \\
= & 0^h\ 24^m\ 8^s.144 & & & &
\end{array}
$$

333. By means of the foregoing formulæ we readily find the precession in right ascension and declination, viz., $\frac{d\alpha}{dt}$ and $\frac{d\delta}{dt}$, at any given instant $1800 + t$.

We have
$$(A' - A) = (\alpha' - \alpha) - (z' + z) + (\vartheta' - \vartheta). \tag{543}$$

If now we make $t' = t$ in the first of (541), we may make $\delta' = \delta$, $\sin(A' - A) = A' - A$, $\sin\theta = \theta$, $\sin A = \sin(\alpha + \vartheta)$; also, $\sin\theta \tan\frac{1}{2}\theta$ will vanish, being an infinitesimal of the second order.

Therefore this equation becomes

$$A' - A = \theta \tan\delta \sin(\alpha + \vartheta). \quad . \quad . \quad . \quad (544)$$

From (538), the same condition existing, viz., $t = t'$, we have

$$\left.\begin{aligned} s' + s &= (\psi' - \psi)\cos\omega_1; \\ \theta &= (\psi' - \psi)\sin\omega_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (545)$$

Combining (543), (544) and (545), writing $d\alpha$, $d\vartheta$, and $d\psi$ for $(\alpha' - \alpha)$, etc., and dividing by dt,

$$\frac{d\alpha}{dt} = -\frac{d\vartheta}{dt} + \frac{d\psi}{dt}\cos\omega_1 - \frac{d\psi}{dt}\sin\omega_1 \tan\delta \sin(\alpha + \vartheta). \quad (546)$$

The last of (542) by a similar process gives

$$\frac{d\delta}{dt} = \frac{d\psi}{dt}\sin\omega_1 \cos(\alpha + \vartheta). \quad . \quad . \quad . \quad . \quad (547)$$

Writing
$$\left.\begin{aligned} m &= -\frac{d\vartheta}{dt} + \frac{d\psi}{dt}\cos\omega_1; \\ n &= \frac{d\psi}{dt}\sin\omega_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad *(548)$$

---

* If we draw in the plane of the equator lines to the mean equinox of $(1800 + t)$ and $(1800 + t + 1)$ years, it will be observed that m represents the angle between them, assuming the rate of change to be uniform during one year. Also, n will be the angle between the two lines drawn to the poles of the equator in the two positions.

From the values of ψ, ω_1, and ϑ—equation (523)—we have

$$\left.\begin{aligned} m &= 46''.0623 + ''.000\,2849t; \\ &= 3^s.07082 + {}^s.000\,01899t; \\ n &= 20''.0607 - ''.000\,0863t; \\ \frac{d\alpha}{dt} &= m + n \sin\alpha \tan\delta; \\ \frac{d\delta}{dt} &= n \cos\alpha. \end{aligned}\right\} \quad . \quad . \quad . \quad (549)$$

We have written α in place of $(\alpha + \vartheta)$, no appreciable error resulting from neglecting ϑ.

These formulæ may be employed for computing the precession between any two dates $1800 + t$ and $1800 + t'$. If the values of $\frac{d\alpha}{dt}$ and $\frac{d\delta}{dt}$ are computed for the middle date, viz., $1800 + \frac{1}{2}(t + t')$, the result will be accurate to terms of the second order in $(t' - t)$ inclusive. We shall return to these formulæ hereafter.

Proper Motion.

334. When the co-ordinates of a star observed at different dates are reduced to the same epoch by means of the precession formulæ, a considerable difference in the values is often found, indicating a motion of the star itself. This change is called *proper motion*, and may be due either to an actual motion of the star in space or to the motion of the solar system, producing an apparent motion of the star. The observed proper motion is in fact the resultant of the two. For our purposes it is not necessary to attempt to separate these components. The proper motions in most cases are very small, requiring many years to produce an appreciable change in the star's place; but there are a few important exceptions to this rule.

In investigating the subject, the path of the star is assumed to coincide with a great circle, and the motion to be uniform. It is not probable that either assumption is true, but such deviations as may exist will be very small.

In order to determine a star's proper motion, its place must be observed on at least two dates which we may call $1800 + t$ and $1800 + t'$. The greater the interval $(t' - t)$ the more accurate will be the results, other things being equal.

Let α and δ = the observed mean right ascension and declination for $1800 + t$;

$\alpha + \Delta\alpha$ and $\delta + \Delta\delta$ = the values given by reducing the values observed at $1800 + t'$ to the first date by the application of the precession only.

Then $\Delta\alpha$ and $\Delta\delta$ will be the changes in α and δ due to proper motion in the interval $(t' - t)$.

Let μ and μ' = the annual proper motion in right ascension and declination respectively.

Then
$$\mu = \frac{\Delta\alpha}{t' - t}; \quad \mu' = \frac{\Delta\delta}{t' - t}. \quad . \quad . \quad . \quad . \quad (550)$$

These values will be referred to the mean equator of $1800 + t$. If we had reduced the co-ordinates for this date to $1800 + t'$ we should have obtained the proper motions referred to the equator of the latter date:

$$\mu = \frac{\Delta\alpha'}{t' - t} \quad \text{and} \quad \mu' = \frac{\Delta\delta'}{t' - t}. \quad . \quad . \quad . \quad (551)$$

These values for stars near the pole may differ very considerably from the first.

335. *Problem* I. To reduce the right ascension and declination of a star from the epoch $1800 + t$ to $1800 + t'$, the proper motion being known.

First. Suppose the proper motion given in reference to the mean equator of $1800 + t$, the solution is as follows:

Add to the right ascension for $1800 + t$ the effect of proper motion for the interval $(t' - t)$, viz., $\mu(t' - t)$; similarly add to the declination $\mu'(t' - t)$. With these values of the right ascension and declination the precession is computed as before by formulæ (542).

Second. The proper motion being given for the mean equator of $1800 + t'$.

Reduce the star's place to $1800 + t'$ by formulæ (542), and add to the results $\mu(t' - t)$ and $\mu'(t' - t)$ respectively.

336. *Problem* II. Having given the proper motion in right ascension and declination, referred to the mean equator of $1800 + t$, to derive the values in reference to the equator of $1800 + t'$.

Equations (539), giving the values of α' and δ' in terms of α and δ, are as follows:

$$\left.\begin{aligned}
\cos\delta' \sin(\alpha' - z' + \vartheta') &= \cos\delta \sin(\alpha + z + \vartheta);\\
\cos\delta' \cos(\alpha' - z' + \vartheta') &= \cos\delta \cos(\alpha + z + \vartheta)\cos\theta - \sin\delta \sin\theta;\\
\sin\delta' &= \cos\delta \cos(\alpha + z + \vartheta)\sin\theta + \sin\delta \cos\theta.
\end{aligned}\right\} \quad (552)$$

We also have

$$\left.\begin{aligned}
\cos\delta \sin(\alpha + z + \vartheta) &= \cos\delta' \sin(\alpha' - z' + \vartheta');\\
\cos\delta \cos(\alpha + z + \vartheta) &= \cos\delta' \cos(\alpha' - z' + \vartheta')\cos\theta + \sin\delta' \sin\theta;\\
\sin\delta &= -\cos\delta' \cos(\alpha' - z' + \vartheta')\sin\theta + \sin\delta' \cos\theta.
\end{aligned}\right\} \quad (553)$$

The proper motion which changes the position of the star itself produces no change in the quantities z, z', ϑ, ϑ', or θ, as these quantities merely serve to fix the positions of the

reference planes. Therefore, proper motion alone being considered, these quantities will be constants, α, α', δ, δ' being variable.

Differentiating the first two of (552) on this hypothesis, we have

$$\begin{aligned}
&\cos\delta'\cos(\alpha'-z'+\vartheta')\,d\alpha'-\sin\delta'\sin(\alpha'-z'+\vartheta')\,d\delta' \\
&\qquad=\cos\delta\cos(\alpha+z+\vartheta)\,d\alpha-\sin\delta\sin(\alpha+z+\vartheta)\,d\delta; \\
&-\cos\delta'\sin(\alpha'-z'+\vartheta')\,d\alpha'-\sin\delta'\cos(\alpha'-z'+\vartheta')\,d\delta' \\
&\qquad=-\cos\delta\sin(\alpha+z+\vartheta)\cos\theta d\alpha-\sin\delta\cos(\alpha+z+\vartheta)\cos\theta d\delta-\cos\delta\sin\theta d\delta.
\end{aligned}$$

Multiply the first of these by $\cos(\alpha'-z'+\vartheta')$, the second by $\sin(\alpha'-z'+\vartheta')$, subtracting and reducing by (552) and (553); then multiply the first by $\sin(\alpha'-z'+\vartheta')$, the second by $\cos(\alpha'-z'+\vartheta')$, add, and reduce. We find

$$\left.\begin{aligned}
\Delta\alpha'&=\Delta\alpha\,[\cos\theta+\sin\theta\tan\delta'\cos(\alpha'-z'+\vartheta')]+\frac{\Delta\delta}{\cos\delta}\sin\theta\frac{\sin(\alpha'-z'+\vartheta')}{\cos\delta'}; \\
\Delta\delta'&=-\Delta\alpha\sin\theta\sin(\alpha'-z'+\vartheta')+\frac{\Delta\delta}{\cos\delta}\cos\delta'\,[\cos\theta+\sin\theta\tan\delta'\cos(\alpha'-z'+\vartheta')].
\end{aligned}\right\}\quad(554)$$

$d\alpha$, $d\delta$, $d\alpha'$, and $d\delta'$ have been changed to $\Delta\alpha$, $\Delta\delta$, etc.

These equations solve the problem above enunciated with all necessary precision; $\Delta\alpha$, $\Delta\delta$, etc., being so small that it is unnecessary to consider terms of the higher orders. They may be used for the entire proper motion between the two dates t and t' or for the annual proper motion.

337. *Problem* III. The proper motion being given in reference to the mean equator of $1800+t'$ to derive the values of $\Delta\alpha$ and $\Delta\delta$ in reference to the mean equator of $1800+t$.

Differentiating equations (553) and reducing by (552) and (553) in a manner similar to that explained above, we have

$$\left.\begin{aligned}
\Delta\alpha&=\Delta\alpha'\,[\cos\theta-\sin\theta\tan\delta\cos(\alpha+z+\vartheta)]-\frac{\Delta\delta'}{\cos\delta'}\sin\theta\frac{\sin(\alpha+z+\vartheta)}{\cos\delta}; \\
\Delta\delta&=\Delta\alpha'\sin\theta\sin(\alpha+z+\vartheta)+\frac{\Delta\delta'}{\cos\delta'}\cos\delta\,[\cos\theta-\sin\theta\tan\delta\cos(\alpha+z+\vartheta)].
\end{aligned}\right\}\quad(555)$$

Example.

In the example Art. 332 we have found by applying the precession to the catalogue place of Polaris the mean position for 1900.0, as follows:

$$\alpha' - \Delta\alpha' = 1^h\ 22^m\ 23^s.46; \qquad \delta' - \Delta\delta' = 88°\ 46'\ 26''.77.$$

From Newcomb's catalogue we find for 1900*

$$\alpha' = 1^h\ 22^m\ 33^s.76; \qquad \delta' = 88°\ 46'\ 26''.66.$$

Therefore $\Delta\alpha' = +\ 10^s.30;$ $\qquad \Delta\delta' = -\ ''.11.$

$t' - t = 75$ years. Therefore

$$\mu = +\ ^s.1373; \qquad \mu' = -\ .''00147.$$

These values are referred to the mean equator of 1900. If we wish to reduce them to the equator of 1825 we employ formulæ (555). From the values of $(\alpha + s + \vartheta)$ and θ, Art. 332, we find

$$\Delta\alpha' [\cos\theta - \sin\theta \tan\delta \cos(\alpha + s + \vartheta)] = \quad 7^s.742$$
$$\dagger - \frac{\Delta\delta'}{15} \frac{\sin\theta \sin(\alpha + s + \vartheta)}{\cos\delta' \cos\delta} = \quad .023$$
$$\Delta\alpha = +\ 7^s.765 \qquad \text{Therefore } \mu = +^s.1035$$

Also, $\qquad \dagger\ 15\ \Delta\alpha' \sin\theta \sin(\alpha + s + \vartheta) = +''.2924$

$$\frac{\Delta\delta'}{\cos\delta'} \cos\delta [\cos\theta - \sin\theta \tan\delta \cos(\alpha + s + \vartheta)] = -\ .1096$$
$$\Delta\delta = +''.1828 \qquad \mu' = +''.0244$$

The above treatment of the problem is due to Bessel.

---

* This is, of course, not an observed place, but it answers equally well for illustrating the method.

† $\Delta\alpha'$ being given in time and $\Delta\delta'$ in arc.

Proper Motion on the Arc of a Great Circle.

338. Let ρ = the annual motion on the arc of a great circle;
χ = the angle which this great circle forms with the hour-circle of the star. When the star is on the meridian, χ will be measured from the north towards the east.

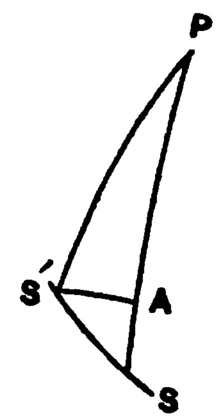

FIG. 70.

In the figure P is the pole, S and S' the first and second positions of the star respectively.

$$\left.\begin{array}{lll} SS' = \rho; & PSS' = \chi; & SA = \Delta\delta = \rho\cos\chi; \\ S'A = \Delta\alpha\cos\delta = \rho\sin\chi; & & \rho^2 = \Delta\delta^2 + \Delta\alpha^2\cos^2\delta. \end{array}\right\} \quad (556)$$

Expansion into Series.

339. The foregoing problem of reducing the mean place of a star from one epoch to another is treated in a very convenient and elegant manner by expansion into series in terms of the time.

If we let α_0 and δ_0 = the right ascension and declination for any time T,
α and δ = the right ascension and declination for any time $T + t$,

we have by Maclaurin's formula

$$\left.\begin{array}{l} \alpha = \alpha_0 + \left[\dfrac{d\alpha}{dt}\right]t + \dfrac{1}{2}\left[\dfrac{d^2\alpha}{dt^2}\right]t^2 + \dfrac{1}{2.3}\left[\dfrac{d^3\alpha}{dt^3}\right]t^3 + \text{etc.} \\ \delta = \delta_0 + \left[\dfrac{d\delta}{dt}\right]t + \dfrac{1}{2}\left[\dfrac{d^2\delta}{dt^2}\right]t^2 + \dfrac{1}{2.3}\left[\dfrac{d^3\delta}{dt^3}\right]t^3 + \text{etc.} \end{array}\right\} \quad (557)$$

When precession and proper motion are both considered,

the changes in α and δ are functions of these two independent variables, and $\left[\frac{d\alpha}{dt}\right]$, $\left[\frac{d\delta}{dt}\right]$, etc., are the total differential coefficients with respect to both precession and proper motion.

If we write $d_p\alpha$, $d_p\delta$ to indicate a variation due to precession, and $d_\mu\alpha$, $d_\mu\delta$ to indicate changes due to proper motion, we have

$$\left[\frac{d\alpha}{dt}\right] = \frac{d_p\alpha}{dt} + \frac{d_\mu\alpha}{dt}; \qquad \left[\frac{d\delta}{dt}\right] = \frac{d_p\delta}{dt} + \frac{d_\mu\delta}{dt}; \quad . \quad (558)$$

$$\left[\frac{d^2\alpha}{dt^2}\right] = \frac{d_p^2\,\alpha}{dt^2} + 2\,\frac{d_p d_\mu\,\alpha}{dt^2} + \frac{d_\mu^2\alpha}{dt^2};$$

and similarly for the other coefficients.

Equations (549) give us $\frac{d_p\alpha}{dt}$ and $\frac{d_p\delta}{dt}$, viz.,

$$\left.\begin{aligned} \frac{d_p\alpha}{dt} &= m + n \sin\alpha \tan\delta; \\ \frac{d_p\delta}{dt} &= n \cos\alpha. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (559)$$

Differentiating these, we have

$$\left.\begin{aligned} \frac{d_p^2\alpha}{dt^2} &= \frac{dm}{dt} + \frac{n^2}{2}\sin 2\alpha + \left[\frac{dn}{dt}\sin\alpha + mn\cos\alpha\right]\tan\delta + n^2\sin 2\alpha\tan^2\delta; \\ \frac{d_p^2\delta}{dt^2} &= -\,mn\sin\alpha + \frac{dn}{dt}\cos\alpha - n^2\sin^2\alpha\tan\delta; \\ \frac{d_p^3\alpha}{dt^3} &= \frac{mn^2}{2} + \frac{3}{2}mn^2\cos 2\alpha + \frac{3}{2}n\frac{dn}{dt}\sin 2\alpha \\ &\quad + \left[(2n^2 - m^2 + 3n^2\cos 2\alpha)\,n\sin\alpha + \left(2m\frac{dn}{dt} + n\frac{dm}{dt}\right)\cos\alpha\right]\tan\delta \\ &\quad + \left[3mn^2\cos 2\alpha + 3n\frac{dn}{dt}\sin 2\alpha\right]\tan^2\delta + 2n^3\sin\alpha\,(1 + 2\cos 2\alpha)\tan^3\delta; \\ \frac{d_p^3\delta}{dt^3} &= -\left(2m\frac{dn}{dt} + n\frac{dm}{dt}\right)\sin\alpha - (m^2 + n^2\sin^2\alpha)\,n\cos\alpha \\ &\quad - \left(\frac{3}{2}mn^2\sin 2\alpha + 3n\frac{dn}{dt}\sin^2\alpha\right)\tan\delta - 3n^3\sin^2\alpha\cos\alpha\tan^2\delta. \end{aligned}\right\} \quad (560)$$

340. Let us now consider proper motion.

ρ, χ, μ, and μ' have the same significance as before, Articles 334 and 338.

α' and δ' = the right ascension and declination at end of time t, proper motion alone being considered.

In the triangle formed by the pole and the two positions of the star we have

$$PS = 90^\circ - \delta; \qquad PS' = 90^\circ - \delta'; \qquad SS' = t\rho;$$
$$S'PS = \alpha' - \alpha; \qquad S'SP = \chi.$$

FIG. 71.

Therefore

$$\left.\begin{aligned} \sin\delta' &= \sin\delta\cos\rho t + \cos\delta\sin\rho t\cos\chi; \\ \cos\delta'\cos(\alpha'-\alpha) &= \cos\delta\cos\rho t - \sin\delta\sin\rho t\cos\chi; \\ \cos\delta'\sin(\alpha'-\alpha) &= \sin\rho t\sin\chi. \end{aligned}\right\} (561)$$

Also, $\rho\sin\chi = \mu\cos\delta$; $\rho\cos\chi = \mu'$; $\rho^2 = (\mu^2\cos^2\delta + \mu'^2)$.

Differentiating the first of (561) with respect to δ' and t, we find

$$\cos\delta'\frac{d\delta'}{dt} = -\rho\sin\delta\sin\rho t + \cos\delta\cos\rho t.\rho\cos\chi.$$

Substituting for $\rho\cos\chi$ its value μ', and making $t = 0$, we have

$$\frac{d_\mu\delta}{dt} = \mu'.$$

Differentiating a second and third time and reducing in a

similar manner, we have the following partial differential coefficients with respect to μ'.

$$\left[\frac{d_\mu\delta}{dt}\right] = \mu'; \quad \left[\frac{d_\mu{}^2\delta}{dt^2}\right] = -\frac{1}{2}\mu^2 \sin 2\delta; \quad \left[\frac{d_\mu{}^3\delta}{dt^3}\right] = -\mu^2\mu'(1 + 2\sin^2\delta). \quad (562)$$

In a similar manner, by differentiating the third of (561), making $t = 0$, and reducing, we find

$$\left[\frac{d_\mu\alpha}{dt}\right] = \mu; \quad \left[\frac{d_\mu{}^2\alpha}{dt^2}\right] = 2\mu\mu'\tan\delta; \left[\frac{d_\mu{}^3\alpha}{dt^3}\right] = -2[\mu^3\sin^2\delta - (1 + 3\tan^2\delta)\mu\mu'^2]. (563)$$

341. For the terms $\frac{d_p d_\mu \alpha}{dt^2}$ and $\frac{d_p d_\mu \delta}{dt^2}$ we differentiate (559) with respect to μ and μ', viz.,

$$\frac{d_p d_\mu \alpha}{dt^2} = n \cos\alpha \tan\delta \frac{d_\mu\alpha}{dt} + n \sin\alpha \sec^2\delta \frac{d_\mu\delta}{dt}; \qquad \frac{d_p d_\mu \delta}{dt^2} = -n \sin\alpha \frac{d_\mu\alpha}{dt}.$$

Substituting for $\frac{d_\mu\alpha}{dt}$ and $\frac{d_\mu\delta}{dt}$ the values given above, we have

$$\left.\begin{aligned} \frac{d_p d_\mu \alpha}{dt^2} &= n\mu \cos\alpha \tan\delta + n\mu' \sin\alpha \sec^2\delta; \\ \frac{d_p d_\mu \delta}{dt^2} &= -n\mu \sin\alpha. \end{aligned}\right\} \quad . \quad (564)$$

Therefore, from (558), (560), (562), (563), and (564),

$$\left.\begin{aligned} \left[\frac{d\alpha}{dt}\right] &= m + n \sin \tan\delta + \mu; \\ \left[\frac{d\delta}{dt}\right] &= n \cos\alpha + \mu'; \end{aligned}\right\} \quad . \quad . \quad . \quad (565)$$

$$\left.\begin{aligned}\left[\frac{d^2\alpha}{dt^2}\right] &= \frac{dm}{dt} + \frac{n^2}{2}\sin 2\alpha + 2n\mu'\sin\alpha + \left[\frac{dn}{dt}\sin\alpha + (m+2\mu)\,n\cos\alpha + 2\mu\mu\right]\tan\delta \\ &\qquad + 2n\sin\alpha(n\cos\alpha + \mu')\tan^2\delta; \\ \left[\frac{d^2\delta}{dt^2}\right] &= -(m+2\mu)n\sin\alpha + \frac{dn}{dt}\cos\alpha - \frac{1}{2}\mu^2\sin 2\delta - n^2\sin^2\alpha\tan\delta.\end{aligned}\right\} (565)_1$$

Also we have

$$\left[\frac{d^3\alpha}{dt^3}\right] = \frac{d_p^3\alpha}{dt^3} + 3\frac{d_p^2 d_\mu\alpha}{dt^3} + 3\frac{d_p d_\mu^2\alpha}{dt^3} + \frac{d_\mu^3\alpha}{dt^3}. \quad (566)$$

Differentiating the first of (560) with respect to μ, we find

$$\begin{aligned}\frac{d_p^2 d_\mu\alpha}{dt^3} &= n^2\cos 2\alpha\frac{d_\mu\alpha}{dt} + \left[\frac{dn}{dt}\cos\alpha\frac{d_\mu\alpha}{dt} - mn\sin\alpha\frac{d_\mu\alpha}{dt}\right]\tan\delta \\ &\qquad + \left[\frac{dn}{dt}\sin\alpha + mn\cos\alpha\right]\sec^2\delta\frac{d_\mu\delta}{dt} \\ &\qquad + 2n^2\cos 2\alpha\tan^2\delta\frac{d_\mu\alpha}{dt} + 2n^2\sin 2\alpha\tan\delta\sec^2\delta\frac{d_\mu\delta}{dt}.\end{aligned}$$

In like manner, differentiating the first of (559) twice with respect to μ, we find

$$\begin{aligned}\frac{d_p d_\mu^2\alpha}{dt^3} &= -n\sin\alpha\tan\delta\left(\frac{d_\mu\alpha}{dt}\right)^2 + n\cos\alpha\sec^2\delta\frac{d_\mu\delta}{dt}\,\frac{d_\mu\alpha}{dt} \\ &\qquad + n\cos\alpha\tan\delta\frac{d_\mu^2\alpha}{dt^2} + n\cos\alpha\sec^2\delta\frac{d_\mu\alpha}{dt}\,\frac{d_\mu\delta}{dt} \\ &\qquad + 2n\sin\alpha\tan\delta\sec^2\delta\left(\frac{d_\mu\delta}{dt}\right)^2 + n\sin\alpha\sec^2\delta\frac{d_\mu^2\delta}{dt^2}.\end{aligned}$$

Substituting in these equations for $\frac{d_\mu\alpha}{dt}$, etc., their values from (562) and (563), then substituting in (566) these values, also $\frac{d_p^3\alpha}{dt^3}$ and $\frac{d_\mu^3\alpha}{dt^3}$ from (560) and (563), we have the required

value of the third differential coefficient. $\left[\frac{d^3\delta}{dt^3}\right]$ is found in a similar manner. They are as follows:

$$\left.\begin{aligned}
\left[\frac{d^3\alpha}{dt^3}\right] = {} & \frac{mn^2}{2} + 2\mu\mu'^2 + 3\frac{dn}{dt}\mu' \sin\alpha + 3n\mu'(m + 2\mu)\cos\alpha + \tfrac{3}{2}(m + 2\mu)n^2 \cos 2\alpha \\
& + \tfrac{3}{2}n\frac{dn}{dt}\sin 2\alpha - 2\mu^3 \sin^2\delta \\
& + \Big[(2n^2 - m^2 - 6\mu^2 + 6\mu'^2 - 3m\mu + 3n^2 \cos 2\alpha)n \sin\alpha \\
& \qquad + \left(2m\frac{dn}{dt} + n\frac{dm}{dt} + 3\frac{dn}{dt}\mu\right)\cos\alpha + 6n^2\mu' \sin 2\alpha\Big]\tan\delta \\
& + \Big[6\mu\mu'^2 + 3\frac{dn}{dt}\mu' \sin\alpha + (12\mu + 3m)n\mu' \cos\alpha \\
& \qquad + 3n\frac{dn}{dt}\sin 2\alpha + (3m + 6\mu)n^2 \cos 2\alpha\Big]\tan^2\delta \\
& + [(2n^2 + 6\mu'^2)n \sin\alpha + 6n^2\mu' \sin 2\alpha + 4n^3 \sin\alpha \cos 2\alpha]\tan^3\delta; \\
\left[\frac{d^3\delta}{dt^3}\right] = {} & -\mu^2\mu' - (2m + 3\mu)\frac{dn}{dt}\sin\alpha - (m^2 + 3\mu^2 + 3m\mu)n \cos\alpha - n\frac{dm}{dt}\sin\alpha \\
& - n^3 \sin^2\alpha \cos\alpha - 3n^2\mu' \sin^2\alpha - 2\mu^2\mu' \sin^2\delta \\
& - \left[6n\mu\mu' \sin\alpha + \tfrac{3}{2}(m + 2\mu)n^2 \sin 2\alpha + 3n\frac{dn}{dt}\sin^2\alpha\right]\tan\delta \\
& - 3n^2(n \cos\alpha + \mu')\sin^2\alpha \tan^2\delta.
\end{aligned}\right\} \quad (567)$$

342. These expressions for the third differential coefficients are too complicated for use in practical computation. A series of tables is given by Argelander* by means of which that part may be readily derived which depends on precession only. These tables are convenient when the proper motion is so small that it may be disregarded. They are given for the epoch 1850, and Bessel's constants are employed.

If the third differential coefficients are required, they may be obtained very conveniently by computing the values of the second differential coefficients for two dates fifty years before and after the given one and proceeding according to the method of Art. 50.

If we make $f(T) = \frac{d^2\alpha}{dt^2}$, then $f(T - w)$ and $f(T + w)$ will

* See Untersuchungen über die Eigenbewegungen von 250 Sternen, p. 145.

be the values for dates fifty years before and after the date T. Then the first of (101) gives

$$\frac{d^2\alpha}{dt^2} = \frac{1}{50} f'(T), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (568)$$

the notation being that of formula (101), and the unit of time being one year.

343. If now we require the precession formulæ for any given date, as 1875.0, we obtain them by substituting for m and n the values given by (549). m will generally be expressed in time and n in arc. It will be convenient to give the formulæ for the second differential coefficients the following form:

$$\left[\frac{d^2\alpha}{dt^2}\right] = \left(\frac{dm}{dt} - \frac{m}{n}\frac{dn}{dt}\right) + \frac{dn}{dt}\frac{1}{n}\left(\frac{d\alpha}{dt} - \mu\right) + n \sin 1'' \left(\frac{d\alpha}{dt} + \mu\right) \cos\alpha \tan\delta$$
$$+ \frac{n}{15} \sin 1'' \left(\frac{d\delta}{dt} + \mu'\right) \sin\alpha \sec^2\delta + 2\mu\mu' \sin 1'' \tan\delta;$$

$$\left[\frac{d^2\delta}{dt^2}\right] = \frac{dn}{dt}\frac{1}{n}\left(\frac{d\delta}{dt} - \mu'\right) - 15n \sin 1'' \left(\frac{d\alpha}{dt} + \mu\right) \sin\alpha - \frac{(15)^2 \sin 1''}{2} \mu^2 \sin 2\delta.$$

m, $\frac{d\alpha}{dt}$, and μ will be expressed in time; n, $\frac{d\delta}{dt}$, and μ' in arc.

We then have the following formulæ for 1875.0:

$$\left.\begin{aligned}
\left[\frac{d\alpha}{dt}\right] &= 3^s.07225 + [0.126115] \sin\alpha \tan\delta + \mu; \\
\left[\frac{d^2\alpha}{dt^2}\right] &= 0.0000322 - [4.63380]\left(\frac{d\alpha}{dt} - \mu\right) + [5.98778]\left(\frac{d\alpha}{dt} + \mu\right) \cos\alpha \tan\delta \\
&\quad + [4.81169]\left(\frac{d\delta}{dt} + \mu'\right) \sin\alpha \sec^2\delta + [4.9866]\mu\mu' \tan\delta; \\
\left[\frac{d\delta}{dt}\right] &= [1.302206] \cos\alpha + \mu'; \\
\left[\frac{d^2}{dt^2}\right] &= -[4.63380]\left(\frac{d\delta}{dt} - \mu'\right) - [7.16387]\left(\frac{d\alpha}{dt} + \mu\right) \sin\alpha - [6.7367]\mu^2 \sin 2\delta
\end{aligned}\right\} (569)$$

The numerical quantities enclosed in brackets are logarithms as usual.

A numerical example illustrating the application of the foregoing formulæ is given in Art. 347.

Star Catalogues and Mean Places of Stars.

344. The various catalogues of stars which are in use may be divided into two classes, viz., *compilations* and those derived from *original observation.*

Among the most important of the first class are the *British Association Catalogue*, Newcomb's *Catalogue of* 1098 *Standard Clock and Zodiacal Stars*, Boss' *Catalogue of* 500 *Stars*, and Safford's Catalogue. These catalogues are of very different degrees of excellence. The British Association Catalogue (often written B. A. C.) contains the right ascensions and north-polar distances of 8377 stars reduced to the mean equator of January 1, 1850. The places of many of these are, however, not well determined, errors of from 5″ to 10″ in north-polar distance, and of corresponding magnitude in right ascension, not being uncommon. It is a very convenient catalogue for use in preliminary work, bnt the co-ordinates of the stars should be taken from other authorities when accuracy is required.

The places given in Newcomb's and Boss' catalogues, on the other hand, have been derived with great care from all of the more reliable authorities, and are entitled to great confidence.

The following are among the most reliable of the other class of catalogues, viz., those derived from original observation:

Bradley's Observations reduced by Bessel. Epoch of catalogue 1755.

Bradley's Observations reduced by Auwers. Epoch 1755.
Piazzi. Precipuarum Stellarum Inerrantium Positiones Mediæ. Epoch 1800.
Groombridge. A Catalogue of Circumpolar Stars, deduced from the Observations of Stephen Groombridge. Epoch 1810.
Struve. Positiones Mediæ. Epoch 1830.
Argelander. DXL Stellarum Fixarum Positiones Mediæ. Epoch 1830.
Airy. First Cambridge Catalogue. Epoch 1830.
Robinson. Armagh Catalogue of 5345 Stars. Epoch 1840.
Gilliss. Observations made at Santiago, Chili. Epoch 1850.
Pulkowa. Catalogue in Vol. I, Pulkowa Observations. Epoch 1845.
Greenwich. The various catalogues from observations at the Greenwich observatory.
Radcliffe. Several catalogues from observations made at the Radcliffe observatory, Oxford.
Washington. Catalogues derived from observations at the Naval Observatory, Washington, D. C,

Besides these there are valuable catalogues published by the observatories of Brussels, Paris, Cambridge, England, Cambridge, U. S., Edinburgh, Vienna, and others.

These catalogues give the right ascension and declination (or north-polar distance) of the stars referred to the mean equator of the date of the catalogue. Generally the data for reducing the star to the mean equator of any other date are also given. These are commonly given under the headings *precession* and *secular variation;* the proper motion is sometimes given when its value is known.

The quantities called *precession* are simply the values of $\frac{d\alpha}{dt}$ and $\frac{d\delta}{dt}$ for the date of the catalogue, precession only

being considered. The *secular variations* are the changes which take place in these quantities in 100 years; i.e., the values of $100\,\frac{d^2\alpha}{dt^2}$ and $100\,\frac{d^2\delta}{dt^2}$.

Let p_a = the annual precession in right ascension $= \frac{d\alpha}{dt}$;

s_a = the secular variation $= 100\,\frac{d^2a}{dt^2}$;

α_0 = the right ascension for epoch T, the date of the catalogue;

α = the right ascension for epoch $T + t$.

Then
$$\alpha = \alpha_0 + t\left(p_a + \frac{s_a}{100}\frac{t}{2}\right). \quad . \quad . \quad . \quad . \quad (570)$$

The declination will be given by a similar process. If proper motion is given, this must also be included in formula (570). In some catalogues the proper motion is included with the precession, when this is generally given under the heading *annual motion*, and it corresponds exactly to $\frac{d\alpha}{dt}$ and $\frac{d\delta}{dt}$ given by formulæ (565).

345. When a star's place is required with extreme accuracy it should be sought for in as many original authorities as may be available, and the values of the co-ordinates given by the various catalogues combined by the method of least squares to determine the most probable values of these co-ordinates with the proper motion. There are different methods for working out the details of this process, the following being perhaps more frequently employed than any other:

Suppose we require the mean place for 1875.0, together with proper motion. If the star has been well observed at

epochs separated by a considerable interval, the latter may be determined; otherwise not.

We first derive the approximate right ascension and declination for 1875.0 by reducing to that date the place as given in one or more of the best modern catalogues, using for this purpose the annual motion and secular variation of the catalogue. For this preliminary place the Greenwich catalogues will generally give a value of the right ascension within $^s.2$ or $^s.3$, and of the declination within $2''$ or $3''$ of the truth.

We then compute accurate values of $\frac{d\alpha}{dt}$, $\frac{d\delta}{dt}$, $\frac{d^2\alpha}{dt^2}$, and $\frac{d^2\delta}{dt^2}$ for 1875.0 by formulæ (569); and if great precision is required, $\frac{d^3\alpha}{dt^3}$ and $\frac{d^3\delta}{dt^3}$, as explained in Art. 342. Our assumed co-ordinates are then to be corrected by comparing them with the places given in the various catalogues. For this purpose the assumed right ascension and declination are reduced to the date of each catalogue.

Let α_1 = the assumed right ascension for 1875.0;
α_1' = the value of α_1 reduced to the epoch of catalogue, $1875 - t$;
α_2 = right ascension given by catalogue;
μ = the annual proper motion.

The difference $(\alpha_2 - \alpha_1')$, supposing for the present α_2 to be free from error, will consist of two parts, viz., the error in the assumed value of α_1 and the change due to proper motion in the interval t. Therefore

$$x - \mu t = (\alpha_2 - \alpha_1') \quad . \quad . \quad . \quad . \quad . \quad (571)$$

is an equation for determining the proper motion μ and the correction to the assumed right ascension x. Each catalogue will give us an equation of this form; from these the most

probable values of x and μ are derived by least squares. A similar series of equations will also be obtained for the declination.

346. The above is an outline of the method; practically it is much complicated by the fact that the different catalogues are of very different degrees of accuracy, and in the same catalogue the weight will depend on the number of observations made on the star. It is impossible to give any infallible rule for the assignment of weights; practically much must depend on the judgment of the investigator. In general, however, the more recent catalogues are entitled to much greater weight than the older ones. Methods and instruments are constantly improving, and in consequence a much higher precision is possible now than was the case a hundred years ago. The old catalogues are, however, indispensable in investigation of proper motion.

The following table shows the weights assigned by Newcomb to the different authorities employed in deriving the right ascensions of the catalogue referred to above:

Number of Observations.	1	2	3	4	5	7	10	15	20	25	30	40	50	60	80	100
Bessel's Bradley	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Auwers' Bradley	$\frac{1}{10}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	1	2	2	3	3	3	4	4
Piazzi	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	1	1	1	1	1	1	1	1
Struve, 1825	[illegible]	1	1	1	2	2	3	3	4	4	5	5	6	6	7	8
Argelander, 1830	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Pond	[illegible]	1	1	1	1	1	2	2	2	2	3	3	4	4	5	6
Johnson	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Airy, Cambridge, 1830	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Gilliss, 1840	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Airy, Greenwich, 1840	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Armagh, 1840	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	2	3	3	4	4	5	6
Pulkowa, 1845	1	2	2	3	4	5	7	7	8	10	15	20	20	25	25	30
Radcliffe, 1845	[illegible]	1	1	1	1	1	2	2	2	2	3	3	4	4	5	6
Airy, Greenwich, 1845	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Airy, Greenwich, 1850	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Pulkowa, 1850	1	2	2	3	4	5	7	7	8	10	15	15	20	25	25	30
Airy, Greenwich, 1860	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Yarnall, Washington, 1860	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Airy, Greenwich, 1864	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Engleman, Leipzig, 1866	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Airy, Greenwich, 1870	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Washington, 1870	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"

Boss gives a similar table of weights for the declination equations. See Report of the U. S. Northern Boundary Commission, p. 566.

If an approximate value of the proper motion is also known it may be employed in computing the differential coefficients by formulæ (569), when we shall have in equation (571), instead of μ, the correction to the assumed value of μ, viz., $\Delta\mu$.

Example.

347. For the purpose of illustrating the foregoing formulæ and methods let us derive the mean co-ordinates and proper motion of the star B. A. C. 2786 * for the epoch 1875.0. The following tabular statement shows the values of the co-ordinates given by the various authorities consulted. It probably explains itself sufficiently.

Catalogue.	Epoch of Catalogue.	Mean Epoch of Observation.	No. of Observations in α.	Catalogue Right Ascension.	Mean Epoch of Observation in δ.	No. of Observations in δ.	Catalogue Declination.
Bradley	1755		5	$8^h\ 5^m\ 8^s.03$		4	$27°\ 59'\ 22''.6$
Piazzi	1800		7	8 7 53 .15		8	27 51 13 .0
Gould's D'Agelet	1800	1783.3		8 7 53 .3	1783.3		27 51 22 .0
Weiss' Bessel	1825	1826.2	2	8 9 24 .70	1826.2	2	27 46 34 .0
Argelander	1830		8	8 9 43 .43		8	27 45 40 .3
Taylor	1835		6	8 10 1 .96		4	27 44 46 .86
Armagh	1840	1830.2	1	8 10 19 .99	1853.3	5	27 43 43 .31
Brussels	1856	1856.1	6	8 11 18 .56	1856.2	1	27 40 49 .37
"	1858	1858.1	4	8 11 25 .98	1858.1	4	27 40 26 .8
"	1860	1860.1	1	8 11 33 .13	1860.1	2	27 40 5 .5
Cape of Good Hope	1860	1857.1	2	8 11 33 .38	1857.1	2	27 40 4 .37
Greenwich	1860	1857.7	8	8 11 33 .28	1857.7	8	27 40 4 .12
Radcliffe	1860	1855.0	5	8 11 33 .29	1856.3	7	27 40 4 .2
Greenwich	1864	1863.7	6	8 11 47 .88	1863.7	10	27 39 18 .96
"	1868	1868.2	3	8 12 2 .53	1868.2	9	27 38 33 .76
"	1869	1869.2	1	8 12 6 .22	1869.2	3	27 38 23 .10
"	1870				1870.2	6	27 38 11 .63
"	1871				1871.2	6	27 38 0 .30
"	1872				1872.2	4	27 37 49 .02
Washington	1872	1872.2	3	8 12 17 .08	1872.2	3	27 37 48 .5

We first require an approximate value of the star's place for 1875.0, which we may readily derive from the four catalogues which give the co-ordinates for 1860.0, viz., Brussels, Cape of Good Hope, Greenwich, and Radcliffe. Thus we find

$$1860 \quad \alpha = 8^h\ 11^m\ 33^s.27;$$
$$\delta = 27°\ 40'\ 4''.5.$$

* This is the number of the star in the British Association catalogue.

For reducing these to 1875 we take from the Greenwich catalogue the following quantities:

In right ascension, precession $= + 3^s.661$;
secular variation $= - \quad .017$.

In declination, precession $= - 10''.89$;
secular variation $= - \quad .44$;
proper motion $\mu' = - \quad .38$.

Therefore

$$1875 \quad \alpha = 8^h\, 11^m\, 33^s.27 + 15(3.661 - 7.5 \times .00017) \qquad = 8^h\, 12^m\, 28^s.17;$$
$$\delta = 27^\circ\, 40'\, 4''.5 \;\; + 15(-10''.89 - 7.5 \times .0044 - .38) = 27^\circ\, 37'\, 15''.0.$$

We may reasonably expect these to prove very close approximations to the final values. With these values of α and δ, and the above value of μ', we next compute $\frac{d\alpha}{dt}$, $\frac{d\delta}{dt}$, $\frac{d^2\alpha}{dt^2}$, and $\frac{d^2\delta}{dt^2}$ by (569). This computation is given in full.

Constant =	0.126115	Constant =	4.63380_n	Constant =	4.63380_n
$\sin\alpha =$	9.923012	$\frac{d\alpha}{dt} - \mu =$	.56326	$\frac{d\delta}{dt} - \mu' =$	1.03968_n
$\tan\delta =$	9.718710				
log =	9.767837	log =	5.19706_n	log =	5.67348
Nat. No. =	0.58592	Nat. No.	− .000 015 7	Nat. No. =	+.000 047 15
$m =$	3.07225	Constant =	5.98778	Constant =	7.16387_n
$\frac{d\alpha}{dt} - \mu =$	3.65817	$\frac{d\alpha}{dt} + \mu =$	.56326	$\frac{d\alpha}{dt} + \mu =$	.56326
$\log n =$	1.302206	$\cos\alpha =$	9.73748_n	$\sin\alpha =$	9.92301
$\cos\alpha =$	9.737476_n	$\tan\delta =$	9.71871		
log =	1.039682_n	log =	6.00723_n	log =	7.65014_n
Nat. No. =	− 10.95676	Nat. No. =	− .000 101 7	Nat. No. =	−.004 468 30
$\mu' =$	− .38	Constant =	4.81169		
$\frac{d\delta}{dt} =$	− 11.3368	$\frac{d\delta}{dt} + \mu' =$	$1\ 06881_n$	$\frac{d^2\delta}{dt^2} =$	−.004 421 2
		$\sin\alpha =$	9.92301		
		$\sec^2\delta =$	.10510		
		log =	5.90861_n		
		Nat. No. =	− .000 081 0		
		Constant =	+ .000 032 2		
		$\frac{d^2\alpha}{dt^2} =$	− .000 166 2		

For determining the third differential coefficients, we find for the dates 1825 and 1925 respectively:

$$1825 \quad \frac{d^3\alpha}{dt^3} = -.000\ 164\ 5; \quad \frac{d^3\delta}{dt^3} = -.004\ 471\ 5.$$

$$1925 \quad \frac{d^3\alpha}{dt^3} = -.000\ 167\ 9; \quad \frac{d^3\delta}{dt^3} = -.004\ 370\ 0.$$

We therefore find, by (568),

$$\frac{d^3\alpha}{dt^3} = -.000\ 000\ 034; \quad \frac{d^3\delta}{dt^3} = +.000\ 001\ 014.$$

Substituting the above values of the differential coefficients in Maclaurin's formula, and making t minus, since we shall want to apply it to dates previous to 1875, we have

$$\alpha = \alpha_0 - t[3^s.65817 + t(.000\ 083\ 1 - t.000\ 000\ 006)];$$
$$\delta = \delta_0 + t[11''.3368 - t(.002\ 211 \quad + t.000\ 000\ 17)].$$

By means of these formulæ we next reduce the above assumed right ascension and declination to the epoch of each of the authorities where our star is found.

The differences between these computed values and the observed values are given in the following table. The "correction for μ'" there given is applied to those catalogues where the epoch of observation differs considerably from the epoch of the catalogue. For example, *Gould's D'Agelet:* The mean epoch of observation is 1783; the catalogue places are given for 1800. We have assumed $\mu' = -''.38$, which in 17 years produces a change in δ of $-6''.46$. This is, in this case, the "correction for μ'."

Numbers.	Authority.	Right Ascension.					Declination.					
		Mean Year.	No. Observations.	Weight.	$O-C$. n.	$O-C$. v.	Mean Year.	No. Observations.	Weight.	Correction for μ'.	$O-C$. n.	$O-C$. v.
1	Bradley	1755	5	.1	+.03	+.04	1755	4	.2		− .69	− .53
2	Piazzi	1800	7	.5	−.19	−.18	1800	8	.1		+ .25	+ .46
3	Gould's D'Agelet	1783		.05	−.04	−.03	1783		.05	−6″.46	+2.79	+2.99
4	Weiss' Bessel	1826	2	.1	−.35	−.34	1826	2	.1	+ .38	−1.91	−1.67
5	Argelander	1830	8	2.0	+.05	+.06	1830	8	2.0		− .36	− .11
6	Taylor	1835	6	.5	+.25	+.26	1835	4	.2		+1.94	+2.19
7	Armagh	1830	1	.5	−.04	−.03	1853	5	.3	+4 .94	− .82	− .54
8	Brussels	1856	6				1856	1				
9	"	1858	4	2.0	−.07	−.06	1858	4	.8		+ .14	+ .42
10	"	1860	1				1860	2				
11	Cape of Good Hope	1860	2	.5	+.10	+.11	1860	2	.7		− .18	+ .10
12	Greenwich	1860	8	6.0	−.0	+.01	1860	8	2.0		− .43	− .15
13	Radcliffe	1860	5	1.0	+.01	+.02	1860	7	.5		− .35	− .07
14	Greenwich	1864	6	4.0	−.04	−.03	1864	10	2.0		− .48	− .19
15	"	1868	3	3.0	−.01	+.01	1868	9				
16	"	1869	1				1869	3				
17	"						1870	6	2.0		− .06	+ .23
18	"						1871	6				
19	"						1872	4				
20	Washington	1872	3	1.0	−.11	−.09	1872	3	.8		− .49	− .19

The weights have been assigned in accordance with the systems of Newcomb and Boss for the most part.

The quantities n are now the absolute terms of the system of equations of condition of the form

$$\sqrt{p}(\varDelta\alpha - t\mu = n) \qquad \text{and} \qquad \sqrt{p}(\varDelta\delta - t\varDelta\mu' = n).$$

From these we derive the following normal equations in the usual manner, with the values of the unknown quantities:

$$\begin{aligned}
21.250\varDelta\alpha - 4.045\mu &= - \ .304; \\
- \ 4.045\varDelta\alpha + 1.365\mu &= + \ .055; \\
\varDelta\alpha &= - \ .015 \quad \pm .0197; \\
\mu &= - \ .00005 \pm .00078.
\end{aligned}$$

$$\begin{aligned}
11.750\varDelta\delta - 2.416\varDelta\mu' &= - \ 3.263; \\
- \ 2.416\varDelta\delta + \ .987\varDelta\mu' &= + \ .615; \\
\varDelta\delta &= - \ .301 \quad \pm .122; \\
\varDelta\mu' &= - \ .00114 \pm .00420.
\end{aligned}$$

Applying these corrections to the assumed values of α, δ, and μ', we have finally, as the most probable values,

$$\alpha = 8^h\ 12^m\ 28^s.155 \pm .0197; \qquad \mu = -\ ^s.00005 \pm .00078;$$
$$\delta = 27^\circ\ 37'\ 14''.70 \pm .122; \qquad \mu' = -\ ''.3811 \pm .0042.$$

Nutation.

348. Nutation has already been defined as the name applied to the periodic part of the precession. The components of the attractive force of the sun and moon, which tend to draw the equator into coincidence with the ecliptic, are not constant with respect to either of those bodies. The component has a maximum value when the attracting body is in the plane passing through the earth's axis and perpendicular to the equator, and it is zero when the body is in the plane of

the equator. The orbit of the moon and apparent orbit of the sun are ellipses, so that the distances of these bodies from the earth are constantly changing. The angle between the plane of the moon's orbit and the equator is variable; so in a less degree is that between the equator and ecliptic, or apparent orbit of the sun. All of these circumstances produce periodic terms in the movement called precession.

It will be seen that the law or laws governing this matter are intricate and difficult to investigate; their discussion belongs to the department of Physical Astronomy. Various investigators have given more or less attention to the determination of the constants which enter into the formulæ; the values which are most extensively employed at present are those of Peters.

349. Since nutation is simply a motion of the equator, the ecliptic remaining unchanged, it follows that it will produce no effect upon the latitudes of stars. The longitudes will be changed, also the obliquity of the ecliptic.

Let $\Delta\lambda$ and $\Delta\omega$ = the nutation in longitude and obliquity respectively.

Then, according to Peters, for 1800.0:

$$\left.\begin{aligned}\Delta\lambda &= -17''.2405 \sin \text{☊} + ''.2073 \sin 2\text{☊} - ''.2041 \sin 2\text{☾} + ''.0677 \sin(\text{☾} - \Gamma') \\ &\quad - 1''.2692 \sin 2\odot + ''.1279 \sin(\odot - \Gamma) - ''.0213 \sin(\odot + \Gamma); \\ \Delta\omega &= 9''.2231 \cos \text{☊} - ''.0897 \cos 2\text{☊} + ''.0886 \cos 2\text{☾} + ''.5509 \cos 2\odot \\ &\quad + ''.0093 \cos(\odot + \Gamma).\end{aligned}\right\} \quad (572)$$

Where ☊ = the mean longitude of the ascending node of the moon's orbit;*
☾ = the moon's true longitude;
$\odot$ = the sun's true longitude;
Γ = true longitude of the sun's perigee;
Γ' = true longitude of the moon's perigee.

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* That is, the point where the moon passes from below the ecliptic to above.

The coefficients of the above formulæ vary slowly with the time, so that, according to Peters, the values for 1900 will be

$$\left.\begin{aligned}\Delta\lambda =& -17''.2577 \sin \Omega + ''.2073 \sin 2\Omega - ''.2041 \sin 2☾ + ''.0677 \sin(☾ - \Gamma') \\ & - 1''.2693 \sin 2\odot + ''.1275 \sin(\odot - \Gamma) - ''.0213 \sin(\odot + \Gamma); \\ \Delta\omega' =& + 9''.2240 \cos \Omega - ''.0896 \cos 2\Omega + ''.0885 \cos 2☾ + ''.5506 \cos 2\odot \\ & + ''.0092 \cos(\odot + \Gamma).\end{aligned}\right\}^{*} \quad (573)$$

The numerical values of $\Delta\lambda$, and the true obliquity, $= \omega + \Delta\omega$, are given in the ephemeris for every tenth day throughout the year. $\Delta\lambda$ is there called the *equation of the equinoxes*, and is additive algebraically to the longitude referred to the mean equinox in order to obtain the longitude referred to the true equinox.

350. *To determine the nutation in right ascension and declination.* Since the terms of the formulæ are always small, a sufficiently accurate result will be obtained by neglecting the squares and higher powers of these quantities. In other words, we may employ differential formulæ, viz.,

$$\left.\begin{aligned}\Delta\alpha &= \frac{d\alpha}{d\lambda}\Delta\lambda + \frac{d\alpha}{d\omega}\Delta\omega; \\ \Delta\delta &= \frac{d\delta}{d\lambda}\Delta\lambda + \frac{d\delta}{d\omega}\Delta\omega.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (574)$$

For the values of the differential coefficients we employ the equations obtained by applying the general formulæ of trigonometry to the triangle formed by joining the poles of the equator and ecliptic with each other and with the star.

* In No. 2387, *Astronomische Nachrichten*, Oppolzer gives formulæ for these quantities carried out so as to include all terms which are appreciable in the fourth decimal place.

In Fig. 72, P is the pole of the ecliptic, P' of the equator, S any star.

$$PP' = \omega, \quad PS = 90^\circ - \beta, \quad P'S = 90^\circ - \delta,$$
$$SPP' = 90^\circ - \lambda, \quad SP'P = 90^\circ + \alpha.$$

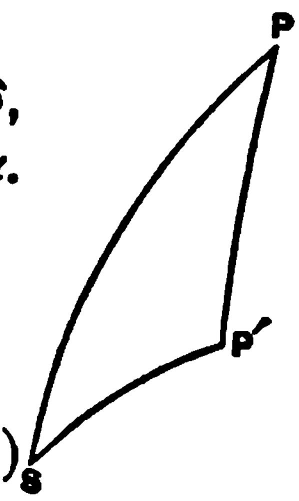

FIG. 72.

Therefore

$$\left.\begin{aligned} \cos\delta\cos\alpha &= \cos\beta\cos\lambda; \\ \cos\delta\sin\alpha &= \cos\beta\sin\lambda\cos\omega - \sin\beta\sin\omega; \\ \sin\delta &= \cos\beta\sin\lambda\sin\omega + \sin\beta\cos\omega. \end{aligned}\right\} (575)$$

Differentiating these equations, considering β as constant, since it is not affected by nutation,

$$\left.\begin{aligned} \cos\delta\sin\alpha d\alpha + \cos\alpha\sin\delta d\delta &= \cos\beta\sin\lambda d\lambda; \\ \cos\delta\cos\alpha d\alpha - \sin\alpha\sin\delta d\delta &= \cos\beta\cos\lambda\cos\omega d\lambda \\ &\quad - (\cos\beta\sin\lambda\sin\omega + \sin\beta\cos\omega)d\omega; \\ \cos\delta d\delta = \cos\beta\cos\lambda\sin\omega d\lambda &+ (\cos\beta\sin\lambda\cos\omega - \sin\beta\sin\omega)d\omega. \end{aligned}\right\} (576)$$

From the second and third of (575) we derive

$$\cos\beta\sin\lambda = \cos\delta\sin\alpha\cos\omega + \sin\delta\sin\omega.$$

Reducing (576) by this and the first of (575), we have

$$\left.\begin{aligned} \cos\delta\sin\alpha d\alpha + \cos\alpha\sin\delta d\delta &= (\cos\delta\sin\alpha\cos\omega + \sin\delta\sin\omega)d\lambda; \\ \cos\delta\cos\alpha d\alpha - \sin\alpha\sin\delta d\delta &= \cos\delta\cos\alpha\cos\omega d\lambda - \sin\delta d\omega; \\ d\delta &= \cos\alpha\sin\omega d\lambda + \sin\alpha d\omega. \end{aligned}\right\} (577)$$

From these we derive

$$\left.\begin{aligned} \frac{d\alpha}{d\lambda} &= \cos\omega + \sin\omega\sin\alpha\tan\delta; & \frac{d\delta}{d\lambda} &= \cos\alpha\sin\omega; \\ \frac{d\alpha}{d\omega} &= -\cos\alpha\tan\delta; & \frac{d\delta}{d\omega} &= \sin\alpha. \end{aligned}\right\} (578)$$

Substituting (572) and (578) in (574), we have*

$$\left.\begin{array}{l}
\Delta\alpha = -(15''.8148 + 6''.8650 \sin\alpha\tan\delta)\sin\text{☊} - 9.2231\cos\alpha\tan\delta\cos\text{☊} \\
\qquad 15\ .8321 \qquad 6\ .8683 \qquad\qquad\qquad\qquad 9.2240 \\
\quad + \ (.1902 + .0825\sin\alpha\tan\delta)\sin 2\text{☊} + .0897\cos\alpha\tan\delta\cos 2\text{☊} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad .0895 \\
\quad - \ (.1872 + .0813\sin\alpha\tan\delta)\sin 2\text{☾} - .0886\cos\alpha\tan\delta\cos 2\text{☾} \\
\qquad\qquad\qquad .0812 \qquad\qquad\qquad\qquad .0885 \\
\quad + \ (.0621 + .0270\sin\alpha\tan\delta)\sin(\text{☾} - \Gamma) \\
\quad + .000\,154\cos 2\alpha\tan^2\delta\sin 2\text{☊} - .000\,160\sin 2\alpha\tan^2\delta\cos 2\text{☊} \\
\quad - \ (1.1642 + .5054\sin\alpha\tan\delta)\sin 2\odot - .5509\cos\alpha\tan\delta\cos 2\odot \\
\qquad 1.1644 \qquad 5052 \qquad\qquad\qquad\qquad\qquad 5506 \\
\quad + \ (.1173 + .0509\sin\alpha\tan\delta)\sin(\odot - \Gamma) \\
\qquad 1170 \qquad 0507 \\
\quad -(.0195 + .0085\sin\alpha\tan\delta)\sin(\odot + \Gamma) - .0093\cos\alpha\tan\delta\cos(\odot + \Gamma); \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0092 \\
\\
\Delta\delta = -\ 6''.8650\cos\alpha\sin\ \text{☊} + 9''.2231\sin\alpha\cos\ \text{☊} \\
\qquad 6\ .8683 \qquad\qquad\qquad 9\ .2240 \\
\quad + \ ''.0825\cos\alpha\sin 2\text{☊} - \ ''.0897\sin\alpha\cos 2\text{☊} \\
\qquad\qquad\qquad\qquad\qquad .0895 \\
\quad - \ .0813\cos\alpha\sin 2\text{☾} + \ .0886\sin\alpha\cos 2\text{☾} \\
\qquad 0812 \qquad\qquad\qquad 0885 \\
\quad + \ .0270\cos\alpha\sin(\text{☾} - \Gamma) \\
\quad -.000\,077\sin 2\alpha\tan\delta\sin 2\text{☊} - (.000\,023 + .000\,060\cos 2\alpha)\tan\delta\cos 2\text{☊} \\
\quad - \ .5054\cos\alpha\sin 2\odot + \ .5509\sin\alpha\cos 2\odot \\
\qquad .5052 \qquad\qquad\qquad 5506 \\
\quad + \ .0509\cos\alpha\sin(\odot - \Gamma) \\
\qquad 0507 \\
\quad - \ .0085\cos\alpha\sin(\odot + \Gamma) + .0093\sin\alpha\cos(\odot + \Gamma). \\
\qquad\qquad\qquad\qquad\qquad .0092
\end{array}\right\} (579)$$

In case of those coefficients which change appreciably during the century the value for 1900.0 is written below that for 1800.0.

Tables have been prepared for facilitating the computation of the above formulæ, but they do not require special consideration here. For our purposes the necessary corrections

* See Peters' *Numerus Constans Nutationis.* Also *Astronomische Nachrichten*, No. 486.

are computed in a simple manner, as explained in Articles 354 and following.

Aberration.

351. Aberration is an apparent displacement of a star's position, resulting from the circumstance that the velocity of light is not infinitely great in comparison with the velocity of the earth. Two essentially different classes of phenomena result from this cause:

First. The observer, who must partake of all the motions of the earth itself, does not see the object in its true position, since the observed direction of a ray of light is determined not by the absolute direction of motion of the undulations coming from the object to the eye, but by the relative motion with respect to the observer. This apparent change of position is called the *aberration of the fixed stars.*

Second. The observer does not see the body in its true position at the instant when the light enters the eye, but in the position which it occupied when the light left the body. This is called *planetary aberration.* This latter we shall not have occasion to consider, as it belongs to another department of astronomy.

The aberration of the fixed stars is determined by the velocity and direction of the motion of the point on the earth's surface occupied by the observer. Of these motions there are three, viz., that due to the diurnal revolution of the earth on its axis, to its annual revolution about the sun, and to the motion of the earth with the sun in space.

The first of these motions produces *diurnal aberration,* which has already been considered so far as is necessary for our purposes.* The last motion it is not important to con-

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* See Articles 173 and 303.

sider, as it affects the place of the star by a constant quantity; further, it is not sufficiently well determined for the purpose, even if it were desirable to consider it. It only remains, therefore, to investigate the change produced by the earth's motion in its orbit, called *annual aberration.*

352. Let the velocity of the ray of light coming from a star and of the earth respectively be considered with respect to three co-ordinate axes, the equator being the plane of X, Y.

Let $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ = the components of the earth's velocity in the direction of the three axes (the measure of the velocity being the space passed over in 1 second);

k = the velocity of light = distance traversed in 1 sidereal second;

α and δ = true right ascension and declination of the star;

= the co-ordinates of point where the ray of light pierces the celestial sphere;

ξ, η, ζ = components of velocity of the ray of light in direction of the three co-ordinate axes.

Then

$$\xi = -k\cos\delta\cos\alpha; \quad \eta = -k\cos\delta\sin\alpha; \quad \zeta = -k\sin\delta. \tag{580}$$

These are minus, since the light moves in a direction opposite to that in which the star is seen.

Let the same symbols affected by accents represent the corresponding quantities affected by aberration. Then

$$\xi' = -k'\cos\delta'\cos\alpha'; \quad \eta' = -k'\cos\delta'\sin\alpha'; \quad \zeta' = -k'\sin\delta'. \tag{581}$$

α' and δ' are then the apparent right ascension and declination of the star, and ξ', η', ζ' are the components of the velocity relatively to the earth.

Since then the relative velocities are equal to the differences of the actual velocities,

$$\left.\begin{aligned} k' \cos\delta' \cos\alpha' &= k \cos\delta \cos\alpha + \frac{dx}{dt}; \\ k' \cos\delta' \sin\alpha' &= k \cos\delta \sin\alpha + \frac{dy}{dt}; \\ k' \sin\delta' \qquad &= k \sin\delta \qquad\quad + \frac{dz}{dt}. \end{aligned}\right\} \quad . \quad (582)$$

Let $\frac{k'}{k} = \varkappa$. Then we readily derive from these equations the following:

$$\left.\begin{aligned} \varkappa \cos\delta' \sin(\alpha'-\alpha) &= -\frac{1}{k}\left[\frac{dx}{dt}\sin\alpha - \frac{dy}{dt}\cos\alpha\right]; \\ \varkappa \cos\delta' \cos(\alpha'-\alpha) &= \cos\delta + \frac{1}{k}\left[\frac{dx}{dt}\cos\alpha + \frac{dy}{dt}\sin\alpha\right]; \\ \varkappa \sin(\delta'-\delta) &= -\frac{1}{k}\left[\frac{dx}{dt}\sin\delta\cos\alpha + \frac{dy}{dt}\sin\delta\sin\alpha - \frac{dz}{dt}\cos\delta\right] \\ &\quad -\frac{1}{2k^2}\left[\frac{dx}{dt}\sin\alpha - \frac{dy}{dt}\cos\alpha\right]^2 \tan\delta; \\ \varkappa \cos(\delta'-\delta) &= 1 + \frac{1}{k}\left[\frac{dx}{dt}\cos\delta\cos\alpha + \frac{dy}{dt}\cos\delta\sin\alpha + \frac{dz}{dt}\sin\delta\right] \\ &\quad + \frac{1}{2k^2}\left[\frac{dx}{dt}\sin\alpha - \frac{dy}{dt}\cos\alpha\right]^2. \end{aligned}\right\} \quad (583)$$

The first two of these are exact; the last two are exact to terms of the second order inclusive.

Dividing the first by the second and the third by the fourth, we have, neglecting terms of the third and higher orders,

$$\left.\begin{aligned}
\alpha' - \alpha &= -\frac{1}{k}\sec\delta\left[\frac{dx}{dt}\sin\alpha - \frac{dy}{dt}\cos\alpha\right] \\
&\quad + \frac{1}{k^2}\sec^2\delta\left[\frac{dx}{dt}\sin\alpha - \frac{dy}{dt}\cos\alpha\right]\times\left[\frac{dx}{dt}\cos\alpha + \frac{dy}{dt}\sin\alpha\right]; \\
\delta' - \delta &= -\frac{1}{k}\left[\frac{dx}{dt}\sin\delta\cos\alpha + \frac{dy}{dt}\sin\delta\sin\alpha - \frac{dz}{dt}\cos\delta\right] \\
&\quad - \frac{1}{2k^2}\left[\frac{dx}{dt}\sin\alpha - \frac{dy}{dt}\cos\alpha\right]^2\tan\delta \\
&\quad + \frac{1}{k^2}\left[\frac{dx}{dt}\sin\delta\cos\alpha + \frac{dy}{dt}\sin\delta\sin\alpha - \frac{dz}{dt}\cos\delta\right] \\
&\quad \times\left[\frac{dx}{dt}\cos\delta\cos\alpha + \frac{dy}{dt}\cos\delta\sin\alpha + \frac{dz}{dt}\sin\delta\right].
\end{aligned}\right\} \quad (584)$$

Let R = the radius vector of the earth;
$\odot$ = the sun's geocentric longitude; then $-\odot$ = the earth's heliocentric longitude;
ω = the obliquity of the ecliptic.

Then x, y, z being the earth's rectangular co-ordinates,

$$x = -R\cos\odot;\ y = -R\sin\odot\cos\omega;\ z = -R\sin\odot\sin\omega. \quad (585)$$

From these we have

$$\left.\begin{aligned}
\frac{dx}{dt} &= R\frac{d\odot}{dt}\sin\odot - \frac{dR}{dt}\cos\odot; \\
\frac{dy}{dt} &= -R\frac{d\odot}{dt}\cos\odot\cos\omega - \frac{dR}{dt}\sin\odot\cos\omega; \\
\frac{dz}{dt} &= -R\frac{d\odot}{dt}\cos\odot\sin\omega - \frac{dR}{dt}\sin\odot\sin\omega.
\end{aligned}\right\} \quad (586)$$

By means of these equations we have the values of $\alpha' - \alpha$ and $\delta' - \delta$ in terms of the sun's distance and longitude, but they are not in a convenient form for practical application unless we are satisfied with an approximation obtained by

regarding the earth's orbit as a circle and the motion uniform. In this case we make $\frac{dR}{dt} = 0$ and $\frac{d\odot}{dt} =$ the mean apparent angular velocity of the sun in longitude.

353. The true velocity of the earth in any part of its orbit may be taken into account as follows: The orbit being an ellipse, its polar equation will be

$$R = \frac{a(1 - e^2)}{1 + e \cos(\odot - \Gamma)}, \quad . \quad . \quad . \quad . \quad (587)$$

a being the semi-major axis, e the eccentricity, and $-(\odot - \Gamma)$ the angle between the major axis and radius vector measured from the perihelion ($\odot$ and Γ having the same significance as in Art. 349).

Let $F =$ the area of the ellipse $= \pi a^2 \sqrt{1 - e^2}$;
$T =$ the time of one revolution of the earth $=$ one sidereal year;
$df =$ an element of area between two consecutive radii vectores;
$dt =$ time required to describe df.

Then by Kepler's first law, viz.—the areas described by the radius vector are proportional to the times—we have

$$\frac{F}{T} = \frac{df}{dt}, \quad \text{or} \quad \frac{\pi a^2 \sqrt{1 - e^2}}{T} = \frac{1}{2} R^2 \frac{d\odot}{dt}, \quad . \quad (588)$$

since the element of area $df = \frac{1}{2}R^2 d(\odot - \Gamma) = \frac{1}{2}R^2 d\odot$.

Therefore

$$R\frac{d\odot}{dt} = \left(\frac{2\pi}{T}\right)\frac{a}{\sqrt{1 - e^2}}[1 + e\cos(\odot - \Gamma)]. \quad . \quad (589)$$

By differentiating (587) we find

$$\frac{dR}{dt} = \left(\frac{2\pi}{T}\right)\frac{a}{\sqrt{1 - e^2}} e \sin (\odot - \Gamma). \quad . \quad . \quad (590)$$

But $\left(\frac{2\pi}{T}\right)$ is equal to the mean angular velocity of the earth in its orbit about the sun; or, what is the same, the apparent angular velocity of the mean sun about the earth. Calling this velocity n, we have, from (586), (589), and (590),

$$\left.\begin{aligned}
\frac{dx}{dt} &= \frac{an}{\sqrt{1 - e^2}} (\sin \odot + e \sin \Gamma); \\
\frac{dy}{dt} &= -\frac{an}{\sqrt{1 - e^2}} \cos \omega (\cos \odot + e \cos \Gamma); \\
\frac{dz}{dt} &= -\frac{an}{\sqrt{1 - e^2}} \sin \omega (\cos \odot + e \cos \Gamma).
\end{aligned}\right\} \quad . \quad (591)$$

The quantity $\dfrac{an}{k\sqrt{1 - e^2}} = \kappa$, say, is called the *constant of aberration.*

Substituting in (584) these values of the differential coefficients, we therefore have

$$\left.\begin{aligned}
\alpha' - \alpha = &- \kappa \sec \delta [\sin \odot \sin \alpha + \cos \odot \cos \alpha \cos \omega] \\
&- \frac{\kappa^2}{4} \sin 1'' \sec^2 \delta [(1 + \cos^2 \omega) \sin 2\alpha \cos 2\odot - 2 \cos \omega \cos 2\alpha \sin 2\odot] \\
&- \kappa e \sec \delta [\sin \Gamma \sin \alpha + \cos \Gamma \cos \alpha \cos \omega] + \frac{\kappa^2}{4} \sin 1'' \sec^2 \delta \sin 2\alpha \sin^2 \omega; \\
\delta' - \delta = &- \kappa [\sin \delta \cos \alpha \sin \odot - (\cos \omega \sin \delta \sin \alpha - \sin \omega \cos \delta) \cos \odot] \\
&- \frac{\kappa^2}{8} \sin 1'' \tan \delta \left\{ [(1 + \cos^2 \omega) \cos 2\alpha - \sin^2 \omega] \cos 2\odot + 2 \cos \omega \sin 2\odot \sin 2\alpha \right\} \\
&- e\kappa [\sin \delta \cos \alpha \sin \Gamma - (\cos \omega \sin \delta \sin \alpha - \sin \omega \cos \delta) \cos \Gamma] \\
&- \frac{\kappa^2}{8} \sin 1'' \tan \delta [(1 + \cos^2 \omega) - \sin^2 \omega \cos 2\alpha].
\end{aligned}\right\} (592)$$

The last two terms in each are constant, or are only subject to a slow secular change; they will therefore be combined with the mean right ascension and declination of the star, and will require no further consideration in this connection, as we are only concerned with the periodic terms.

The most commonly received value of the constant κ is that of Struve, who found from a very carefully executed series of observations at the observatory of Pulkova $\kappa = 20''.4451$. (Recently Nyréne finds from a still more exhaustive investigation $20''.492$.) For 1875.0 the mean value of the obliquity of the ecliptic is $\omega = 23° 27' 19''$.

Substituting these values in (592), and dropping the constant terms, we have finally

$$\left.\begin{aligned}
\alpha' - \alpha = &- 20''.4451 \sec\delta[\sin\odot\sin\alpha + \cos\odot\cos\alpha\cos\omega] \\
&- \quad .0009330 \sec^2\delta \sin 2\alpha \cos 2\odot \\
&+ \quad .0009295 \sec^2\delta \cos 2\alpha \sin 2\odot; \\
\delta' - \delta = &- 20''.4451 \sin\delta\cos\alpha\sin\odot \\
&+ 20\ .4451 \cos\odot\,[\sin\delta\sin\alpha\cos\omega - \cos\delta\sin\omega] \\
&- \quad .0004648 \tan\delta \sin 2\alpha \sin 2\odot \\
&+ \quad [.0000401 - .0004665 \cos 2\alpha] \tan\delta \cos 2\odot.
\end{aligned}\right\} \quad (593)$$

Reduction to Apparent Place.

354. We have now deduced the essential formulæ for reducing a star from mean to apparent place or the converse. The place as given in the star catalogue will be the mean place for the beginning of the year of the catalogue. The reduction of this place to the mean place at any other date has been explained and illustrated with sufficient fulness. In applying the formulæ as we have done we obtain the mean place for the beginning of the year, to which we reduce the star's co-ordinates. If now we wish to reduce this *mean* place to the *apparent* place at a time τ from the beginning of the year (τ being expressed as a fraction of a year), we must add to the mean right ascension and declination the

precession and proper motion for the time τ, as given by formulæ (565); the result is the *mean* place at time τ. To this mean place the nutation being added as given by (579), we have the *true* place; finally adding the aberration (593), we have the required apparent right ascension and declination of the star.

The following are the formulæ written out in full, omitting those terms in the *nutation* and *aberration* which are ordinarily inappreciable:

$$
\left.\begin{array}{ll}
\alpha' - \alpha = & (m + n \sin\alpha \tan\delta)\,\tau + \tau\mu \\
& -\,(15''.8148 + 6''.8650 \sin\alpha\tan\delta)\sin\Omega \\
& \quad\;\; 15\;\,.8321 \quad\;\, 6\;\,.8683 \\
& +\,(\quad\;\; .1902 + \quad .0825 \sin\alpha\tan\delta)\sin 2\Omega \\
& -\,(\quad\;\; .1872 + \quad .0813 \sin\alpha\tan\delta)\sin 2☾ \\
& +\,(\quad\;\; .0621 + \quad .0270 \sin\alpha\tan\delta)\sin(☾ - \Gamma) \\
& -\,(\;1\;.1642 + \quad .5054 \sin\alpha\tan\delta)\sin 2\odot \\
& +\,(\quad\;\; .1173 + \quad .0509 \sin\alpha\tan\delta)\sin(\odot - \Gamma) \\
& -\,(\quad\;\; .0195 + \quad .0085 \sin\alpha\tan\delta)\sin(\odot + \Gamma) \\
& -\quad 9\;.2231 \cos\alpha\tan\delta\cos\Omega + .0897 \cos\alpha\tan\delta\cos 2\Omega \\
& \quad\;\;\; 9\;.2240 \\
& -\quad\;\; .0886 \cos\alpha\tan\delta\cos 2☾ - .5509 \cos\alpha\tan\delta\cos 2\odot \\
& -\quad\;\; .0093 \cos\alpha\tan\delta\cos(\odot + \Gamma) \\
& -\; 20\;.4451 \cos\omega\sec\delta\cos\alpha\cos\odot \\
& -\; 20\;.4451 \qquad\quad \sec\delta\sin\alpha\sin\odot; \\
\delta' - \delta = & \tau n \cos\alpha + \tau\mu' \\
& -\; 6''.8650 \cos\alpha\sin\Omega + 9''.2231 \sin\alpha\cos\Omega \\
& \quad 6\;.8683 \qquad\qquad\qquad 9\;.2240 \\
& +\quad .0825 \cos\alpha\sin 2\Omega - \;\; .0897 \sin\alpha\cos 2\Omega \\
& -\quad .0813 \cos\alpha\sin 2☾ + \;\; .0886 \sin\alpha\cos 2☾ \\
& +\quad .0270 \cos\alpha\sin(☾ - \Gamma) \\
& -\quad .5054 \cos\alpha\sin 2\odot + \;\; ''.5509 \sin\alpha\cos 2\odot \\
& +\quad .0509 \cos\alpha\sin(\odot - \Gamma) \\
& -\quad .0085 \cos\alpha\sin(\odot + \Gamma) + ''.0093 \sin\alpha\cos(\odot + \Gamma) \\
& -\; 20''.4451 \cos\omega\cos\odot\,(\tan\omega\cos\delta - \sin\alpha\sin\delta) \\
& -\; 20\;.4451 \cos\alpha\sin\delta\sin\odot.
\end{array}\right\} \quad (594)
$$

The values of the constants are determined for 1800.0. Where the change is appreciable the value for 1900.0 is written below.

355. The formulæ as written above are complicated and very inconvenient for practical application. If no method could be devised for abridging the work, star reduction would be such a formidable undertaking that but little progress would be possible in this direction. The method in common use, however, originally proposed by Bessel, reduces the labor to a small fraction of that required for applying the formulæ directly.

It will be observed that the first part of $(\alpha' - \alpha)$ consists of a number of terms which have a factor of the general form $(m' + n' \sin \alpha \tan \delta)$, the constants m' and n' in each case having nearly the same ratio to each other as m to n in the precession formulæ, viz., 2.3 approximately. Therefore let

$$\left.\begin{aligned}
6.8650 &= ni; & 15.8148 &= mi + k;\\
.0825 &= ni'; & .1902 &= mi' + k';\\
.0813 &= ni''; & .1872 &= mi'' + k'';\\
.0270 &= ni'''; & .0621 &= mi''' + k''';\\
.5054 &= ni^{\text{iv}}; & 1.1642 &= mi^{\text{iv}} + k^{\text{iv}};\\
.0509 &= ni^{\text{v}}; & .1173 &= mi^{\text{v}} + k^{\text{v}};\\
.0085 &= ni^{\text{vi}}; & .0195 &= mi^{\text{vi}} + k^{\text{vi}}.
\end{aligned}\right\} \qquad (595)$$

By introducing these values equations (594) may be written

$$\begin{aligned}
\alpha' = \alpha &+ \tau\mu + [\tau - i \sin \Omega + i' \sin 2\Omega - i'' \sin 2☾ + i''' \sin (☾ - \Gamma) - i^{\text{iv}} \sin 2\odot \\
&+ i^{\text{v}} \sin (\odot - \Gamma) - i^{\text{vi}} \sin (\odot + \Gamma)] \times [m + n \sin \alpha \tan \delta] \\
&- [9''.2231 \cos \Omega - 0''.0897 \cos 2\Omega + 0''.0886 \cos 2☾ + 0''.5509 \cos 2\odot \\
&\qquad\qquad + 0''.0093 \cos (\odot + \Gamma)] \cos \alpha \tan \delta \\
&- 20''.4451 \cos \omega \sec \delta \cos \alpha \cos \odot - 20''.4451 \sec \delta \sin \alpha \sin \odot \\
&- k \sin \Omega + k' \sin 2\Omega - k'' \sin 2☾ + k''' \sin (☾ - \Gamma) - k^{\text{iv}} \sin 2\odot \\
&\qquad\qquad + k^{\text{v}} \sin (\odot - \Gamma) - k^{\text{vi}} \sin (\odot + \Gamma); \\
\delta' = \delta &+ \tau\mu' + [\tau - i \sin \Omega + i' \sin 2\Omega - i'' \sin 2☾ + i''' \sin (☾ - \Gamma) \\
&- i^{\text{iv}} \sin 2\odot + i^{\text{v}} \sin (\odot - \Gamma) - i^{\text{vi}} \sin (\odot + \Gamma)] \times n \cos \alpha \\
&+ [9''.2231 \cos \Omega - 0''.0897 \cos 2\Omega + 0''.0886 \cos 2☾ + .5509 \cos 2\odot \\
&\qquad\qquad + 0.0093 \cos (\odot + \Gamma)] \sin \alpha \\
&- 20''.4451 \cos \omega \cos \odot (\tan \omega \cos \delta - \sin \alpha \sin \delta) \\
&\qquad\qquad - 20''.4451 \cos \alpha \sin \delta \sin \odot.
\end{aligned}$$

It will be observed that the corrections to the mean values of α and δ consist of terms made up of two classes of factors, the first class independent of the star's place and varying with the time, the other class depending on the star's place and varying so slowly that they may be regarded as constant for a considerable time. Writing them in accordance with Bessel's original notation,

$$\left.\begin{array}{l}
{}^{*}A = \tau - i\sin\Omega + i'\sin 2\Omega - i''\sin 2☾ + i'''\sin(☾ - \Gamma) - i^{\text{iv}}\sin 2\odot \\
\qquad + i^{\text{v}}\sin(\odot - \Gamma) - i^{\text{vi}}\sin(\odot + \Gamma); \\
B = -9''.2231\cos\Omega + ''.0897\cos 2\Omega - .0886\cos 2☾ - .5509\cos 2\odot \\
\qquad - .0093\cos(\odot + \Gamma); \\
C = -20''.4451\cos\omega\cos\odot; \\
D = -20''.4451\sin\odot; \\
E = -h\sin\Omega + h'\sin 2\Omega - h''\sin 2☾ + h'''\sin(☾ - \Gamma) - h^{\text{iv}}\sin 2\odot \\
\qquad + h^{\text{v}}\sin(\odot - \Gamma) - h^{\text{vi}}\sin(\odot + \Gamma); \\
\qquad a = \frac{1}{15}(m + n\sin\alpha\tan\delta);\dagger \qquad a' = n\cos\alpha; \\
\qquad b = \frac{1}{15}\cos\alpha\tan\delta; \qquad b' = -\sin\alpha; \\
\qquad c = \frac{1}{15}\cos\alpha\sec\delta; \qquad c' = \tan\omega\cos\delta - \sin\alpha\sin\delta; \\
\qquad d = \frac{1}{15}\sin\alpha\sec\delta; \qquad d' = \cos\alpha\sin\delta.
\end{array}\right\} \quad (596)$$

Then our formulæ become

$$\left.\begin{array}{l}
\alpha' = \alpha + \tau\mu + Aa + Bb + Cc + Dd + E; \\
\delta' = \delta + \tau\mu' + Aa' + Bb' + Cc' + Dd'.
\end{array}\right\} \quad (597)$$

A, B, C, D, E being the same for all stars are computed in advance for every day throughout the year, and the values given in the nautical almanac and the similar publications of other countries; so for our purposes we need only take them from these sources.

In some star catalogues a, b, c, d and a', b', c', d' are given in connection with the star's place. For the purposes of an accurate reduction, however, these become obsolete in a few years, as m, n, α, δ, and ω are all subject to slow secular

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* See Art. 358.

† These are divided by 15, since the right ascension is generally given in time.

changes. It will be advisable to recompute them if much time has elapsed.

Example. Required the apparent place of α Lyræ, 1884, November 10, for upper transit, Washington.

$$\text{Mean } \alpha = 18^h\ 33^m\ 0^s.678 \qquad \text{Mean } \delta = 38°\ 40'\ 34''.40$$
$$\mu = {}^s.0179 \qquad \mu' = +\ ''.2726$$
$$\tau = 0.863$$
$$\left.\begin{array}{l}\tfrac{1}{15}m = 3^s.0724\\ n = 20''.0534\end{array}\right\} \text{by formulæ (549).}$$

Then

	$\log a = 0.3039$	$\log b = 7.8842$	$\log c = 8.0884$	$\log d = 8.9269_n$
N. A. p. 284,	$\log A = 9.9602$	$\log B = 0.9619$	$\log C = 1.0894$	$\log D = 1.1883$
	$\log a' = 0.4592$	$\log b' = 9.9955$	$\log c' = 9.9809$	$\log d' = 8.9528$
	$\log Aa = 0.2641$	$\log Bb = 8.8461$	$\log Cc = 9.1778$	$\log Dd = 0.1152_n$
	$\log Aa' = 0.4194$	$\log Bb' = 0.9574$	$\log Cc' = 1.0703$	$\log Dd' = 0.1411$

Mean place	$\alpha =$	$18^h\ 33^m\ 0^s.678$	$\delta =$	$38°\ 40'\ 34''.40$
	$Aa =$	1 .837	$Aa' =$	2 .63
	$Bb =$	.070	$Bb' =$	9 .07
	$Cc =$	.150	$Cc' =$	11 .76
	$Dd = -$	1 .304	$Dd' =$	1 .38
	$E =$	.001		
	$\tau\mu =$	.016	$\tau\mu' =$	.23
Apparent place	$\alpha' =$	$18^h\ 33^m\ 1^s.448$	$\delta =$	$38°\ 40'\ 59''.47$

356. The above form of reduction is most convenient when a considerable number of apparent places is required, or when the star catalogue gives reliable values of the constants a, b, c, d, etc. If these quantities are not given and only one or two apparent places are required, a different form may be given to equations (597) which will be more convenient. This transformation, also due to Bessel, is as follows:

Write

$$f = mA + E; \qquad i = C \tan \omega;$$
$$g \cos G = nA; \qquad h \cos H = D;$$
$$g \sin G = B; \qquad h \sin H = C.$$

Then we have

$$\left.\begin{array}{l}\alpha' = \alpha + \tau\mu + f \qquad + g \sin (G + \alpha) \tan \delta + h \sin (H + \alpha) \sec \delta;\\ \delta' = \delta + \tau\mu' + i \cos \delta + g \cos (G + \alpha) \qquad + h \cos (H + \alpha) \sin \delta.\end{array}\right\} \quad (598)$$

The values of τ, f, G, H, $\log g$, $\log h$, and $\log i$ are also given in the ephemeris for every day of the year.

As an example, let these formulæ be applied to determine the apparent place of α Lyræ on the date given above.

We have	$\alpha = 18^h\ 33^m.0$	$\delta = 38°\ 40'.6$
page 291 of ephemeris,	$G = \ \ 1\ \ 46\ .3$	$^*G + \alpha = 20^h\ 19^m.3$
	$H = \ \ 2\ \ 34\ .2$	$^*H + \alpha = 21\ \ \ 7\ .2$

	$\log \frac{1}{15} = 8.8239$	$\log \frac{1}{15} = 8.8239$
page 291 of ephemeris,	$\log g = 1.3109$	$\log h = 1.2952$
	$^*\sin(G + \alpha) = 9.9142_n$	$^*\sin(H + \alpha) = 9.8373$
	$\tan \delta = 9.9033$	$\sec \delta = \ .1075$
	$\log(g) = 9.9523_n$	$\log(h) = \ .0639_n$
	$\log g = 1.3109$	$\log h = 1.2952$
	$\cos(G + \alpha) = 9.7570$	$\cos(H + \alpha) = 9.8610$
	$\log(g') = 1.0679$	$\sin \delta = 9.7958$
page 291 of ephemeris,	$\log i = 0.7273$	$\log(h') = 0.9520$
	$\cos \delta = 9.8925$	
	$\log(i) = 0.6198$	

$\alpha =$	$18^h\ 33^m\ 0^s.678$	$\delta =$	$38°\ 40'\ 34''.40$
$f =$	$2\ .804$	$(g') =$	$11\ .70$
$(g) = -$	$.895$	$(h') =$	$8\ .95$
$(h) = -$	$1\ .158$	$(i) =$	$4\ .17$
$\tau\mu =$	$.016$	$\tau\mu' =$	$.23$
$\alpha' =$	$18^h\ 33^m\ 1^s.445$	$\delta' =$	$38°\ 40'\ 59''.45$

357. *Note.* Certain of the small terms which have been neglected in the preceding formulæ will sometimes be appreciable for stars near the pole where great accuracy is required.

1st. *The Precession for Time τ.* We have only used the term depending on the first power of τ. The values of the second differential coefficients are given by equations (565). The numerical values being substituted, the only terms which can be appreciable are

$$\left.\begin{aligned} \Delta(\alpha' - \alpha) &= + {}^s.000\,003\tau^2 \sin\alpha \tan\delta - 0^s.000\,149\tau^2 \cos\alpha \tan\delta \\ &\quad - .000\,065\tau^2 \sin 2\alpha \tan^2\delta; \\ \Delta(\delta' - \delta) &= + \ .000\,975\tau^2 \sin^2\alpha \tan\delta. \end{aligned}\right\} \qquad (599)$$

2d. In the formulæ for *aberration* (593), rigorously α, δ, $\odot$, and ω are not the *mean* values of these quantities as there assumed, but the *true* values. They

* A table giving logarithmic sines and cosines with the argument expressed in time is convenient. If this is not available, $(G + a)$ and $(H + a)$ must be reduced to arc.

should therefore be corrected for *nutation*. The necessary corrections to $(\alpha' - \alpha)$ and $(\delta' - \delta)$ as given by (593) may be determined by differential formulæ.

Since $(\alpha' - \alpha) = f(\alpha, \delta, \odot, \omega)$, and similarly for $(\delta' - \delta)$,

$$\left.\begin{aligned} \Delta(\alpha' - \alpha) &= \frac{d(\alpha'-\alpha)}{d\alpha}\Delta\alpha + \frac{d(\alpha'-\alpha)}{d\delta}\Delta\delta + \frac{d(\alpha'-\alpha)}{d\odot}\Delta\odot + \frac{d(\alpha'-\alpha)}{d\omega}\Delta\omega; \\ \Delta(\delta' - \delta) &= \frac{d(\delta'-\delta)}{d\alpha}\Delta\alpha + \frac{d(\delta'-\delta)}{d\delta}\Delta\delta + \frac{d(\delta'-\delta)}{d\odot}\Delta\odot + \frac{d(\delta'-\delta)}{d\omega}\Delta\omega. \end{aligned}\right\} \quad (600)$$

Where $\Delta\alpha$, $\Delta\delta$, etc., represent the corrections for nutation given by (572) and (579).

Practically the terms in $\Delta\odot$ and $\Delta\omega$ will never be appreciable, and of the values of $\Delta\alpha$ and $\Delta\delta$ we need only retain the following terms:

$$\left.\begin{aligned} \Delta\alpha &= -[6''.865 \sin\alpha \sin \Omega + 9''.2235 \cos\alpha \cos \Omega] \tan\delta; \\ \Delta\delta &= -6''.865 \cos\alpha \sin \Omega + 9''.2235 \sin\alpha \cos \Omega. \end{aligned}\right\} \quad . \quad (601)$$

Differentiating (593) with respect to α and δ, neglecting the smaller terms,

$$\frac{d(\alpha' - \alpha)}{d\alpha} = -20''.4451 \sec\delta[\cos\alpha \sin\odot - \sin\alpha \cos\odot \cos\omega];$$

$$\frac{d(\alpha' - \alpha)}{d\delta} = -20''.4451 \sec\delta \tan\delta[\sin\alpha \sin\odot + \cos\alpha \cos\odot \cos\omega];$$

$$\frac{d(\delta' - \delta)}{d\alpha} = 20''.4451 [\sin\delta \sin\alpha \sin\odot + \sin\delta \cos\alpha \cos\odot \cos\omega];$$

$$\begin{aligned} \frac{d(\delta' - \delta)}{d\delta} = &-20''.4451 \cos\delta \cos\alpha \sin\odot \\ &+ 20''.4451 \cos\odot [\cos\delta \sin\alpha \cos\omega + \sin\delta \sin\omega]. \end{aligned}$$

Substituting in (600) and retaining only terms multiplied by $\tan\delta$ or $\sec\delta$, we find

$$\left.\begin{aligned} \Delta(\alpha'-\alpha) &= \frac{1}{15}\,\frac{20''.4451}{2}\sin 1''\tan\delta\sec\delta \begin{Bmatrix} -(6''.865+9''.2235\cos\omega)\sin 2\alpha\cos(\odot+\Omega); \\ +(6''.865\cos\omega+9''.2235)\cos 2\alpha\sin(\odot+\Omega); \\ +(6''.865-9''.2235\cos\omega)\sin 2\alpha\cos(\odot-\Omega); \\ -(6''.865\cos\omega-9''.2235)\cos 2\alpha\sin(\odot-\Omega); \end{Bmatrix} \\ \Delta(\delta'-\delta) &= \frac{20''.4451}{4}\sin 1''\sin\delta\tan\delta \begin{Bmatrix} -(6''.865+9''.2235\cos\omega)\cos 2\alpha\cos(\odot+\Omega); \\ -(6''.865\cos\omega+9''.2235)\sin 2\alpha\sin(\odot+\Omega); \\ +(6''.865-9''.2235\cos\omega)\cos 2\alpha\cos(\odot-\Omega); \\ +(6''.865\cos\omega-9''.2235)\sin 2\alpha\sin(\odot-\Omega); \\ +(6''.865-9''.2235\cos\omega)\cos(\odot+\Omega); \\ -(6''.865+9''.2235\cos\omega)\cos(\odot-\Omega). \end{Bmatrix} \end{aligned}\right\} \quad (602)$$

These expressions reduce to the following:

$$\left.\begin{array}{l}
\Delta(\alpha'-\alpha)=\left\{\begin{array}{l}
-\ ^{s}.000\,05065 \sin 2\alpha \cos(\odot+\Omega)\\
+\ .000\,05129 \cos 2\alpha \sin(\odot+\Omega)\\
-\ .000\,00527 \sin 2\alpha \cos(\odot-\Omega)\\
+\ .000\,00966 \cos 2\alpha \sin(\odot-\Omega)
\end{array}\right\}\tan\delta\sec\delta;\\
\\
\Delta(\delta'-\delta)=\left\{\begin{array}{l}
-\ ''.000\,3799 \cos 2\alpha \cos(\odot+\Omega)\\
-\ .000\,3847 \sin 2\alpha \sin(\odot+\Omega)\\
-\ .000\,0395 \cos 2\alpha \cos(\odot-\Omega)\\
-\ .000\,0725 \sin 2\alpha \sin(\odot-\Omega)\\
-\ .000\,0391 \cos(\odot+\Omega)\\
-\ .000\,3799 \cos(\odot-\Omega)
\end{array}\right\}\sin\delta\tan\delta.
\end{array}\right\} \quad . \quad . \quad (603)$$

3d. In a few cases of double stars the mean place of the star requires a correction for orbital motion. The corrections to the right ascension and declination will have the form

$$\Delta\alpha = a + bt + k\sin(n + x);$$
$$\Delta\delta = a' + b't + k'\sin(n + x');$$

the quantities entering into the formulæ depending on the elements of the star's orbit.

358. The foregoing comprises all that is necessary for reducing stars from mean to apparent place, or from apparent to mean place. In the latter case the corrections will be applied with the opposite signs to those given by formulæ (597) or (598). Since 1834 the factors A, B, C, D have been published by the British Nautical Almanac, and in the American Ephemeris since its first publication, 1855. In the British Almanac and previous to 1865 in the American Ephemeris the notation is not Bessel's which we have given, but that of Baily, viz., A is interchanged with C, and B with D.* Particu-

* This unnecessary and confusing change of notation was introduced by Baily for no better reason than the following: "I have thought it desirable that we should as much as possible make them serve the purpose of an *artificial memory*. It is on this account that I have made *AB* represent the quantity by which ABerration is determined; *C* the quantity by which preCession is determined; and *D* the quantity by which the Deviation, or (as it is now more generally called) the nutation, is determined."—*British Association Catalogue, p.* 34, *note*.

lar attention must therefore be given to the notation, otherwise errors will be very likely to occur. Since 1865 the notation of Bessel has been employed in the American Ephemeris.

For any date from 1750 to 1850 the logarithms of A, B, C, D may be taken from Bessel's *Tabulæ Regiomontanæ*. Bessel's constants are employed and the smaller terms are neglected; they will, however, give all necessary precision in the few cases where it will be found necessary to employ them. A convenient table by Hubbard for correcting them so as to make the values conform to the constants of Struve and Peters will be found in Gould's *Astronomical Journal*, vol. iv. p. 142. Bessel's tables are computed for every tenth day of the *fictitious year*. Their employment involves a subject the consideration of which we have not found necessary heretofore, viz.,

The Fictitious Year.

359. We have heretofore spoken of the year without specifying very definitely which of the various periods called a year was to be understood. The common year is not well adapted to the requirements of astronomy, since the length is not the same in all cases, each fourth year containing one more day than the other three. The Julian year of 365¼ days is better, but its length does not exactly correspond to the movements of the earth in its orbit.

In the reduction of star places Bessel obviates the difficulties which would follow from the employment of either of the above periods by employing a fictitious year to begin at the instant when the longitude of the mean sun is 280°. This instant will of course not coincide with the transit of the sun over the meridian of Greenwich or Washington, but from

the known mean motion of the sun the Greenwich or Washington time may be found at which the mean longitude is 280°, and consequently the meridian over which the sun is passing at this instant. This is sometimes called the *normal meridian*, and may then be employed as the prime meridian from which to reckon longitudes throughout the year precisely as the meridians of Greenwich and Washington are used. Since the sun's mean right ascension equals the mean longitude, the sidereal time at this meridian corresponding to the beginning of the year will be $18^h\,40^m$ (= 280°). If then we imagine a point on the celestial equator whose right ascension is $18^h\,40^m$, the sidereal day throughout the fictitious year may be regarded as beginning at the instant when this point crosses the meridian, just as in the common method the sidereal day begins when the vernal equinox crosses the meridian. By adopting this device a uniformity and simplicity is introduced into those quantities which are functions of τ. This is also the date to which the mean places of stars are reduced in the star catalogues. When the elements of reduction are taken from the Nautical Almanac or American Ephemeris no attention need be given to this matter, as it is already provided for.

Bessel calls the instant when the sun's mean longitude equals 280° Jan. 0.0 of the fictitious year. This corresponds to Dec. 31.0 of the usual method of reckoning; that is, according to Bessel's method Jan. 1, 2, 3, etc., indicate 1, 2, 3, etc., days from the beginning of the year, while in the common method the beginning of the 1st, 2d, etc., days is understood.

We shall now show the relation between the beginning of the fictitious and common years, afterwards returning to the *Tabulæ Regiomontanæ*.

360. During one complete century the period of the common year is the same as that of the Julian year. Suppose now for the moment that at 1800.0 the fictitious year began

with the date Jan. 0.0 of the common year, and that the length of the tropical year coincided with that of the Julian. Then for any other date $1800 + t$ we should have

$$\text{Beginning of year} = \text{Jan. } 0.0 + \tfrac{1}{4}f, \quad . \quad . \quad (604)$$

where f is the remainder after dividing the number of the year by 4. In case of a leap-year, where the number of the year is exactly divisible by 4, f must be made equal to 4, since the intercalary day is not introduced until the end of February.

If we choose, in accordance with Bessel, as our prime meridian that of Paris, the above formula involves two erroneous assumptions: first, the beginning of the year from which we reckon will not coincide with Jan. 0.0; and second, the length of the tropical year is not that of the Julian. We shall use the constants of Bessel in order to have our results those of the *Tabulæ Regiomontanæ*.

For mean noon at Paris, viz., 1800, Jan. 0.0, Bessel finds for the sun's mean longitude

$$279° \; 54' \; 1''.36,$$

and for the mean daily motion of the sun in longitude

$$3548''.3302 + ''.000\,000\,6902t,\text{*}$$

where t = number of years elapsed since 1800.

For the meridian of Paris we must add to (604) the time required for the sun to move $358''.64$, viz., 0.10107289 day.

It remains to correct (604) for the difference between the

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* It will be observed from the expression for the mean daily motion that the length of the year is not constant; the variation, however, amounts only to $0^s.595$ per century.

Julian and tropical years. The tropical motion of the sun in one Julian year is, according to Bessel,

$$360^\circ\ 00'\ 27''.605844 + 0''.0002443 61t.^*$$

Therefore the mean tropical motion in t years will be

$$[360^\circ\ 00'\ 27''.605844]t + 0''.00012218t^2.$$

The time required for the mean sun to pass over the distance $27''.605844t$, expressed as a fraction of a day, will be $.0077799535t + 0.000\,000\,034433t^2$. Therefore the complete formula for the Paris mean time of the beginning of any fictitious year will be

$$\text{Jan. } 0.0 + 0.^d10107289 - 0.^d0077799535t - 0.000\,000\,034433t^2 + \tfrac{1}{4}f. \quad (605)$$

To reduce any mean solar date at Paris to the date of the fictitious year the above quantity must be subtracted. Therefore let

$$k = -0.10107289 + 0.0077799535t + 0.000\,000\,034433t^2 - \tfrac{1}{4}f.$$

k is then the longitude east from Paris of the meridian where the fictitious year begins, or of the normal meridian.

Let d = the longitude west of Paris of any meridian, expressed as a fraction of a day.

Then the reduction which must be applied to any mean solar date at this meridian to reduce it to the normal meridian is $k + d$.

361. Let us now return to the *Tabulæ Regiomontanæ*. The logarithms of A, B, C, D, τ, and the quantity E are there given for every tenth day of the fictitious year from 1750 to 1850; the intervals being *sidereal* instead of *mean solar* days, an arrangement which is a little more convenient in star re-

* This quantity divided by 365.25 is the mean daily motion already given.

duction, for the reason that, the star being generally observed on the meridian, its right ascension is at once the sidereal time of observation. In order to apply the tables we must first convert this sidereal time to the corresponding sidereal time at the normal meridian.

It will be remembered that the sidereal day of the fictitious year at any meridian begins at $18^h\ 40^m$ sidereal time; therefore at this meridian itself the tables are applicable for this instant of local time. For any other meridian at the instant $18^h\ 40^m$ local sidereal time the argument of the tables will be $k + d$.

At any other sidereal time g at this last meridian the argument will be

$$k + d + \frac{g - 18^h\ 40^m}{24^h},$$

which must be less than unity and positive. Or we may write

$$g' = \frac{g + 5^h\ 20^m}{24^h}$$

as the quantity to be added to $k + d$, omitting one whole day when $g + 5^h\ 20^m =$ or $> 24^h$.

If, as before assumed, we regard the sidereal day of the fictitious year as beginning when the right ascension of the meridian is $18^h\ 40^m$, then as long as the right ascension of the sun is less than this quantity it will cross the meridian before the point on the equator having this right ascension, and the day of the fictitious year will be the same as the common date. When the sun's right ascension is equal to $18^h\ 40^m$ (the sun being on the meridian) the two days begin together, and when it is greater than $18^h\ 40^m$ the sidereal day of the fictitious year begins before the common day, and therefore one day must be added to the common reckoning for the

date of the fictitious year. Therefore the argument of the table will be

$$k + d + g' + i,$$

in which $i = 0$ from beginning of the year to where the right ascension of the mean sun equals the sidereal time, after which $i = 1$.

The *Tabulæ Regiomontanæ* then give the following quantities:

Table I gives k for the longitude of Paris expressed in hours, minutes, and seconds, and also as a fraction of a day, for every year from 1750 to 1849.

Table II gives d, the west longitude from Paris of a number of the principal cities of Europe. (Better values can, however, be found in the ephemeris.)

Auxiliary table, p. 16, gives $g' = \dfrac{g + 5^h\ 20^m}{24}$.

Table VIII, pp. 17–116 inclusive, gives $\log A$, $\log B$, $\log C$, $\log D$, $\log \tau$, and E.

For C and D table IX may be employed. It requires no special explanation here.

Example. Required the logarithms of A, B, C, D, τ, for 1825, July $1^d\ 10^h$, Greenwich sidereal time.

Table I for 1825,	$k = -\ .157$	
Table II for 1825,	$d = +\ .007$	
Page 16,	$g' = +\ .639$	
	$i = \ \ .000$	
Argument = July	1.489	
Page 92, table VIII,	$\log A = 9.9224$	
	$\log B = 0.3026$	$E = +\ ''.05$
	$\log \tau = 9.6975$	
table IX,	$\log C = \ .4817$	
	$\log D = 1.3006_n$	

The quantities have been interpolated directly from the tables; log C and log D are given more accurately by table IX. If thought desirable, the interpolation may be carried out to second differences, but this will not often be necessary.

As an example of a case where $i = 1$ let it be required to find the above quantities for 1825, Dec. $1^d\ 10^h$, Greenwich sidereal time.

As before,	$k =$	$-\ .157$
	$d =$	$+\ .007$
Table VI, right ascension of	$g' =$	$.639$
Mean sun Dec. 1 is $16^h\ 40^m$, therefore	$i =$	1.000
Argument = Dec.		2.489

With this argument we find

$\log A = 0.0867$; $\log B = .4976$; $\log C = .7599$;
$\log D = 1.2772$; $\log \tau = 9.9631$; $E = +.05$.

Various forms of tables for star reductions have been proposed and employed. Some of these are very useful for special purposes, but it is not necessary to enter into the details of their construction in this connection.

362. *Conversion of Mean Solar into Sidereal Time and the converse.* The solution of this problem for any date after the British and American Nautical Almanacs became available in their present form has been treated with all necessary fulness in Articles 94 and 95. For earlier dates other methods must be used. The *Tabulæ Regiomontanæ* gives the data necessary for solving the problem for any date between 1750 and 1850.

We have shown in Art. 94 that the *mean time* at any meridian is equal to the true hour-angle of the second mean sun, which moves uniformly in the equator, and whose mean

right ascension is equal to the mean longitude of the first mean sun, which moves in the ecliptic.

Also, the *sidereal time* is equal to the hour-angle of the true equinox. Therefore in our formula

$$\Theta = \alpha \odot + T. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (199)$$

$\alpha \odot$ must be understood to mean the true right ascension of the second mean sun. This equals the mean right ascension plus the nutation of the vernal equinox in right ascension. The latter is found from the general equations (579), by making $\alpha = 0$, $\delta = 0$ to be $\varDelta\lambda \cos \omega$, and is given in the ephemeris as the "equation of the equinoxes in right ascension." It is included in the sidereal time of mean noon given by the ephemeris. When the ephemeris is available it will therefore require no further notice.

Table VI of the *Tabulæ Regiomontanæ* gives the right ascension of the second mean sun corrected for the solar nutation of the equinox for every mean noon at the fictitious meridian. The fictitious year always begins with the same right ascension of the mean sun, therefore this table is available for every year. The number taken from this table for any date, which must be the date at the normal meridian, is then corrected for lunar nutation in right ascension, which is given by table IV. The result is the sidereal time of mean noon, V_0, at the normal meridian, which may be used in precisely the same way as the sidereal time of mean noon at Washington. (See Articles 94 and 95.) Or writing the formulæ out in full,

$$\Theta = T + \text{table VI} + \text{table IV} + (T + k + d)(\mu - 1); \quad (606)$$

or $V = V_0 + (k + d)(\mu - 1) = \text{VI} + \text{IV} + (k + d)(\mu - 1)$,

$\Theta = T + V + T(\mu - 1)$.

And for converting sidereal into mean solar time,

$$T = \Theta - V - (\Theta - V)\left(1 - \frac{1}{\mu}\right); \quad . \quad . \quad . \quad (607)$$

The notation being that of Articles 94 and 95.

Example. Given 1825, July $1^d\ 7^h\ 25^m$, Greenwich mean solar time. Required the corresponding sidereal time.

By the first of formulæ (606),

$$\begin{array}{rl} T = & 7^h\ 25^m\ \ 0^s.000 \\ \text{Table VI} = & 6\ \ 37\ \ 33\ .099 \\ \text{Table IV} = & \qquad\qquad 1\ .015 \\ (T+k+d)(\mu-1), \text{Table VII} = & \qquad\quad 37\ .606 \\ \hline \Theta = & 14^h\ 3^m\ 11^s.72 \end{array}$$

$$\begin{array}{rl} T = & 7^h\ 25^m\ \ 0^s.000 \\ \text{Table I}, k = & -3\ \ 45\ \ 26\ .1 \\ \text{Table II}, d = & +\quad\ 9\ \ 21\ .6 \\ \hline (T+k+d) = & 3^h\ 48^m\ 55^s.5 \end{array}$$

Example 2. Given 1825, July $1^d\ 14^h\ 3^m\ 11^s.72$, Greenwich sidereal time. Required the corresponding mean solar time.

$$(k+d) = -3^h\ 36^m\ 4^s.5$$

$$\begin{array}{rl} \text{Table VI} = & 6^h\ 37^m\ 33^s.099 \\ \text{Table IV} = & \qquad\qquad 1\ .015 \\ (k+d)(\mu-1), \text{Table VII} = & \qquad -35\ .495 \\ \hline V = & 6^h\ 36^m\ 58^s.619 \\ \Theta = & 14^h\ \ 3^m\ 11^s.720 \\ \hline \Theta - V = & 7^h\ 26^m\ 13^s.101 \\ \text{Table VII} = & -\quad 1\ \ 13\ .101 \\ \hline T = & 7^h\ 25^m\ \ 0^s.0 \end{array}$$

TABLES.

Table I gives values of the function $\frac{2}{\sqrt{\pi}}\int_0^t \varepsilon^{-t^2} dt$ for values of t from 0 to ∞.

Table II A gives the refraction corresponding to different altitudes for a mean state of the atmosphere, viz., barometer 30 inches, thermometer 50°. For any other readings of the barometer and thermometer the factors by which the mean refraction must be multiplied are taken from tables II B, II C, and II D. (See Art. 86.)

Table III A, B, C, and D are Bessel's refraction tables. These will be employed when extreme precision is required. When the altitude is less than 5° no table will give reliable values for the refraction, but it may be found approximately by the supplementary table following III A. (See Art. 86.)

Table IV is intended for use in connection with the refraction table when the barometer is graduated according to the metric system.

Tables V or VI may be used when the thermometer is not graduated according to Fahrenheit's scale.

Table VII requires no explanation.

Table VIII A and B give values of m, log m, n, and log n, where

$$m = \frac{2\sin^2 \frac{1}{2}t}{\sin 1''}; \qquad n = \frac{2\sin^4 \frac{1}{2}t}{\sin 1''}. \qquad \text{(See Art. 146.)}$$

Table VIII C gives the factor to employ in reducing circummeridian altitudes when the chronometer has an appreciable rate, viz., $k = \left(\frac{1}{1 - \frac{r}{86400}}\right)^2$. (See Art. 152.)

TABLE I.

$$f(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}\,dt.$$

t	$f(t)$	t	$f(t)$	t	$f(t)$	t	$f(t)$
.00	.000000	.50	.520500	1.00	.842701	1.50	.966105
.01	.011283	.51	.529244	1.01	.846810	1.51	.967277
.02	.022565	.52	.537899	1.02	.850838	1.52	.968414
.03	.033841	.53	.546464	1.03	.854784	1.53	.969516
.04	.045111	.54	.554939	1.04	.858650	1.54	.970586
.05	.056372	.55	.563323	1.05	.862436	1.55	.971623
.06	.067622	.56	.571616	1.06	.866144	1.56	.972628
.07	.078858	.57	.579817	1.07	.869773	1.57	.973603
.08	.090078	.58	.587923	1.08	.873326	1.58	.974547
.09	.101281	.59	.595937	1.09	.876803	1.59	.975462
.10	.112463	.60	.603856	1.10	880205	1.60	.976348
.11	.123623	.61	.611681	1.11	.883533	1.61	.977207
.12	.134756	.62	.619412	1.12	.886788	1.62	.978038
.13	.145867	.63	.627046	1.13	.889971	1.63	.978843
.14	.156947	.64	.634586	1.14	.893082	1.64	.979622
.15	.167996	.65	.642029	1.15	.896124	1.65	.980376
.16	.179012	.66	.649377	1.16	.899096	1.66	.981105
.17	.189992	.67	.656628	1.17	.902000	1.67	.981810
.18	.200936	.68	.663782	1.18	.904837	1.68	.982493
.19	.211840	.69	.670840	1.19	.907608	1.69	.983153
.20	.222703	.70	.677801	1.20	.910314	1.70	.983791
.21	.233522	.71	.684666	1.21	.912956	1.71	.984407
.22	.244296	.72	.691433	1.22	.915534	1.72	.985003
.23	.255023	.73	.698104	1.23	.918050	1.73	.985578
.24	.265700	.74	.704678	1.24	.920505	1.74	.986135
.25	.276326	.75	.711156	1.25	.922900	1.75	.986672
.26	.286900	.76	.717537	1.26	.925236	1.76	.987190
.27	.297418	.77	.723822	1.27	.927514	1.77	.987691
.28	.307880	.78	.730010	1.28	.929734	1.78	.988174
.29	.318284	.79	.736104	1.29	.931899	1.79	.988641
.30	.328627	.80	.742101	1.30	.934008	1.80	.989090
.31	.338908	.81	.748003	1.31	.936063	1.81	.989525
.32	.349126	.82	.753811	1.32	.938065	1.82	.989943
.33	.359279	.83	.759524	1.33	.940015	1.83	.990347
.34	.369365	.84	.765143	1.34	.941914	1.84	.990736
.35	.379382	.85	.770668	1.35	.943762	1.86	.991473
.36	.389330	.86	.776100	1.36	.945562	1.88	.992156
.37	.399206	.87	.781440	1.37	.947313	1.90	.992790
.38	.409010	.88	.786687	1.38	.949016	1.92	.993378
.39	.418739	.89	.791843	1.39	.950673	1.94	.993923
.40	.428392	.90	.796908	1.40	.952285	1.96	.994426
.41	.437969	.91	.801883	1.41	.953852	1.98	.994892
.42	.447468	.92	.806768	1.42	.955376	2.0	.995323
.43	.456887	.93	.811564	1.43	.956857	2.1	.997021
.44	.466225	.94	.816271	1.44	.958296	2.2	.998137
.45	.475482	.95	.820891	1.45	.959695	2.3	.998857
.46	.484655	.96	.825424	1.46	.961054	2.4	.999312
.47	.493745	.97	.829870	1.47	.962373	2.5	.999593
.48	.502750	.98	.834232	1.48	.963654	3.0	.999978
.49	.511668	.99	.838508	1.49	.964898	3.5	.999999
.50	.520500	1.00	.842701	1.50	.966105	∞	1.000000

MEAN REFRACTION.

Barometer 30 inches. Fahrenheit's Thermometer 50°.

Apparent Altitude.	Mean Refraction.	Apparent Altitude.	Mean Refraction.	Apparent Altitude.	Mean Refraction.	Apparent Altitude.	Mean Refraction.	Apparent Altitude.	Mean Refraction.	Apparent Altitude.	Mean Refraction.	Apparent Altitude.	Mean Refraction.
0°30′	29′19″	8°35′	6′ 8″.5	12°35′	4′15″.3	19°10′	2′46″.1	27°10′	1′53″.1	42°20′	1′ 3″.9	79°00′	0′11″.3
1 0	24 38	8 40	6 5 .2	12 40	4 13 .6	19 20	2 44 .6	27 20	1 52 .3	42 40	1 3 .2	80 0	10 .3
2 0	18 19	8 45	6 2 .0	12 45	4 12 .0	19 30	2 43 .1	27 30	1 51 .5	43 0	1 2 .4	81 0	9 .2
3 0	14 22	8 50	5 58 .8	12 50	4 10 .4	19 40	2 41 .6	27 40	1 50 .7	43 20	1 1 .7	82 0	8 .2
4 0	11 45	8 55	5 55 .7	12 55	4 8 .8	19 50	2 40 .2	27 50	1 50 .0	43 40	1 1 .0	83 0	7 .2
5 0	9 52	9 0	5 52 .6	13 0	4 7 .2	20 0	2 38 .8	28 0	1 49 .2	44 0	1 0 .3	84 0	6 .1
5 5	9 44 .0	9 5	5 49 .6	13 5	4 5 .6	20 10	2 37 .4	28 20	1 47 .7	44 20	0 59 .6	85 0	0 5 .1
5 10	9 36 .2	9 10	5 46 .6	13 10	4 4 .1	20 20	2 36 .0	28 40	1 46 .2	44 40	58 .9	86 0	4 .1
5 15	9 28 .6	9 15	5 43 .6	13 15	4 2 .6	20 30	2 34 .6	29 0	1 44 .8	45 0	58 .2	87 0	3 .1
5 20	9 21 .2	9 20	5 40 .7	13 20	4 1 .0	20 40	2 33 .3	29 20	1 43 .4	45 20	57 .6	88 0	2 .0
5 25	9 14 .0	9 25	5 37 .9	13 25	3 59 .6	20 50	2 32 .0	29 40	1 42 .0	45 40	56 .9	89 0	1 .0
5 30	9 7 .0	9 30	5 35 .1	13 30	3 58 .1	21 0	2 30 .7	30 0	1 40 .6	46 0	56 .2	90 0	0 .0
5 35	9 0 .1	9 35	5 32 .4	13 35	3 56 .6	21 10	2 29 .4	30 20	1 39 .3	46 20	0 55 .6		
5 40	8 53 .4	9 40	5 29 .6	13 40	3 55 .2	21 20	2 28 .1	30 40	1 38 .0	46 40	55 .0		
5 45	8 46 .8	9 45	5 27 .0	13 45	3 53 .7	21 30	2 26 .9	31 0	1 36 .7	47 0	54 .3		
5 50	8 40 .4	9 50	5 24 .3	13 50	3 52 .3	21 40	2 25 .7	31 20	1 35 .5	47 20	53 .7		
5 55	8 34 .2	9 55	5 21 .7	13 55	3 50 .9	21 50	2 24 .5	31 40	1 34 .2	47 40	53 .1		
6 0	8 28 .1	10 0	5 19 .2	14 0	3 49 .5	22 0	2 23 .3	32 0	1 33 .0	48 0	52 .5		
6 5	8 22 .1	10 5	5 16 .7	14 10	3 46 .8	22 10	2 22 .1	32 20	1 31 .8	49 0	0 50 .6		
6 10	8 16 .2	10 10	5 14 .2	14 20	3 44 .2	22 20	2 20 .9	32 40	1 30 .7	50 0	48 .9		
6 15	8 10 .4	10 15	5 11 .7	14 30	3 41 .6	22 30	2 19 .8	33 0	1 29 .5	51 0	47 .2		
6 20	8 4 .8	10 20	5 9 .3	14 40	3 39 .0	22 40	2 18 .7	33 20	1 28 .4	52 0	45 .5		
6 25	7 59 .3	10 25	5 6 .9	14 50	3 36 .5	22 50	2 17 .5	33 40	1 27 .3	53 0	43 .9		
6 30	7 53 .9	10 30	5 4 .6	15 0	3 34 .1	23 0	2 16 .4	34 0	1 26 .2	54 0	42 .3		
6 35	7 48 .7	10 35	5 2 .3	15 10	3 31 .7	23 10	2 15 .4	34 20	1 25 .1	55 0	0 40 .8		
6 40	7 43 .5	10 40	5 0 .0	15 20	3 29 .4	23 20	2 14 .3	34 40	1 24 .1	56 0	39 .3		
6 45	7 38 .4	10 45	4 57 .8	15 30	3 27 .1	23 30	2 13 .3	35 0	1 23 .1	57 0	37 .8		
6 50	7 33 .5	10 50	4 55 .6	15 40	3 24 .8	23 40	2 12 .2	35 20	1 22 .0	58 0	36 .4		
6 55	7 28 .6	10 55	4 53 .4	15 50	3 22 .6	23 50	2 11 .2	35 40	1 21 .0	59 0	35 .0		
7 0	7 23 .8	11 0	4 51 .2	16 0	3 20 .5	24 0	2 10 .2	36 0	1 20 .1	60 0	33 .6		
7 5	7 19 .2	11 5	4 49 .1	16 10	3 18 .4	24 10	2 9 .2	36 20	1 19 .1	61 0	0 32 .3		
7 10	7 14 .6	11 10	4 47 .0	16 20	3 16 .3	24 20	2 8 .2	36 40	1 18 .2	62 0	31 .0		
7 15	7 10 .1	11 15	4 44 .9	16 30	3 14 .2	24 30	2 7 .2	37 0	1 17 .2	63 0	29 .7		
7 20	7 5 .7	11 20	4 42 .9	16 40	3 12 .2	24 40	2 6 .2	37 20	1 16 .3	64 0	28 .4		
7 25	7 1 .4	11 25	4 40 .9	16 50	3 10 .3	24 50	2 5 .3	37 40	1 15 .4	65 0	27 .2		
7 30	6 57 .1	11 30	4 38 .9	17 0	3 8 .3	25 0	2 4 .4	38 0	1 14 .5	66 0	25 .9		
7 35	6 53 .0	11 35	4 36 .9	17 10	3 6 .4	25 10	2 3 .4	38 20	1 13 .6	67 0	0 24 .7		
7 40	6 48 .9	11 40	4 35 .0	17 20	3 4 .6	25 20	2 2 .5	38 40	1 12 .7	68 0	23 .6		
7 45	6 44 .9	11 45	4 33 .1	17 30	3 2 .8	25 30	2 1 .6	39 0	1 11 .9	69 0	22 .4		
7 50	6 41 .0	11 50	4 31 .2	17 40	3 1 .0	25 40	2 0 .7	39 20	1 11 .0	70 0	21 .2		
7 55	6 37 .1	11 55	4 29 .4	17 50	2 59 .2	25 50	1 59 .8	39 40	1 10 .2	71 0	20 .1		
8 0	6 33 .3	12 0	4 27 .5	18 0	2 57 .5	26 0	1 58 .9	40 0	1 9 .4	72 0	18 .9		
8 5	6 29 .6	12 5	4 25 .7	18 10	2 55 .8	26 10	1 58 .1	40 20	1 8 .6	73 0	0 17 .8		
8 10	6 25 .9	12 10	4 23 .9	18 20	2 54 .1	26 20	1 57 .2	40 40	1 7 .8	74 0	16 .7		
8 15	6 22 .3	12 15	4 22 .2	18 30	2 52 .4	26 30	1 56 .4	41 0	1 7 .0	75 0	15 .6		
8 20	6 18 .8	12 20	4 20 .4	18 40	2 50 .8	26 40	1 55 .5	41 20	1 6 .2	76 0	14 .5		
8 25	6 15 .3	12 25	4 18 .7	18 50	2 49 .2	26 50	1 54 .7	41 40	1 5 .4	77 0	13 .5		
8 30	6 11 .9	12 30	4 17 .0	19 0	2 47 .7	27 0	1 53 .9	42 0	1 4 .7	78 0	12 .4		
8°35′	6′ 8″.5	12°35′	4′15″.3	19°10′	2′46″.1	27°10′	1′53″.1	42°20′	1′ 3″.9	79° 0′	0′11″.3		

TABLE II B.

Factor Depending on Barometer.

Inches.	B	log B
27.5	.917	9.9622
27.6	.920	9.9638
27.7	.923	9.9653
27.8	.927	9.9669
27.9	.930	9.9685
28.0	.933	9.9700
28.1	.937	9.9716
28.2	.940	9.9731
28.3	.943	9.9747
28.4	.947	9.9762
28.5	.950	9.9777
28.6	.953	9.9792
28.7	.957	9.9808
28.8	.960	9.9823
28.9	.963	9.9838
29.0	.967	9.9853
29.1	.970	9.9868
29.2	.973	9.9883
29.3	.977	9.9897
29.4	.980	9.9912
29.5	.983	9.9927
29.6	.987	9.9942
29.7	.990	9.9956
29.8	.993	9.9971
29.9	.997	9.9986
30.0	1.000	.0000
30.1	1.003	.0014
30.2	1.007	.0029
30.3	1.010	.0043
30.4	1.013	.0057
30.5	1.017	.0072
30.6	1.020	.0086
30.7	1.023	.0100
30.8	1.027	.0114
30.9	1.030	.0128
31.0	1.033	.0142

TABLE II C.

Factor Depending on Attached Thermometer.

F	t	log t
− 30°	1.007	.0031
− 20	1.006	.0027
− 10	1.005	.0023
0	1.005	.0020
+ 10	1.004	.0016
+ 20	1.003	.0012
30	1.002	.0009
40	1.001	.0005
50	1.000	.0000
60	.999	9.9996
70	.998	9.9992
80	.997	9.9989
90	.996	9.9985
100	.996	9.9981

TABLE II D.

Factor Depending on Detached Thermometer.

F	T	log T	F	T	log T	F	T	log T
− 25°	1.172	.0688	15°	1.073	.0308	55°	.990	9.9958
24	1.169	.0678	16	1.071	.0298	56	.988	9.9949
23	1.166	.0669	17	1.069	.0289	57	.986	9.9941
22	1.164	.0658	18	1.067	.0280	58	.985	9.9933
21	1.161	.0648	19	1.064	.0271	59	.983	9.9924
20	1.158	.0639	20	1.062	.0262	60	.981	9.9916
19	1.156	.0629	21	1.060	.0253	61	.979	9.9908
18	1.153	.0619	22	1.058	.0244	62	.977	9.9899
17	1.151	.0609	23	1.056	.0235	63	.975	9.9891
16	1.148	.0599	24	1.054	.0226	64	.973	9.9883
15	1.145	.0590	25	1.051	.0217	65	.972	9.9875
14	1.143	.0580	26	1.049	.0209	66	.970	9.9866
13	1.140	.0570	27	1.047	.0200	67	.968	9.9858
12	1.138	.0561	28	1.045	.0191	68	.966	9.9850
11	1.135	.0551	29	1.043	.0182	69	.964	9.9842
10	1.133	.0541	30	1.041	.0173	70	.962	9.9834
9	1.130	.0532	31	1.039	.0164	71	.961	9.9825
8	1.128	.0522	32	1.036	.0155	72	.959	9.9817
7	1.125	.0513	33	1.034	.0147	73	.957	9.9809
6	1.123	.0503	34	1.032	.0138	74	.955	9.9801
5	1.120	.0494	35	1.030	.0129	75	.953	9.9793
4	1.118	.0484	36	1.028	.0120	76	.952	9.9785
3	1.115	.0475	37	1.026	.0112	77	.950	9.9777
2	1.113	.0465	38	1.024	.0103	78	.948	9.9769
− 1	1.111	.0456	39	1.022	.0094	79	.946	9.9761
0	1.108	.0446	40	1.020	.0086	80	.945	9.9753
+ 1	1.106	.0437	41	1.018	.0077	81	.943	9.9745
2	1.103	.0428	42	1.016	.0068	82	.941	9.9737
3	1.101	.0418	43	1.014	.0060	83	.939	9.9729
4	1.099	.0409	44	1.012	.0051	84	.938	9.9721
5	1.096	.0400	45	1.010	.0043	85	.936	9.9713
6	1.094	.0390	46	1.008	.0034	86	.934	9.9705
7	1.092	.0381	47	1.006	.0026	87	.933	9.9697
8	1.089	.0372	48	1.004	.0017	88	.931	9.9689
9	1.087	.0363	49	1.002	.0009	89	.929	9.9681
10	1.085	.0353	50	1.000	.0000	90	.928	9.9673
11	1.082	.0344	51	.998	9.9992	91	.926	9.9665
12	1.080	.0335	52	.996	9.9983	92	.924	9.9658
13	1.078	.0326	53	.994	9.9975	93	.923	9.9650
14	1.076	.0317	54	.992	9.9966	94	.921	9.9642
+ 15	1.073	.0308	55	.990	9.9958	95	.919	9.9634

r = (mean refraction) $\times B \times T \times t$.

BESSEL'S REFRACTION TABLE.

Apparent Altitude.	Apparent Zenith Distance.	log α.	Dif.	A.	λ.	Apparent Altitude.	Apparent Zenith Distance.	log α.	Dif.	A.	λ.
5° 0′	85° 0′	1.71020	259	1.0127	1.1229	14° 20′	75° 40′	1.75391	17		1.0212
10	84 50	1.71279	243	1.0121	1.1178	30	30	1.75408	17		1.0208
20	30	1.71522	227	1.0115	1.1130	40	20	1.75425	16		1.0204
30	40	1.71749	212	1.0110	1.1082	50	10	1.75441	16		1.0200
40	20	1.71961	199	1.0105	1.1036	15 0	75 0	1.75457	86		1.0197
50	10	1.72160	186	1.0100	1.0992	16	74	1.75543	72		1.0175
6 0	84 0	1.72346	173	1.0096	1.0951	17	73	1.75615	60		1.0156
10	83 50	1.72519	162	1.0092	1.0914	18	72	1.75675	51		1.0139
20	40	1.72681	151	1.0088	1.0879	19	71	1.75726	45		1.0124
30	30	1.72832	142	1.0084	1.0846	20	70	1.75771	38		1.0111
40	20	1.72974	131	1.0081	1.0815	21	69	1.75809	33		1.0101
50	10	1.73105	124	1.0078	1.0784	22	68	1.75842	29		1.0092
7 0	83 0	1.73229	118	1.0075	1.0754	23	67	1.75871	26		1.0083
10	82 50	1.73347	112	1.0073	1.0725	24	66	1.75897	22		1.0075
20	40	1.73459	105	1.0070	1.0697	25	65	1.75919	20		1.0068
30	30	1.73564	99	1.0067	1.0671	26	64	1.75939	18		1.0063
40	20	1.73663	94	1.0065	1.0646	27	63	1.75957	16		1.0058
50	10	1.73757	88	1.0062	1.0622	28	62	1.75973	15		1.0054
8 0	82 0	1.73845	83	1.0060	1.0600	29	61	1.75988	13		1.0049
10	81 50	1.73928	79	1.0058	1.0579	30	60	1.76001	11		1.0046
20	40	1.74007	76	1.0056	1.0559	31	59	1.76012	11		1.0043
30	30	1.74083	72	1.0054	1.0540	32	58	1.76023	10		1.0040
40	20	1.74155	68	1.0052	1.0523	33	57	1.76033	9		1.0037
50	10	1.74223	65	1.0050	1.0508	34	56	1.76042	8		1.0034
9 0	81 0	1.74288	64	1.0049	1.0493	35	55	1.76050	8		1.0031
10	80 50	1.74352	60	1.0047	1.0479	36	54	1.76058	7		1.0029
20	40	1.74412	56	1.0046	1.0466	37	53	1.76065	6		1.0027
30	30	1.74468	53	1.0045	1.0454	38	52	1.76071	6		1.0026
40	20	1.74521	52	1.0043	1.0442	39	51	1.76077	5		1.0025
50	10	1.74573	50	1.0042	1.0431	40	50	1.76082	5		1.0023
10 0	80 0	1.74623	47	1.0041	1.0420	41	49	1.76087	5		1.0021
10	79 50	1.74670	44	1.0040	1.0409	42	48	1.76092	4		1.0020
20	40	1.74714	43	1.0039	1.0398	43	47	1.76096	4		1.0019
30	30	1.74757	42	1.0038	1.0387	44	46	1.76100	4		1.0019
40	20	1.74799	40	1.0037	1.0377	45	45	1.76104	3		1.0018
50	10	1.74839	37	1.0036	1.0367	46	44	1.76107	4		
11 0	79 0	1.74876	36	1.0035	1.0357	47	43	1.76111	3		
10	78 50	1.74912	35	1.0034	1.0347	48	42	1.76114	3		
20	40	1.74947	34	1.0033	1.0338	49	41	1.76117	2		
30	30	1.74981	32	1.0032	1.0328	50	40	1.76119	3		
40	20	1.75013	30	1.0031	1.0318	51	39	1.76122	2		
50	10	1.75043	29	1.0030	1.0308	52	38	1.76124	2		
12 0	78 0	1.75072	29	1.0030	1.0299	53	37	1.76126	2		
10	77 50	1.75101	28	1.0029	1.0290	54	36	1.76128	2		
20	40	1.75129	26	1.0028	1.0281	65	35	1.76130	2		
30	30	1.75155	25	1.0027	1.0272	56	34	1.76132	2		
40	20	1.75180	25	1.0027	1.0264	57	33	1.76134	2		
50	10	1.75205	24	1.0026	1.0258	58	32	1.76136	2		
13 0	77 0	1.75229	23	1.0026	1.0252	59	31	1.76138	1		
10	76 50	1.75252	22		1.0246	60	30	1.76139	6		
20	40	1.75274	21		1.0241	65	25	1.76145	4		
30	30	1.75295	21		1.0235	70	20	1.76149	3		
40	20	1.75316	20		1.0230	75	15	1.76152	2		
50	10	1.75336	19		1.0225	80	10	1.76154	2		
14 0	76 0	1.75355	18		1.0220	85	5	1.76156	0		
10	75 50	1.75373	18		1.0216	90°	0° 0′	1.76156			
14° 20′	75° 40′	1.75391			1.0212						

TABLE III. A.

Supplement.

Apparent Altitude.	Apparent Zenith Distance.	Logarithm of Refraction.	*A*.	λ.
0° 30′	89° 30′	3.24142	1.0780	1.5789
1 0	89 0	3.16572	1.0593	1 4653
1 30	88 30	3.09723	1.0465	1.3797
2 0	88 0	3.03686	1.0368	1.3141
2 30	87 30	2.98269	1.0298	1.2624
3 0	87 0	2.93174	1.0244	1.2215
3 30	86 30	2 88555	1.0204	1.1888
4 0	86 0	2.84444	1.0172	1.1624
4 30	85 30	2.80590	1.0147	1.1408
5 0	85 0	2.76687	1.0127	1.1229

TABLE III D.

Factor Depending on Detached Thermometer.

F.	Log γ.	*F*.	Log γ.	*F*.	Log γ.
−25°	+.06773	15	+.02969	55	−.00528
−24	.06674	16	.02878	56	−.00612
−23	.06575	17	.02787	57	−.00698
−22	.06476	18	.02697	58	−.00780
−21	.06377	19	.02606	59	−.00863
−20	.06279	20	.02514	60	−.00946
−19	.06181	21	.02426	61	−.01029
−18	.06083	22	.02336	62	−.01112
−17	.05985	23	.02247	63	−.01195
−16	.05887	24	.02157	64	−.01278
−15	.05790	25	.02068	65	−.01360
−14	.05693	26	.01979	66	−.01443
−13	.05596	27	.01890	67	−.01525
−12	.05500	28	.01801	68	−.01607
−11	.05403	29	.01713	69	−.01689
−10	.05307	30	.01624	70	−.01770
− 9	.05211	31	.01536	71	−.01852
− 8	.05115	32	.01448	72	−.01933
− 7	.05020	33	.01360	73	−.02015
− 6	.04924	34	.01273	74	− 02096
− 5	.04829	35	.01185	75	−.02177
− 4	.04734	36	.01098	76	−.02257
− 3	.04640	37	.01011	77	−.02338
− 2	.04545	38	.00924	78	− 02419
− 1	.04451	39	.00837	79	−.02499
− 0	.04357	40	.00750	80	−.02579
+ 1	.04263	41	.00664	81	−.02659
+ 2	.04169	42	.00578	82	−.02738
+ 3	.04076	43	.00492	83	−.02819
+ 4	.03982	44	.00406	84	−.02898
+ 5	.03889	45	.00320	85	−.02978
+ 6	.03796	46	.00234	86	−.03057
+ 7	.03704	47	.00149	87	−.03136
+ 8	.03611	48	+.00064	88	−.03216
+ 9	.03519	49	−.00021	89	−.03294
+10	.03427	50	−.00106	90	−.03373
+11	.03335	51	−.00191	91	−.03452
+12	.03243	52	−.00275	92	−.03530
+13	.03152	53	−.00360	93	−.03609
+14	.03060	54	−.00444	94	−.03687
+15	+.02969	55	−.00528	95	−.03765

TABLE III B.

Factor Depending on Barometer.

Inches.	Log *B*.
27.5	−.03191
27.6	−.03033
27.7	−.02876
27.8	−.02720
27.9	−.02564
28.0	−.02409
28.1	−.02254
28.2	−.02099
28.3	−.01946
28.4	−.01793
28.5	−.01640
28.6	−.01488
28.7	− 01336
28.8	−.01185
28.9	− 01035
29.0	−.00885
29.1	−.00735
29.2	−.00586
29.3	−.00438
29.4	− 00290
29.5	−.00142
29.6	+.00005
29.7	.00151
29.8	.00297
29.9	.00443
30.0	.00588
30.1	.00732
30.2	.00876
30.3	.01020
30.4	.01163
30.5	.01306
30.6	.01448
30.7	.01589
30.8	.01731
30.9	.01871
31.0	.02012

TABLE III C.

Factor Depending on Attached Thermometer.

F.	Log *T*.
− 30°	+.00242
− 20	+.00203
− 10	+.00164
0	+.00125
+ 10	+.00086
20	+.00047
30	+.00008
40	−.00031
50	−.00070
60	−.00109
70	−.00148
80	−.00186
90	−.00225
100	−.00264

$$\log \beta = \log B + \log T.$$

$$\log r = \log \alpha + A \,.\, \log \beta + \lambda \,.\, \log \gamma + \log \tan z.$$

To Convert Centimetres into Inches.

Centi-metres.	English Inches.	Centi-metres.	English Inches.	Centi-metres.	English Inches.
68.0	26.772	73.5	28.938	0.1	.0394
68.5	26.969	74.0	29.134	0.2	.0787
69.0	27.166	74.5	29.331	0.3	.1181
69.5	27.363	75.0	29.528	0.4	.1575
70.0	27.560	75.5	29.725	0.5	.1969
70.5	27.756	76.0	29.922	0.6	.2362
71.0	27.953	76.5	30.119	0.7	.2756
71.5	28.150	77.0	30.316	0.8	.3150
72.0	28.347	77.5	30.512	0.9	.3543
72.5	28.544	78.0	30.709	1.0	.3937
73.0	28.741	78.5	30.906		

TABLE V.

To Convert Reading of Centigrade Thermometer into Fahrenheit's.

C.	F.	C.	F.	C.	F.
−32°	−25°.6	+3°	+37°.4	0°.1	0°.18
−31	−23 .8	4	39 .2	0 .2	0 .36
−30	−22 .0	5	41 .0	0 .3	0 .54
−29	−20 .2	6	42 .8	0 .4	0 .72
−28	−18 .4	7	44 .6	0 .5	0 .90
−27	−16 .6	8	46 .4	0 .6	1 .08
−26	−14 .8	9	48 .2	0 .7	1 .26
−25	−13 .0	10	50 .0	0 .8	1 .44
−24	−11 .2	11	51 .8	0 .9	1 .62
−23	− 9 .4	12	53 6	1 .0	1 .80
−22	− 7 .6	13	55 .4		
−21	− 5 .8	14	57 .2		
−20	− 4 .0	15	59 .0		
−19	− 2 .2	16	60 .8		
−18	− 0 .4	17	62 .6		
−17	+ 1 .4	18	64 .4		
−16	+ 3 .2	19	66 .2		
−15	+ 5 .0	20	68 .0		
−14	+ 6 .8	21	69 .8		
−13	+ 8 .6	22	71 .6		
−12	10 .4	23	73 .4		
−11	12 .2	24	75 .2		
−10	14 .0	25	77 .0		
− 9	15 .8	26	78 .8		
− 8	17 .6	27	80 .6		
− 7	19 .4	28	82 .4		
− 6	21 .2	29	84 .2		
− 5	23 .0	30	86 .0		
− 4	24 .8	31	87 .8		
− 3	26 .6	32	89 .6		
− 2	28 .4	33	91 .4		
− 1	30 .2	34	93 .2		
0	32 .0	35	95 .0		
+ 1	33 .8	36	96 .8		
2	35 .6	37	98 .6		
3	37 .4	38	100 .4		

TABLE VI.

To Convert Reading of Reaumur's Thermometer into Fahrenheit's.

R.	F.	R.	F.	R.	F.
−25°	−24°.25	3°	38°.75	0°.1	0°.225
−24	−22 .0	4	41 .0	0 .2	0 .45
−23	−19 .75	5	43 .25	0 .3	0 .675
−22	−17 .5	6	45 .5	0 .4	0 .90
−21	−15 .25	7	47 .75	0 .5	1 .125
−20	−13 .0	8	50 .0	0 .6	1 .35
−19	−10 .75	9	52 .25	0 .7	1 .575
−18	− 8 .5	10	54 .5	0 .8	1 .80
−17	− 6 .25	11	56 .75	0 .9	2 .025
−16	− 4 .0	12	59 0	1 .0	2 .25
−15	− 1 .75	13	61 .25		
−14	+ 0 .5	14	63 .5		
−13	2 .75	15	65 .75		
−12	5 .0	16	68 .0		
−11	7 .25	17	70 .25		
−10	9 .5	18	72 .5		
− 9	11 .75	19	74 .75		
− 8	14 .0	20	77 .0		
− 7	16 .25	21	79 .25		
− 6	18 .5	22	81 5		
− 5	20 .75	23	83 .75		
− 4	23 .0	24	86 .0		
− 3	25 .25	25	88 .25		
− 2	27 .5	26	90 .5		
− 1	29 .75	27	92 .75		
− 0	32 .0	28	95 .0		
+ 1	34 .25	29	97 .25		
2	36 .5	30	99 .5		
3	38 .75	31	101 .75		

To Convert Hours, Minutes, and Seconds into a Decimal of a Day.

Hour.	Decimal of Day.	Minute.	Decimal of Day.	Second.	Decimal of Day.
1	.041 6667	1	.000 6944	1	.000 0116
2	.083 3333	2	.001 3889	2	.000 0231
3	.125 0000	3	.002 0833	3	.000 0347
4	.166 6667	4	.002 7778	4	.000 0463
5	.208 3333	5	.003 4722	5	.000 0579
6	.250 0000	6	.004 1667	6	.000 0694
7	.291 6667	7	.004 8611	7	.000 0810
8	.333 3333	8	.005 5556	8	.000 0926
9	.375 0000	9	.006 2500	9	.000 1042
10	.416 6667	10	.006 9444	10	.000 1157
11	.458 3333	11	.007 6389	11	.000 1273
12	.500 0000	12	.008 3333	12	.000 1389
13	.541 6667	13	.009 0278	13	.000 1505
14	.583 3333	14	.009 7222	14	.000 1620
15	.625 0000	15	.010 4167	15	.000 1736
16	.666 6667	16	.011 1111	16	.000 1852
17	.708 3333	17	.011 8056	17	.000 1968
18	.750 0000	18	.012 5000	18	.000 2083
19	.791 6667	19	.013 1944	19	.000 2199
20	.833 3333	20	.013 8889	20	.000 2315
21	.875 0000	21	.014 5833	21	.000 2431
22	.916 6667	22	.015 2778	22	.000 2546
23	.958 3333	23	.015 9722	23	.000 2662
24	1.000 0000	24	.016 6667	24	.000 2778
		25	.017 3611	25	.000 2894
		26	.018 0556	26	.000 3009
		27	.018 7500	27	.000 3125
		28	.019 4444	28	.000 3241
		29	.020 1389	29	.000 3356
		30	.020 8333	30	.000 3472
		31	.021 5278	31	.000 3588
		32	.022 2222	32	.000 3704
		33	.022 9167	33	.000 3819
		34	.023 6111	34	.000 3935
		35	.024 3056	35	.000 4051
		36	.025 0000	36	.000 4167
		37	.025 6944	37	.000 4282
		38	.026 3889	38	.000 4398
		39	.027 0833	39	.000 4514
		40	.027 7778	40	.000 4630
		41	.028 4722	41	.000 4745
		42	.029 1667	42	.000 4861
		43	.029 8611	43	.000 4977
		44	.030 5556	44	.000 5093
		45	.031 2500	45	.000 5208
		46	.031 9444	46	.000 5324
		47	.032 6389	47	.000 5440
		48	.033 3333	48	.000 5556
		49	.034 0278	49	.000 5671
		50	.034 7222	50	.000 5787
		51	.035 4167	51	.000 5903
		52	.036 1111	52	.000 6019
		53	.036 8056	53	.000 6134
		54	.037 5000	54	.000 6250
		55	.038 1944	55	.000 6366
		56	.038 8889	56	.000 6481
		57	.039 5833	57	.000 6597
		58	.040 2778	58	.000 6713
		59	.040 9722	59	.000 6829
		60	.041 6667	60	.000 6944

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}.$$

S	0ᵐ m	0ᵐ log m	1ᵐ m	1ᵐ log m	2ᵐ m	2ᵐ log m	3ᵐ m	3ᵐ log m
0	0″.00		1″.96	.29303	7″.85	.89509	17″.67	1.24727
1	0 .00	6.73673	2 .03	.30739	7 .98	.90230	17 .87	1.25208
2	0 .00	7.33879	2 .10	.32151	8 .12	.90945	18 .07	1.25687
3	0 .00	7.69097	2 .16	.33541	8 .25	.91654	18 .27	1.26163
4	0 .01	7.94085	2 .23	.34909	8 .39	.92357	18 .47	1.26636
5	0 .01	8.13467	2 .31	.36255	8 .52	.93055	18 .67	1.27107
6	0 .02	8.29303	2 .38	.37581	8 .66	.93747	18 .87	1.27575
7	0 .02	8.42692	2 .45	.38888	8 .80	.94434	19 .07	1.28041
8	0 .03	8.54291	2 .52	.40174	8 .94	.95115	19 .28	1.28504
9	0 .04	8 64521	2 .60	.41442	9 .08	.95791	19 .48	1.28965
10	0 .05	8.73673	2 .67	.42692	9 .22	.96462	19 .69	1.29423
11	0 .06	8.81951	2 .75	.43925	9 .36	.97127	19 .90	1.29879
12	0 .08	8.89509	2 .83	.45140	9 .50	.97788	20 .11	1.30332
13	0 .09	8.96461	2 .91	.46338	9 .64	.98443	20 .32	1.30783
14	0 .11	9.02898	2 .99	.47519	9 .79	.99094	20 .53	1.31232
15	0 .12	9 08891	3 .07	.48685	9 .94	.99740	20 .74	1.31679
16	0 .14	9.14497	3 .15	.49836	10 .09	1.00381	20 .95	1.32123
17	0 .16	9.19763	3 .23	.50971	10 .24	1.01017	21 .16	1.32566
18	0 .18	9 24727	3 .32	.52092	10 .39	1 01649	21 .38	1.33006
19	0 .20	9 29423	3 .40	.53198	10 .54	1 02276	21 .60	1.33443
20	0 .22	9.33879	3 .49	.54291	10 .09	1.02898	21 .82	1.33878
21	0 .24	9.38117	3 .58	.55370	10 .84	1.03517	22 .03	1.34311
22	0 .26	9.42157	3 .67	.56436	11 .00	1.04131	22 .25	1.34743
23	0 .28	9.46018	3 .76	.57489	11 .15	1.04740	22 .47	1.35172
24	0 .31	9.49715	3 .85	.58529	11 31	1 05345	22 .70	1.35598
25	0 .34	9 53261	3 .94	.59557	11 .47	1.05946	22 .92	1.36022
26	0 .37	9.56067	4 .03	.60573	11 .63	1.06543	23 .14	1 36445
27	0 .40	9.59945	4 .12	.61577	11 .79	1.07136	23 .37	1.36866
28	0 .43	9 63104	4 .22	.62570	11 .95	1.07725	23 .60	1.37285
29	0 .46	9.66152	4 .32	63551	12 .11	1.08310	23 .82	1.37702
30	0 .49	9.69037	4 .42	.64521	12 .27	1 08891	24 .05	1 38116
31	0 .52	9.71945	4 .52	.65481	12 .43	1.09468	24 .28	1.38529
32	0 .56	9 74703	4 .62	.66431	12 .60	1.10042	24 .51	1.38940
33	0 .59	9.77376	4 .72	.67370	12 .76	1.10611	24 .74	1.39348
34	0 .63	9.79968	4 .82	.68299	12 .93	1.11177	24 .98	1.39755
35	0 .67	9.82486	4 .92	.69218	13 .10	1.11739	25 .21	1 40160
36	0 .71	9.84933	5 .03	.70127	13 .27	1.12298	25 .45	1.40563
37	0 .75	9.87313	5 .13	.71027	13 .44	1.12853	25 .68	1.40964
38	0 .79	9.89629	5 .24	.71918	13 .62	1 13404	25 .92	1.41364
39	0 .83	9.91886	5 .34	.72800	13 .79	1.13952	26 .16	1.41761
40	0 .87	9.94085	5 .45	.73673	13 .96	1.14497	26 .40	1.42157
41	0 .91	9 96229	5 .56	.74537	14 .13	1.15038	26 .64	1.42551
42	0 .96	9.98323	5 67	.75393	14 .31	1.15576	26 .88	1.42943
43	1 .01	0 00366	5 .78	.76240	14 .49	1.16110	27 .12	1.43333
44	1 .06	.07363	5 .90	.77080	14 .67	1.16641	27 .37	1 43722
45	1 .10	.04315	6 .01	.77911	14 .85	1.17169	27 .61	1.44109
46	1 .15	.06224	6 .13	.78734	15 .03	1.17694	27 .86	1.44494
47	1 .20	.08092	6 .24	.79550	15 .21	1.18216	28 .10	1.44877
48	1 .26	.09921	6 .36	.80358	15 .39	1.18735	28 .35	1.45259
49	1 .31	.11712	6 .48	.81158	15 .57	1.19250	28 .60	1.45639
50	1 .36	.13467	6 .60	.81952	15 .76	1.19762	28 .85	1.46018
51	1 .42	.15187	6 .72	.82738	15 .95	1.20271	29 .10	1.46395
52	1 .48	.16875	6 .84	.83517	16 .14	1.20778	29 .36	1.46770
53	1 .53	.18528	6 .96	.84288	16 .32	1.21281	29 .61	1.47143
54	1 .59	.20151	7 .09	.85053	16 .51	1.21782	29 .86	1.47515
55	1 .65	.21745	7 .21	.85813	16 .70	1.22280	30 .12	1.47886
56	1 .71	.23310	7 .34	.86564	16 .89	1.22775	30 .38	1.48255
57	1 .77	.24848	7 .46	.87310	17 .08	1.23267	30 .64	1.48622
58	1 .83	.26358	7 .60	.88049	17 .28	1.23756	30 .90	1.48988
59	1 .89	.27843	7 .72	.88782	17 .47	1.24243	31 .16	1.49352
60	1″.96	.29303	7″.85	.89509	17″.67	1.24727	31″.42	1.49714

$$m = \frac{2 \sin^2 \frac{1}{2}t}{\sin 1''}.$$

S	4^m *m*	4^m log *m*	5^m *m*	5^m log *m*	6^m *m*	6^m log *m*	7^m *m*	7^m log *m*
0	31″.42	1.49714	49″.09	1.69096	70″.68	1.84931	96″.20	1.98320
1	31 .68	1.50076	49 .41	1.69385	71 .07	1.85172	96 .66	1.98526
2	31 .94	1.50435	49 .74	1.69673	71 .47	1.85412	97 .12	1.98732
3	32 .20	1.50793	50 .07	1.69960	71 .86	1.85651	97 .58	1.98937
4	32 .47	1.51150	50 .40	1.70246	72 .26	1.85890	98 .04	1.99142
5	32 .74	1.51505	50 .73	1.70531	72 .66	1.86129	98 .50	1.99347
6	33 .01	1.51859	51 .07	1.70815	73 .06	1.86366	98 .97	1.99551
7	33 .27	1.52211	51 .40	1.71099	73 .46	1.86603	99 .43	1.99755
8	33 .54	1.52562	51 .74	1.71382	73 .86	1.86840	99 .90	1.99958
9	33 .81	1.52912	52 .07	1.71663	74 .26	1.87075	100 .37	2.00161
10	34 .09	1.53260	52 .41	1.71944	74 .66	1.87310	100 .84	2.00363
11	34 .36	1.53606	52 .75	1.72223	75 .06	1.87545	101 .31	2.00565
12	34 .64	1.53952	53 .09	1.72502	75 .47	1.87779	101 .78	2.00766
13	34 .91	1.54296	53 .43	1.72780	75 .88	1.88012	102 .25	2.00967
14	35 .19	1.54639	53 .77	1.73057	76 .29	1.88244	102 .72	2.01167
15	35 .46	1.54980	54 .11	1.73333	76 .69	1.88476	103 .20	2.01367
16	35 .74	1.55320	54 .46	1.73608	77 .10	1.88708	103 .67	2.01566
17	36 .02	1.55659	54 .80	1.73883	77 .51	1.88938	104 .15	2.01765
18	36 .30	1.55996	55 .15	1.74157	77 .93	1.89168	104 .63	2.01964
19	36 .58	1.56332	55 .50	1.74429	78 .34	1.89398	105 .10	2.02162
20	36 .87	1.56667	55 .84	1.74701	78 .75	1.89627	105 .58	2.02360
21	37 .15	1.57000	56 .19	1.74972	79 .16	1.89855	106 .06	2.02557
22	37 .44	1.57332	56 .55	1.75242	79 .58	1.90083	106 .55	2.02753
23	37 .72	1.57663	56 .90	1.75511	80 .00	1.90310	107 .03	2.02950
24	38 .01	1.57993	57 .25	1.75780	80 .42	1.90536	107 .51	2.03146
25	38 .30	1.58321	57 .60	1.76048	80 .84	1.90762	107 .99	2.03341
26	38 .59	1.58648	57 .96	1.76314	81 .26	1.90987	108 .48	2.03536
27	38 .88	1.58974	58 .32	1.76580	81 .68	1.91212	108 .97	2.03730
28	39 .17	1.59299	58 .68	1.76846	82 .10	1.91436	109 .46	2.03924
29	39 .46	1.59622	59 .03	1.77110	82 .52	1.91660	109 .95	2.04118
30	39 .76	1.59945	59 .40	1.77373	82 .95	1.91883	110 .44	2.04311
31	40 .05	1.60266	59 .75	1.77636	83 .38	1.92105	110 .93	2.04504
32	40 .35	1.60586	60 .11	1.77898	83 .81	1.92327	111 .43	2.04697
33	40 .65	1.60904	60 .47	1.78160	84 .23	1.92548	111 .92	2.04888
34	40 .95	1.61222	60 .84	1.78420	84 .66	1.92769	112 .41	2.05080
35	41 .25	1.61538	61 .20	1.78680	85 .09	1.92990	112 .90	2.05271
36	41 .55	1.61854	61 .57	1.78938	85 .52	1.93209	113 .40	2.05462
37	41 .85	1.62168	61 .94	1.79197	85 .95	1.93428	113 .90	2.05652
38	42 .15	1.62481	62 .31	1.79454	86 .39	1.93646	114 .40	2.05842
39	42 .45	1.62793	62 .68	1.79710	86 .82	1.93864	114 .90	2.06031
40	42 .76	1.63103	63 .05	1.79967	87 .26	1.94082	115 .40	2.06220
41	43 .06	1.63413	63 .42	1.80221	87 .70	1.94299	115 .90	2.06409
42	43 .37	1.63722	63 .79	1.80476	88 .14	1.94515	116 .40	2.06597
43	43 .68	1.64029	64 .16	1.80729	88 .57	1.94731	116 .90	2.06785
44	43 .99	1.64335	64 .54	1.80982	89 .01	1.94946	117 .41	2.06972
45	44 .30	1.64641	64 .91	1.81234	89 .45	1.95161	117 .92	2.07159
46	44 .61	1.64945	65 .29	1.81486	89 .89	1.95375	118 .43	2.07346
47	44 .92	1.65248	65 .67	1.81736	90 .33	1.95589	118 .94	2.07532
48	45 .24	1.65550	66 .05	1.81986	90 .78	1.95802	119 .45	2.07718
49	45 .55	1.65851	66 .43	1.82236	91 .23	1.96014	119 .96	2.07903
50	45 .87	1.66151	66 .81	1.82484	91 .68	1.96226	120 .47	2.08088
51	46 .18	1.66450	67 .19	1.82732	92 .12	1.96438	120 .98	2.08273
52	46 .50	1.66748	67 .58	1.82979	92 .57	1.96649	121 .49	2.08457
53	46 .82	1.67045	67 .96	1.83225	93 .02	1.96860	122 .01	2.08641
54	47 .14	1.67341	68 .35	1.83471	93 .47	1.97070	122 .53	2.08824
55	47 .46	1.67636	68 .73	1.83716	93 .92	1.97279	123 .05	2.09007
56	47 .79	1.67930	69 .12	1.83960	94 .38	1.97488	123 .57	2.09190
57	48 .11	1.68223	69 .51	1.84204	94 .83	1.97697	124 .09	2.09372
58	48 .43	1.68515	69 .90	1.84447	95 .29	1.97905	124 .61	2.09554
59	48 .76	1.68806	70 .29	1.84690	95 .74	1.98112	125 .13	2.09735
60	49 .09	1.69096	70 .68	1.84931	96 .20	1.98320	125 .65	2.09917

$$m = \frac{2 \sin^2 \frac{1}{2}t}{\sin 1''}.$$

S	8^m		9^m		10^m		11^m	
	m	log m	m	log m	m	log m	m	log m
0	125″.65	2.09917	159″.02	2.20146	196″.32	2.29296	237″.54	2.37574
1	126 .17	2.10098	159 .61	2.20307	196 .97	2.29441	238 .26	2.37705
2	126 .70	2.10278	160 .20	2.20467	197 .63	2.29586	238 .98	2.37836
3	127 .22	2.10458	160 .80	2.20627	198 .28	2.29730	239 .70	2.37967
4	127 .75	2.10637	161 .39	2.20787	198 .94	2.29874	240 .42	2.38098
5	128 .28	2.10817	161 .98	2.20946	199 .60	2.30017	241 .14	2.38229
6	128 .81	2.10995	162 .58	2.21106	200 .26	2.30161	241 .87	2.38360
7	129 .34	2.11174	163 .17	2.21264	200 .92	2.30304	242 .60	2.38490
8	129 .87	2.11352	163 .77	2.21423	201 .59	2.30447	243 .33	2.38619
9	130 .40	2.11530	164 .37	2.21581	202 .25	2.30590	244 .06	2.38749
10	130 .94	2.11707	164 .97	2.21739	202 .92	2.30732	244 .79	2.38879
11	131 .47	2.11884	165 .57	2.21897	203 .58	2.30874	245 .52	2.39009
12	132 .01	2.12061	166 .17	2.22055	204 .25	2.31016	246 .25	2.39138
13	132 .55	2.12237	166 .77	2.22212	204 92	2.31158	246 .98	2.39267
14	133 .09	2.12413	167 .37	2.22369	205 .59	2.31300	247 .72	2.39396
15	133 .63	2.12589	167 .97	2.22525	206 .26	2.31441	248 .45	2.39525
16	134 .17	2.12764	168 .58	2.22682	206 .93	2.31582	249 .19	2.39654
17	134 .71	2.12939	169 .19	2.22838	207 .60	2.31723	249 .93	2.39782
18	135 .25	2.13114	169 .80	2.22994	208 .27	2.31864	250 .67	2.39910
19	135 .80	2.13288	170 .41	2.23150	208 .94	2.32004	251 .41	2.40038
20	136 .34	2.13462	171 .02	2.23304	209 .62	2.32144	252 .15	2.40166
21	136 .88	2.13635	171 .63	2.23459	210 .30	2.32284	252 .89	2.40294
22	137 .43	2.13809	172 .24	2.23614	210 .98	2.32424	253 .63	2.40421
23	137 .98	2.13982	172 .85	2.23768	211 .66	2.32563	254 .37	2.40548
24	138 .53	2.14154	173 .47	2.23922	212 .34	2.32703	255 .12	2.40675
25	139 .08	2.14326	174 .08	2.24076	213 .02	2.32842	255 .87	2.40802
26	139 .63	2.14498	174 .70	2.24230	213 .70	2.32980	256 .62	2.40929
27	140 .18	2.14670	175 .32	2.24383	214 .38	2.33119	257 .37	2.41055
28	140 .74	2.14841	175 .94	2.24536	215 .07	2.33258	258 .12	2.41181
29	141 .29	2.15011	176 .56	2.24689	215 .75	2.33396	258 .87	2.41307
30	141 .85	2.15182	177 .18	2.24842	216 .44	2.33534	259 .62	2.41434
31	142 .40	2.15352	177 .80	2.24994	217 .12	2.33671	260 .37	2.41560
32	142 .96	2.15522	178 .43	2.25146	217 .81	2.33809	261 .12	2.41685
33	143 .52	2.15691	179 .05	2.25297	218 .50	2.33946	261 .88	2.41811
34	144 .08	2.15860	179 .68	2.25449	219 .19	2.34083	262 .64	2.41936
35	144 .64	2.16029	180 .30	2.25600	219 .88	2.34220	263 .39	2.42061
36	145 .20	2.16198	180 .93	2.25751	220 .58	2.34357	264 .15	2.42186
37	145 .76	2.16366	181 .56	2.25902	221 .27	2.34493	264 .91	2.42310
38	146 .33	2.16534	182 .19	2.26052	221 .97	2.34630	265 .68	2.42435
39	146 .89	2.16701	182 .82	2.26202	222 .66	2.34766	266 .44	2.42559
40	147 .46	2.16868	183 .46	2.26352	223 .36	2.34901	267 .20	2.42683
41	148 .03	2.17035	184 .09	2.26501	224 .06	2.35037	267 .96	2.42807
42	148 .60	2.17202	184 .72	2.26651	224 .76	2.35172	268 .73	2.42931
43	149 .17	2.17368	185 .35	2.26800	225 .46	2.35307	269 .49	2.43055
44	149 .74	2.17534	185 .99	2.26949	226 .16	2.35442	270 .26	2.43178
45	150 .31	2.17700	186 .63	2.27097	226 .86	2.35577	271 .02	2.43302
46	150 .88	2.17865	187 .27	2.27246	227 .57	2.35712	271 .79	2.43425
47	151 .45	2.18030	187 .91	2.27394	228 .27	2.35846	272 .56	2.43548
48	152 .03	2.18194	188 .55	2.27542	228 .98	2.35980	273 .34	2.43670
49	152 .61	2.18359	189 .19	2.27689	229 .68	2.36114	274 .11	2.43793
50	153 .19	2.18523	189 .83	2.27836	230 .39	2.36248	274 .88	2.43915
51	153 .77	2.18687	190 .47	2.27984	231 .10	2.36381	275 .65	2.44037
52	154 .35	2.18850	191 .12	2.28130	231 .81	2.36515	276 .43	2.44159
53	154 .93	2.19013	191 .76	2.28277	232 .52	2.36648	277 .20	2.44281
54	155 .51	2.19176	192 .41	2.28423	233 .24	2.36781	277 .98	2.44403
55	156 .09	2.19338	193 .06	2.28569	233 .95	2.36913	278 .76	2.44525
56	156 .67	2.19500	193 .71	2.28715	234 .67	2.37046	279 .55	2.44646
57	157 .25	2.19662	194 .36	2.28861	235 .38	2.37178	280 .33	2.44767
58	157 .84	2.19824	195 .01	2.29006	236 .10	2.37310	281 .12	2.44888
59	158 .43	2.19985	195 .66	2.29151	236 .82	2.37442	281 .90	2.45009
60	159 .02	2.20146	196 .32	2.29296	237 .54	2.37574	282 .68	2.45130

$$m = \frac{2 \sin^2 \frac{1}{2}t}{\sin 1''}.$$

S	12^m m	12^m log m	13^m m	13^m log m	14^m m	14^m log m	15^m m	15^m log m
0	282″.68	2.45130	331″.74	2.52081	384″.74	2.58516	441″.63	2.64506
1	283 .47	2.45250	332 .59	2.52192	385 .65	2.58619	442 .62	2.64603
2	284 .26	2.45371	333 .44	2.52303	386 .56	2.58722	443 .60	2.64699
3	285 .04	2.45491	334 .29	2.52414	387 .48	2.58825	444 .58	2.64795
4	285 .83	2.45611	335 .15	2.52525	388 .40	2.58928	445 .56	2.64891
5	286 .62	2.45731	336 .00	2.52635	389 .32	2.59031	446 .55	2.64987
6	287 .41	2.45850	336 .86	2.52746	390 .24	2.59134	447 .54	2.65083
7	288 .20	2.45970	337 .72	2.52856	391 .16	2.59236	448 .53	2.65179
8	289 .00	2.46089	338 .58	2.52967	392 .09	2.59339	449 .51	2.65274
9	289 .79	2.46209	339 .44	2.53077	393 .01	2.59441	450 .50	2 65370
10	290 .58	2.46328	340 .30	2.53187	393 .94	2.59543	451 .50	2.65466
11	291 .38	2.46446	341 .16	2.53297	394 .86	2 59645	452 .49	2 65561
12	292 .18	2.46565	342 .02	2.53406	395 .79	2.59747	453 .48	2.65656
13	292 .98	2.46684	342 .88	2.53516	396 .72	2.59849	454 .48	2.65751
14	293 .78	2.46802	343 .75	2.53625	397 65	2 59951	455 .47	2 65846
15	294 .58	2.46920	344 .62	2.53735	398 .58	2.60052	456 .47	2.65941
16	295 .38	2.47038	345 .49	2.53844	399 .52	2.60154	457 .47	2.66036
17	296 .18	2.47156	346 .36	2.53953	400 .45	2.60255	458 .47	2.66131
18	296 .99	2.47274	347 .23	2.54062	401 .38	2.60357	459 .47	2.66225
19	297 .79	2.47392	348 .10	2.54170	402 .32	2.60458	460 .47	2.66320
20	298 .60	2.47509	348 .97	2.54279	403 .26	2.60559	461 .47	2.66414
21	299 .40	2.47626	349 .84	2.54387	404 .20	2.60660	462 .48	2.66509
22	300 .21	2.47743	350 .71	2.54496	405 .14	2.60760	463 .48	2.66603
23	301 .02	2.47860	351 .58	2.54604	406 .08	2.60861	464 .48	2.66697
24	301 .83	2.47977	352 .46	2.54712	407 .02	2.60961	465 .49	2.66791
25	302 .64	2.48094	353 .34	2.54820	407 .96	2.61062	466 .50	2.66885
26	303 .46	2.48210	354 .22	2.54928	408 .90	2.61162	467 .51	2.66979
27	304 .27	2.48327	355 .10	2.55035	409 .84	2.61263	468 .52	2.67073
28	305 .09	2.48443	355 .98	2 55143	410 .79	2.61363	469 .53	2.67166
29	305 .90	2.48559	356 .86	2.55250	411 .73	2.61463	470 .54	2.67260
30	306 .72	2.48675	357 .74	2.55358	412 .68	2.61563	471 .55	2.67353
31	307 .54	2.48790	358 .62	2.55465	413 .63	2.61662	472 .57	2.67446
32	308 .36	2.48906	359 .51	2.55572	414 .59	2.61762	473 .58	2.67539
33	309 .18	2.49021	360 .39	2.55679	415 .54	2.61861	474 .60	2.67633
34	310 .00	2 49136	361 .28	2.55785	416 .49	2.61961	475 .62	2.67726
35	310 .82	2.49251	362 .17	2.55892	417 .44	2.62060	476 .64	2.67818
36	311 .65	2.49366	363 .07	2.55999	418 .40	2.62159	477 .65	2.67911
37	312 .47	2.49481	363 .96	2.56105	419 .35	2.62258	478 .67	2.68004
38	313 .30	2.49596	364 .85	2.56211	420 .31	2.62357	479 .70	2.68097
39	314 .12	2.49711	365 .75	2.56317	421 .27	2.62456	480 .72	2.68189
40	314 .95	2.49825	366 .64	2.56423	422 .23	2.62555	481 .74	2.68281
41	315 .78	2.49939	367 .53	2.56529	423 .19	2.62654	482 .77	2 68374
42	316 .61	2.50053	368 .42	2.56635	424 .15	2.62752	483 .79	2.68466
43	317 .44	2.50167	369 .31	2.56740	425 .11	2.62850	484 .82	2.68558
44	318 .27	2.50281	370 .21	2.56846	426 .07	2.62949	485 .85	2.68650
45	319 .10	2.50394	371 .11	2.56951	427 .04	2.63047	486 .88	2.68742
46	319 .94	2.50508	372 .01	2.57056	428 .01	2.63145	487 .91	2.68834
47	320 .78	2.50621	372 .91	2.57161	428 .97	2.63243	488 .94	2.68926
48	321 .62	2.50734	373 .82	2.57266	429 .93	2.63341	489 .97	2.69017
49	322 .45	2.50847	374 .72	2.57371	430 .90	2.63438	491 .01	2.69109
50	323 .29	2.50960	375 .62	2.57476	431 .87	2.63536	492 .05	2.69201
51	324 .13	2.51073	376 .52	2.57580	432 .84	2.63634	493 .08	2.69292
52	324 .97	2.51185	377 .43	2.57685	433 .82	2.63731	494 .12	2.69383
53	325 .81	2.51298	378 .34	2.57789	434 .79	2.63828	495 .15	2 69474
54	326 .66	2.51410	379 .26	2.57893	435 .76	2.63925	496 .19	2.69565
55	327 .50	2.51522	380 .17	2.57997	436 .73	2.64022	497 .23	2.69656
56	328 .35	2.51634	381 .08	2.58101	437 .71	2.64119	498 .28	2.69747
57	329 .19	2.51746	381 .99	2.58205	438 .69	2.64216	499 .32	2.69838
58	330 .04	2.51858	382 .90	2.58309	439 .67	2.64313	500 .37	2 69929
59	330 .89	2.51969	383 .82	2.58412	440 .65	2.64410	501 .41	2.70019
60	331 .74	2.52081	384 .74	2.58516	441 .63	2.64506	502 .46	2.70109

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}.$$

S	16^m m	16^m log m	17^m m	17^m log m	18^m m	18^m log m	19^m m	19^m log m
0	502″.5	2.70109	567″.2	2.75373	635″.9	2.80336	708″.4	2.85029
1	503 .5	2.70200	568 .3	2.75458	637 .0	2.80416	709 .7	2.85105
2	504 .5	2.70291	569 .4	2.75543	638 .2	2.80496	710 .9	2.85181
3	505 .6	2.70381	570 .5	2.75628	639 .4	2.80576	712 .1	2.85257
4	506 .6	2.70471	571 .6	2.75713	640 .6	2.80656	713 .4	2.85333
5	507 .7	2.70561	572 .8	2.75798	641 .7	2.80736	714 .6	2.85409
6	508 .8	2.70651	573 .9	2.75883	642 .9	2.80816	715 .9	2.85485
7	509 .8	2.70741	575 .0	2.75967	644 .1	2.80896	717 .1	2.85561
8	510 .9	2.70830	576 .1	2.76052	645 .3	2.80976	718 .4	2.85636
9	511 .9	2.70920	577 .2	2.76136	646 .5	2.81056	719 .6	2.85712
10	513 .0	2.71010	578 .4	2.76220	647 .7	2.81135	720 .9	2.85787
11	514 .0	2.71099	579 .5	2.76304	648 .9	2.81215	722 .1	2.85863
12	515 .1	2.71188	580 .6	2.76388	650 .0	2.81295	723 .4	2.85938
13	516 .1	2.71278	581 .7	2.76472	651 .2	2.81375	724 .6	2.86014
14	517 .2	2.71367	582 .9	2.76556	652 .4	2.81454	725 .9	2.86089
15	518 .3	2.71456	584 .0	2.76640	653 .6	2.81533	727 .2	2.86164
16	519 .3	2.71545	585 .1	2.76724	654 .8	2.81612	728 .4	2.86239
17	520 .4	2.71634	586 .2	2.76808	656 .0	2.81691	729 .7	2.86314
18	521 .5	2.71723	587 .4	2.76892	657 .2	2.81770	730 .9	2.86389
19	522 .5	2.71811	588 .5	2.76976	658 .4	2.81849	732 .2	2.86464
20	523 .6	2.71900	589 .6	2.77059	659 .6	2.81928	733 .5	2.86539
21	524 .7	2.71989	590 .8	2.77143	660 .8	2.82007	734 .7	2.86614
22	525 .7	2.72077	591 .9	2.77226	662 .0	2.82086	736 .0	2.86689
23	526 .8	2.72165	593 .0	2.77309	663 .2	2.82165	737 .3	2.86763
24	527 .9	2.72254	594 .2	2.77392	664 .4	2.82244	738 .5	2.86838
25	529 .0	2.72342	595 .3	2.77476	665 .6	2.82322	739 .8	2.86912
26	530 .0	2.72430	596 .5	2.77559	666 .8	2.82401	741 .1	2.86987
27	531 .1	2.72518	597 .6	2.77642	668 .0	2.82479	742 .3	2.87061
28	532 .2	2.72606	598 .7	2.77724	669 .2	2.82558	743 .6	2.87136
29	533 .3	2.72694	599 .9	2.77807	670 .4	2.82636	744 .9	2.87210
30	534 .3	2.72781	601 .0	2.77890	671 .6	2.82714	746 .2	2.87284
31	535 .4	2.72869	602 .2	2.77973	672 .8	2.82792	747 .4	2.87358
32	536 .5	2.72957	603 .3	2.78056	674 .1	2.82870	748 .7	2.87432
33	537 .6	2.73044	604 .5	2.78138	675 .3	2.82948	750 .0	2.87506
34	538 .7	2.73132	605 .6	2.78220	676 .5	2.83026	751 .3	2.87580
35	539 .7	2.73219	606 .8	2.78302	677 .7	2.83104	752 .6	2.87654
36	540 .8	2.73306	607 .9	2.78385	678 .9	2.83182	753 .8	2.87728
37	541 .9	2.73393	609 .1	2.78467	680 .1	2.83260	755 .1	2.87802
38	543 .0	2.73480	610 .2	2.78549	681 .3	2.83337	756 .4	2.87876
39	544 .1	2.73567	611 .4	2.78631	682 .6	2.83414	757 .7	2.87949
40	545 .2	2.73654	612 .5	2.78713	683 .8	2.83492	759 .0	2.88023
41	546 .3	2.73741	613 .7	2.78795	685 .0	2.83570	760 .2	2.88096
42	547 .4	2.73827	614 .8	2.78877	686 .2	2.83648	761 .5	2.88170
43	548 .5	2.73914	616 .0	2.78958	687 .4	2.83725	762 .8	2.88243
44	549 .5	2.74001	617 .2	2.79040	688 .7	2.83802	764 .1	2.88317
45	550 .6	2.74087	618 .3	2.79121	689 .9	2.83879	765 .4	2.88390
46	551 .7	2.74173	619 .5	2.79203	691 .1	2.83957	766 .7	2.88463
47	552 .8	2.74259	620 .6	2.79284	692 .4	2.84034	768 .0	2.88536
48	553 .9	2.74346	621 .8	2.79366	693 .6	2.84111	769 .3	2.88610
49	555 .0	2.74432	623 .0	2.79447	694 .8	2.84188	770 .6	2.88683
50	556 .1	2.74518	624 .1	2.79528	696 .0	2.84264	771 .9	2.88756
51	557 .2	2.74604	625 .3	2.79609	697 .3	2.84341	773 .1	2.88828
52	558 .3	2.74690	626 .5	2.79690	698 .5	2.84418	774 .5	2.88901
53	559 .4	2.74775	627 .6	2.79771	699 .7	2.84495	775 .7	2.88974
54	560 .5	2.74861	628 .8	2.79852	701 .0	2.84571	777 .1	2.89047
55	561 .7	2.74047	630 .0	2.79933	702 .2	2.84648	778 .4	2.89119
56	562 .8	2.75032	631 .2	2.80014	703 .5	2.84724	779 .7	2.89192
57	563 .9	2.75118	632 .3	2.80094	704 .7	2.84801	781 .0	2.89265
58	565 .0	2.75203	633 .5	2.80175	705 .9	2.84877	782 .3	2.89337
59	566 .1	2.75288	634 .7	2.80255	707 .1	2.84953	783 .6	2.89409
60	567 .2	2.75373	635 .9	2.80336	708 .4	2.85029	784 .9	2.89481

$$m = \frac{2 \sin^2 \frac{1}{2}t}{\sin 1''}.$$

S	20m m	20m log m	21m m	21m log m	22m m	22m log m	23m m	23m log m
0	784″.9	2.89481	865″.3	2.93717	949″.6	2.97755	1037″.8	3.01613
1	786.2	2.89554	866.6	2.93786	951.0	2.97820	1039.3	3.01675
2	787.5	2.89626	868.0	2.93855	952.4	2.97886	1040.8	3.01738
3	788.8	2.89698	869.4	2.93923	953.8	2.97952	1042.3	3.01801
4	790.1	2.89770	870.8	2.93992	955.3	2.98017	1043.8	3.01864
5	791.4	2.89842	872.1	2.94061	956.7	2.98083	1045.3	3.01926
6	792.7	2.89914	873.5	2.94129	958.2	2.98148	1046.8	3.01989
7	794.0	2.89986	874.9	2.94198	959.6	2.98214	1048.3	3.02052
8	795.4	2.90058	876.3	2.94266	961.1	2.98279	1049.8	3.02114
9	796.7	2.90130	877.6	2.94335	962.5	2.98344	1051.3	3.02177
10	798.0	2.90202	879.0	2.94403	963.9	2.98410	1052.8	3.02239
11	799.3	2.90274	880.4	2.94471	965.4	2.98475	1054.3	3.02302
12	800.7	2.90346	881.8	2.94540	966.9	2.98540	1055.9	3.02364
13	802.0	2.90417	883.2	2.94608	968.3	2.98605	1057.4	3.02426
14	803.3	2.90489	884.6	2.94676	969.8	2.98670	1058.9	3.02489
15	804.6	2.90560	886.0	2.94744	971.2	2.98735	1060.4	3.02551
16	806.0	2.90632	887.4	2.94812	972.7	2.98800	1062.0	3.02613
17	807.3	2.90703	888.8	2.94880	974.1	2.98865	1063.5	3.02675
18	808.6	2.90774	890.2	2.94948	975.5	2.98930	1065.0	3.02737
19	809.9	2.90845	891.6	2.95016	977.0	2.98995	1066.5	3.02799
20	811.3	2.90917	893.0	2.95084	978.5	2.99060	1068.1	3.02861
21	812.6	2.90988	894.4	2.95152	979.9	2.99125	1069.6	3.02923
22	813.9	2.91058	895.8	2.95219	981.4	2.99189	1071.1	3.02985
23	815.2	2.91129	897.2	2.95287	982.9	2.99254	1072.6	3.03047
24	816.6	2.91200	898.6	2.95355	984.4	2.99319	1074.2	3.03109
25	817.9	2.91271	900.0	2.95422	985.8	2.99383	1075.7	3.03171
26	819.2	2.91342	901.4	2.95490	987.3	2.99448	1077.2	3.03232
27	820.5	2.91413	902.8	2.95557	988.8	2.99512	1078.7	3.03294
28	821.9	2.91484	904.2	2.95625	990.3	2.99576	1080.3	3.03356
29	823.2	2.91555	905.6	2.95692	991.8	2.99641	1081.8	3.03417
30	824.6	2.91625	907.0	2.95759	993.2	2.99705	1083.3	3.03479
31	825.9	2.91696	908.4	2.95827	994.7	2.99769	1084.8	3.03540
32	827.3	2.91766	909.8	2.95894	996.2	2.99834	1086.4	3.03602
33	828.6	2.91837	911.2	2.95961	997.6	2.99898	1087.9	3.03663
34	829.9	2.91907	912.6	2.96028	999.1	2.99962	1089.5	3.03725
35	831.2	2.91977	914.0	2.96095	1000.6	3.00026	1091.0	3.03787
36	832.6	2.92048	915.5	2.96162	1002.1	3.00090	1092.6	3.03848
37	833.9	2.92118	916.9	2.96229	1003.5	3.00154	1094.1	3.03909
38	835.3	2.92188	918.3	2.96296	1005.0	3.00218	1095.7	3.03970
39	836.6	2.92258	919.7	2.96362	1006.5	3.00282	1097.2	3.04031
40	838.0	2.92328	921.1	2.96429	1008.0	3.00346	1098.8	3.04092
41	839.3	2.92398	922.5	2.96496	1009.4	3.00409	1100.3	3.04153
42	840.7	2.92468	923.9	2.96563	1010.9	3.00473	1101.9	3.04214
43	842.0	2.92538	925.3	2.96630	1012.4	3.00537	1103.4	3.04275
44	843.4	2.92608	926.8	2.96696	1013.9	3.00600	1105.0	3.04336
45	844.7	2.92677	928.2	2.96763	1015.4	3.00664	1106.5	3.04397
46	846.1	2.92747	929.6	2.96829	1016.9	3.00728	1108.1	3.04458
47	847.5	2.92817	931.0	2.96896	1018.4	3.00791	1109.6	3.04519
48	848.9	2.92886	932.4	2.96962	1019.9	3.00855	1111.2	3.04580
49	850.2	2.92956	933.8	2.97028	1021.4	3.00918	1112.7	3.04641
50	851.6	2.93026	935.2	2.97095	1022.8	3.00981	1114.3	3.04701
51	852.9	2.93096	936.6	2.97161	1024.3	3.01045	1115.8	3.04762
52	854.3	2.93164	938.1	2.97227	1025.8	3.01108	1117.4	3.04823
53	855.7	2.93233	939.5	2.97293	1027.3	3.01171	1118.9	3.04883
54	857.1	2.93303	940.9	2.97359	1028.8	3.01234	1120.5	3.04944
55	858.4	2.93372	942.3	2.97425	1030.3	3.01298	1122.0	3.05004
56	859.8	2.93441	943.8	2.97491	1031.8	3.01361	1123.6	3.05065
57	861.1	2.93510	945.2	2.97557	1033.3	3.01424	1125.1	3.05125
58	862.5	2.93579	946.6	2.97623	1034.8	3.01487	1126.7	3.05185
59	863.9	2.93648	948.1	2.97689	1036.3	3.01550	1128.3	3.05246
60	865.3	2.93717	949.6	2.97755	1037.8	3.01613	1129.9	3.05306

$$m = \frac{2 \sin^2 \frac{1}{2}t}{\sin 1''}.$$

S	24^m		25^m		26^m		27^m	
	m	log m	m	log m	m	log m	m	log m
0	1129″.9	3.05306	1225″.9	3.08848	1325″.9	3.12252	1429″.7	3.15526
1	1131 .4	3.05366	1227 .5	3.08906	1327 .6	3.12307	1431 .4	3.15580
2	1133 .0	3.05426	1229 .2	3.08964	1329 .3	3.12363	1433 .2	3.15633
3	1134 .6	3.05487	1230 .8	3.09022	1331 .0	3.12418	1435 .0	3.15686
4	1136 .2	3.05547	1232 .5	3.09079	1332 .7	3.12474	1436 .7	3.15740
5	1137 .8	3.05607	1234 .1	3.09137	1334 .4	3.12529	1438 .5	3.15793
6	1139 .3	3.05667	1235 .7	3.09195	1336 .1	3.12585	1440 .3	3.15847
7	1140 .9	3.05727	1237 .3	3.09252	1337 .8	3.12640	1442 .1	3.15900
8	1142 .5	3.05787	1239 .0	3.09310	1339 .5	3.12695	1443 .9	3.15953
9	1144 .0	3.05847	1240 .6	3.09367	1341 .2	3.12751	1445 .6	3.16007
10	1145 .6	3.05907	1242 .3	3.09425	1342 .9	3.12806	1447 .4	3.16060
11	1147 .2	3.05966	1243 .9	3.09482	1344 .6	3.12861	1449 .2	3.16113
12	1148 .8	3.06026	1245 .6	3.09540	1346 .3	3.12916	1451 .0	3.16166
13	1150 .4	3.06086	1247 .2	3.09597	1348 .0	3.12971	1452 .8	3.16220
14	1152 .0	3.06146	1248 .9	3.09655	1349 .7	3.13026	1454 .5	3.16273
15	1153 .6	3.06205	1250 .5	3.09712	1351 .4	3.13081	1456 .3	3.16326
16	1155 .2	3.06265	1252 .2	3.09769	1353 .2	3.13136	1458 .1	3.16379
17	1156 .8	3.06324	1253 .8	3 09826	1354 .9	3.13191	1459 .9	3.16432
18	1158 .3	3.06384	1255 .5	3.09883	1356 .6	3.13246	1461 .6	3.16485
19	1159 .9	3.06444	1257 .1	3.09941	1358 .3	3.13301	1463 .4	3.16538
20	1161 .5	3.06503	1258 .8	3.09998	1360 .1	3.13356	1465 .2	3.16591
21	1163 .1	3.06562	1260 .5	3.10055	1361 .8	3.13411	1466 .9	3.16643
22	1164 .7	3.06622	1262 .2	3.10112	1363 .5	3.13466	1468 .7	3.16696
23	1166 .3	3.06681	1263 .8	3.10169	1365 .2	3.13521	1470 .5	3.16749
24	1167 .9	3.06740	1265 .5	3.10226	1367 .0	3.13576	1472 .3	3.16802
25	1169 .5	3.06800	1267 .1	3.10283	1368 .7	3.13631	1474 .1	3.16855
26	1171 .1	3.06859	1268 .8	3.10340	1370 .4	3.13686	1475 .9	3.16907
27	1172 .7	3.06918	1270 .5	3.10396	1372 .1	3.13740	1477 .7	3.16960
28	1174 .3	3.06977	1272 .1	3.10453	1373 .9	3.13795	1479 .5	3.17013
29	1175 .9	3.07036	1273 .7	3.10510	1375 .6	3.13850	1481 .3	3.17066
30	1177 .5	3.07095	1275 .4	3.10567	1377 .3	3.13904	1483 .1	3.17118
31	1179 .1	3.07154	1277 .1	3.10623	1379 .0	3.13959	1484 .9	3.17170
32	1180 .7	3.07213	1278 .8	3.10680	1380 .8	3.14013	1486 .7	3.17223
33	1182 .3	3.07272	1280 .4	3.10737	1382 .5	3.14068	1488 .5	3.17275
34	1183 9	3.07331	1282 .1	3.10793	1384 .2	3.14122	1490 .3	3.17327
35	1185 .5	3.07389	1283 .8	3.10850	1385 .9	3 14177	1492 .1	3.17380
36	1187 .1	3.07448	1285 .5	3.10906	1387 .7	3.14231	1493 .9	3.17433
37	1188 .7	3.07507	1287 .1	3.10963	1389 .4	3.14285	1495 .7	3.17485
38	1190 .3	3.07566	1288 .8	3.11019	1391 .2	3.14340	1497 .5	3.17538
39	1191 .9	3.07625	1290 .5	3.11076	1392 .9	3.14394	1499 .3	3.17590
40	1193 .5	3.07683	1292 .2	3.11132	1394 .7	3.14448	1501 .1	3.17642
41	1195 .1	3.07742	1293 .8	3.11188	1396 .4	3.14502	1502 .9	3.17694
42	1196 .7	3.07801	1295 .5	3.11245	1398 .2	3.14557	1504 .7	3 17746
43	1198 .3	3.07859	1297 .2	3.11301	1399 .9	3.14611	1506 .5	3.17799
44	1199 .9	3.07918	1298 .9	3.11357	1401 .7	3.14665	1508 .4	3.17851
45	1201 .5	3 07976	1300 .5	3.11413	1403 .4	3.14719	1510 .2	3.17903
46	1203 .1	3.08035	1302 .2	3.11469	1405 .2	3.14773	1512 .0	3.17955
47	1204 .7	3.08093	1303 .9	3.11525	1406 .9	3.14827	1513 .8	3.18007
48	1206 .4	3.08151	1305 .6	3.11582	1408 .7	3.14881	1515 .6	3.18059
49	1208 .0	3.08210	1307 .3	3.11638	1410 .4	3.14935	1517 .4	3.18111
50	1209 .6	3.08268	1309 .0	3.11694	1412 .2	3.14989	1519 .2	3.18163
51	1211 .2	3.08326	1310 .7	3.11750	1413 .9	3.15043	1521 .0	3.18215
52	1212 .9	3.08384	1312 .4	3.11805	1415 .7	3.15096	1522 .9	3.18267
53	1214 .5	3.08442	1314 .1	3.11861	1417 .4	3.15150	1524 .7	3 18319
54	1216 .1	3.08501	1315 .7	3.11917	1419 .2	3.15204	1526 .5	3.18371
55	1217 .7	3.08559	1317 .4	3.11973	1420 .9	3.15258	1528 .3	3.18422
56	1219 .4	3.08617	1319 .1	3.12029	1422 .7	3.15312	1530 .2	3.18474
57	1221 .0	3.08675	1320 .8	3.12085	1424 .4	3.15365	1532 .0	3.18526
58	1222 .6	3.08733	1322 .5	3.12140	1426 .2	3.15419	1533 .8	3.18578
59	1224 .2	3.08791	1324 .2	3.12196	1427 .9	3.15472	1535 .6	3.18629
60	1225 .9	3.08848	1325 .9	3.12252	1429 .7	3.15526	1537 .5	3.18681

$$m = \frac{2\sin^2 \frac{1}{2}t}{\sin 1''}.$$

S	28ᵐ		29ᵐ		30ᵐ		31ᵐ	
	m	log *m*	*m*	log *m*	*m*	log *m*	*m*	log *m*
0	1537″.5	3.18681	1649″.1	3.21725	1764″.6	3.24665	1884″.0	3.27509
1	1539 .3	3.18733	1651 .0	3.21775	1766 .6	3.24713	1886 .1	3.27556
2	1541 .1	3.18784	1652 .9	3.21825	1768 .5	3.24761	1888 .1	3.27602
3	1542 .9	3.18836	1654 .8	3.21875	1770 .5	3.24810	1890 .1	3.27649
4	1544 .8	3.18887	1656 .7	3.21924	1772 .5	3.24858	1892 .1	3.27695
5	1546 .6	3.18939	1658 .6	3.21974	1774 .4	3.24906	1894 .2	3.27742
6	1548 .4	3.18990	1660 .5	3.22024	1776 .4	3.24954	1896 .2	3.27788
7	1550 .2	3.19042	1662 .4	3.22073	1778 .4	3.25002	1898 .2	3.27835
8	1552 .1	3.19093	1664 .3	3.22123	1780 .3	3.25050	1900 .3	3.27881
9	1553 .9	3.19145	1666 .2	3.22172	1782 .3	3.25098	1902 .3	3.27928
10	1555 .8	3.19196	1668 .1	3.22222	1784 .3	3.25146	1904 .3	3.27974
11	1557 .6	3.19247	1670 .0	3.22272	1786 .2	3.25194	1906 .4	3.28020
12	1559 .5	3.19299	1671 .9	3.22321	1788 .2	3.25242	1908 .4	3.28067
13	1561 .3	3.19350	1673 .8	3.22371	1790 .2	3.25289	1910 .4	3.28113
14	1563 .2	3.19401	1675 .7	3.22420	1792 .1	3.25337	1912 .5	3.28159
15	1565 .0	3.19452	1677 .6	3.22470	1794 .1	3.25385	1914 .5	3.28206
16	1566 .9	3.19503	1679 .5	3.22519	1796 .1	3.25433	1916 .5	3.28252
17	1568 .7	3.19554	1681 .4	3.22568	1798 .1	3.25480	1918 .6	3.28298
18	1570 .5	3.19606	1683 .3	3.22618	1800 .0	3.25528	1920 .6	3.28344
19	1572 .4	3.19657	1685 .2	3.22667	1802 .0	3.25576	1922 .7	3.28390
20	1574 .3	3.19708	1687 .2	3.22716	1804 .0	3.25624	1924 .7	3.28437
21	1576 .1	3.19759	1689 .1	3.22766	1806 .0	3.25671	1926 .8	3.28483
22	1578 .0	3.19810	1691 .0	3.22815	1808 .0	3.25719	1928 .8	3.28529
23	1579 .8	3.19861	1692 .9	3.22864	1809 .9	3.25766	1930 .9	3.28575
24	1581 .7	3.19912	1694 .8	3.22913	1811 .9	3.25814	1932 .9	3.28621
25	1583 .5	3.19962	1696 .7	3.22963	1813 .9	3.25862	1934 .9	3.28667
26	1585 .3	3.20013	1698 .6	3.23012	1815 .9	3.25909	1937 .0	3.28713
27	1587 .2	3.20064	1700 .5	3.23061	1817 .9	3.25957	1939 .1	3.28759
28	1589 .1	3.20115	1702 .5	3.23110	1819 .9	3.26004	1941 .1	3.28805
29	1590 .9	3.20166	1704 .4	3.23159	1821 .9	3.26051	1943 .2	3.28851
30	1592 .7	3.20216	1706 .3	3.23208	1823 .8	3.26099	1945 .2	3.28897
31	1594 .6	3.20267	1708 .2	3.23257	1825 .8	3.26146	1947 .3	3.28943
32	1596 .5	3.20318	1710 .2	3.23306	1827 .8	3.26194	1949 .3	3.28988
33	1598 .3	3.20369	1712 .1	3.23355	1829 .8	3.26241	1951 .4	3.29034
34	1600 .2	3.20419	1714 .0	3.23404	1831 .8	3.26288	1953 .4	3.29080
35	1602 .1	3.20470	1715 .9	3.23453	1833 .8	3.26336	1955 .5	3.29126
36	1604 .0	3.20520	1717 .9	3.23501	1835 .8	3.26383	1957 .6	3.29172
37	1605 .9	3.20571	1719 .8	3.23550	1837 .8	3.26430	1959 .6	3.29217
38	1607 .7	3.20621	1721 .7	3.23599	1839 .8	3.26477	1961 .7	3.29263
39	1609 .6	3.20672	1723 .6	3.23648	1841 .8	3.26524	1963 .8	3.29309
40	1611 .5	3.20722	1725 .6	3.23697	1843 .8	3.26571	1965 .8	3.29354
41	1613 .3	3.20772	1727 .5	3.23745	1845 .8	3.26619	1967 .9	3.29400
42	1615 .2	3.20822	1729 .5	3.23794	1847 .8	3.26666	1970 .0	3.29446
43	1617 .1	3.20873	1731 .5	3.23843	1849 .8	3.26713	1972 .0	3.29491
44	1619 .0	3.20924	1733 .4	3.23891	1851 .8	3.26760	1974 .1	3.29537
45	1620 .8	3.20974	1735 .3	3.23940	1853 .8	3.26807	1976 .2	3.29582
46	1622 .7	3.21024	1737 .2	3.23988	1855 .8	3.26854	1978 .2	3.29628
47	1624 .6	3.21075	1739 .2	3.24037	1857 .8	3.26901	1980 .3	3.29673
48	1626 .5	3.21125	1741 .2	3.24086	1859 .8	3.26948	1982 .4	3.29719
49	1628 .3	3.21175	1743 .1	3.24134	1861 .9	3.26995	1984 .5	3.29764
50	1630 .2	3.21225	1745 .1	3.24182	1863 .9	3.27042	1986 .5	3.29810
51	1632 .1	3.21275	1747 .0	3.24231	1865 .9	3.27088	1988 .6	3.29855
52	1634 .0	3.21325	1749 .0	3.24279	1867 .9	3.27135	1990 .7	3.29900
53	1635 .9	3.21375	1750 .9	3.24328	1869 .9	3.27182	1992 .8	3.29946
54	1637 .7	3.21425	1752 .8	3.24376	1871 .9	3.27229	1994 .8	3.29991
55	1639 .6	3.21475	1754 .8	3.24424	1873 .9	3.27276	1996 .9	3.30036
56	1641 .5	3.21525	1756 .8	3.24473	1876 .0	3.27322	1999 .0	3.30082
57	1643 .4	3.21575	1758 .7	3.24521	1878 .0	3.27369	2001 .1	3.30127
58	1645 .3	3.21625	1760 .7	3.24569	1880 .0	3.27416	2003 .2	3.30172
59	1647 .2	3.21675	1762 .6	3.24617	1882 .0	3.27462	2005 .3	3.30217
60	1649 .1	3.21725	1764 .6	3.24665	1884 .0	3.27509	2007 .4	3.30262

TABLE VIII B.

$$n = \frac{2 \sin^4 \frac{1}{2} t}{\sin 1''}.$$

t	n	log n	t	n	log n
0m 0s	0''.00		22m 0s	2''.19	0.3396
1 0	0 .00	4.9706	10	2 .25	0.3527
2 0	0 .00	6.1747	20	2 .32	0.3657
3 0	0 .00	6.8791	30	2 .39	0.3786
4 0	0 .00	7.3788	40	2 .46	0.3915
5 0	0 .01	7.7665	50	2 .54	0.4042
6 0	0 .01	8.0832	23 0	2 .61	0.4168
7 0	0 .02	8.3509	10	2 .69	0.4293
8 0	0 .04	8.5829	20	2 .77	0.4418
9 0	0 .06	8.7875	30	2 .85	0.4541
10 0	0 .09	8.9705	40	2 .93	0.4664
11 0	0 .13	9.1360	50	3 .01	0.4786
12 0	0 .19	9.2871	24 0	3 .10	0.4907
12 30	0 .23	9.3580	10	3 .18	0.5027
13 0	0 .27	9.4262	20	3 .27	0.5146
13 30	0 .31	9.4917	30	3 .36	0.5264
14 0	0 .36	9.5549	40	3 .45	0.5382
14 30	0 .41	9.6158	50	3 .55	0.5499
15 0	0 .47	9.6747	25 0	3 .64	0.5615
10	0 .49	9.6939	10	3 .74	0.5730
20	0 .52	9.7128	20	3 .84	0.5845
30	0 .54	9.7316	30	3 .94	0.5959
40	0 .56	9.7502	40	4 .05	0.6072
50	0 .59	9.7686	50	4 .15	0.6184
16 0	0 .61	9.7867	26 0	4 .26	0.6296
10	0 .64	9.8047	10	4 .37	0.6407
20	0 .67	9.8225	20	4 .48	0.6517
30	0 .69	9.8402	30	4 .60	0.6626
40	0 .72	9.8576	40	4 .72	0.6735
50	0 .75	9.8749	50	4 .83	0.6843
17 0	0 .78	9.8920	27 0	4 .96	0.6951
10	0 .81	9.9089	10	5 .08	0.7057
20	0 .84	9.9257	20	5 .20	0.7164
30	0 .88	9.9423	30	5 .33	0.7269
40	0 .91	9.9588	40	5 .46	0.7374
50	0 .95	9.9751	50	5 .60	0.7478
18 0	0 .98	9.9913	28 0	5 .73	0.7582
10	1 .02	0.0072	10	5 .87	0.7685
20	1 .06	0.0231	20	6 .01	0.7787
30	1 .09	0.0388	30	6 .15	0.7889
40	1 .13	0.0544	40	6 .30	0.7990
50	1 .18	0.0698	50	6 .44	0.8090
19 0	1 .22	0.0851	29 0	6 .59	0.8190
10	1 .26	0.1003	10	6 .75	0.8290
20	1 .30	0.1153	20	6 .90	0.8389
30	1 .35	0.1302	30	7 .06	0.8487
40	1 .40	0.1450	40	7 .22	0.8585
50	1 .44	0.1597	50	7 .38	0.8682
20 0	1 .49	0.1742	30 0	7 .55	0.8778
10	1 .54	0.1886	10	7 .72	0.8874
20	1 .60	0.2029	20	7 .89	0.8970
30	1 .65	0.2170	30	8 .06	0.9065
40	1 .70	0.2311	40	8 .24	0.9160
50	1 .76	0.2450	50	8 .42	0.9254
21 0	1 .82	0.2589	31 0	8 .60	0.9347
10	1 .87	0.2726	10	8 .79	0.9440
20	1 .93	0.2862	20	8 .98	0.9533
30	1 .99	0.2997	30	9 .17	0.9625
40	2 .06	0.3131	40	9 .37	0.9716
50	2 .12	0.3264	50	9 .57	0.9807
22m 0s	2 .19	0.3396	32m 0s	9 .77	0.9898

TABLE VIII C.

$$k = \left[\frac{1}{1 - \frac{r}{86400}}\right]^2.$$

Rate.	log k.
− 30s	9.999 6985
29	7085
28	7186
27	7286
26	7387
25	7487
24	7588
23	7688
22	7789
21	7889
20	7990
19	8090
18	8191
17	8291
16	8392
15	8492
14	8593
13	8693
12	8794
11	8894
10	8995
9	9095
8	9196
7	9296
6	9397
5	9497
4	9598
3	9698
2	9799
− 1	9899
0	0.000 0000
+ 1	0101
2	0201
3	0302
4	0402
5	0503
6	0603
7	0704
8	0804
9	0905
10	1005
11	1106
12	1206
13	1307
14	1407
15	1508
16	1608
17	1709
18	1809
19	1910
20	2010
21	2111
22	2212
23	2312
24	2412
25	2513
26	2613
27	2714
28	2814
29	2915
+ 30	0.000 3015